TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

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Lecture 6: Calculus

6.1. Calculus generalizes the process of **taking differences** and **taking sums**. Differences measure **change**, sums explore how quantities **accumulate**. The procedure of taking differences has a limit called **derivative**. The activity of taking sums leads to the **integral**. Sum and difference are dual to each other and related in an intimate way. In this lecture, we look first at the simplest possible setup, where functions are evaluated on integers and where we do not take any limits.

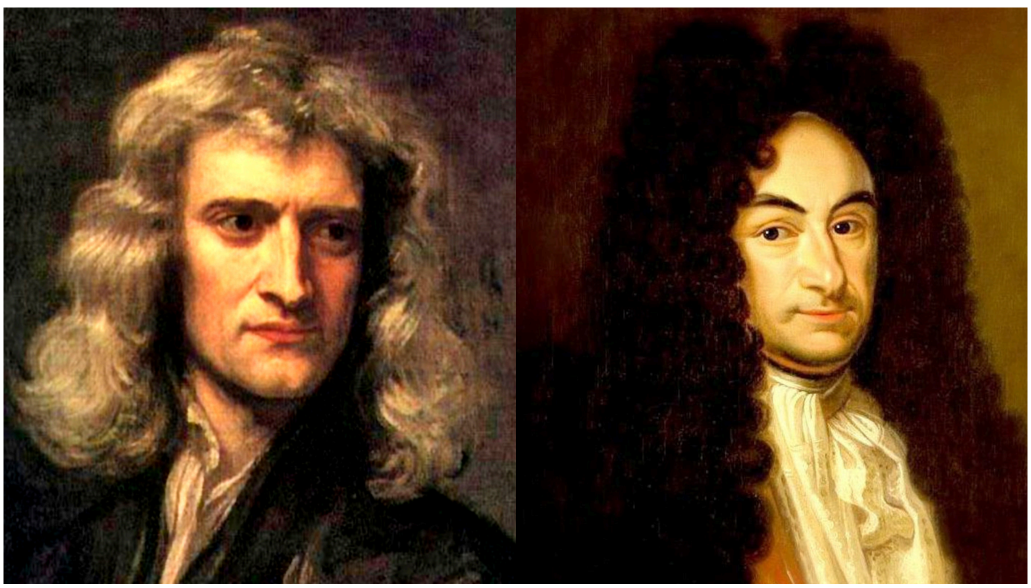


FIGURE 1. Newton and Leibniz

6.2. Several dozen thousand years ago, numbers were represented by units like

$$1, 1, 1, 1, 1, 1, \dots .$$

for example carved in the Ishango bone. It took thousands of years until numbers were represented with symbols like

$$0, 1, 2, 3, 4, \dots .$$

Using the modern concept of **function**, we can say $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$ and mean that the **function** f assigns to an **input** like 1001 an **output** like $f(1001) = 1001$. Define $Df(x) = f(x+1) - f(x)$, the **difference** between two function values. We see that the function $f(x) = x$ satisfies $Df(x) = 1$ for all x. We can also formalize the summation process. If $g(x) = 1$ is the function which is constant 1, then $Sg(x) = g(0) + g(1) + \cdots + g(x-1) = 1 + 1 + \cdots + 1 = x$. We see that $Df = g$ and $Sg = f$.

6.3. If we start with $f(x) = x$ and apply **summation** on that function we get

$$Sf(x) = f(0) + f(1) + f(2) + \cdots + f(x-1) .$$

In our example, we get the values:

$$0, 1, 3, 6, 10, 15, 21, \dots .$$

The new function $g(x) = Sf(x)$ satisfies $g(1) = 1, g(2) = 3, g(2) = 6$, etc. These numbers are called **triangular numbers**. From g we can get back f by taking difference:

$$Dg(x) = g(x+1) - g(x) = f(x) .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$ which indeed is $f(5)$. Finding a formula for the sum $Sf(x)$ is not so easy. Can you do it? When **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, a Mr. Büttner gave the class the task to sum up the first 100 numbers $1+2+\cdots+100$. Gauss found the answer immediately by pairing things up: to add up $1+2+3+\cdots+100$, he would write this as $(1+100)+(2+99)+\cdots+(50+51)$, leading to 50 terms of 101 to get for $x = 101$ the value $g(x) = x(x-1)/2 = 5050$. Taking differences again is easier $Dg(x) = g(x+1) - g(x) = x(x+1)/2 - x(x-1)/2 = x = f(x)$.

6.4. Lets add now the triangular numbers up compute $h = Sg$. We get the sequence

$$0, 1, 4, 10, 20, 35, \dots$$

called the **tetrahedral numbers**. One can stack $h(x)$ balls to build a tetrahedron of side length x. For example, $h(4) = 20$ golf balls are needed to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(n) = n(n-1)(n-2)/6}$. Here is the fundamental theorem of calculus, which is the core of calculus:

$$\boxed{SDf(n) = f(n) - f(0), \quad DSf(n) = f(n) .}$$

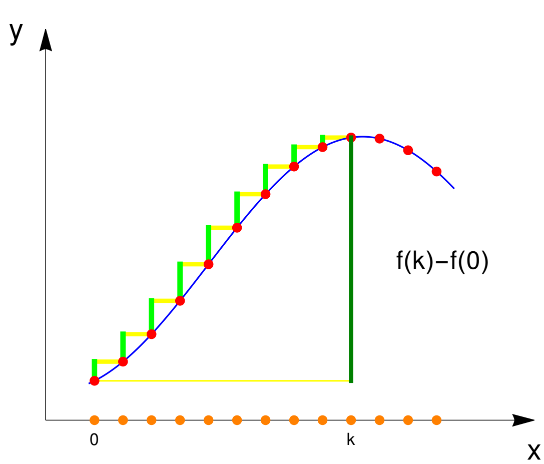
Proof.

$$SDf(n) = \sum_{k=0}^{n-1} [f(k+1) - f(k)] = f(n) - f(0) ,$$

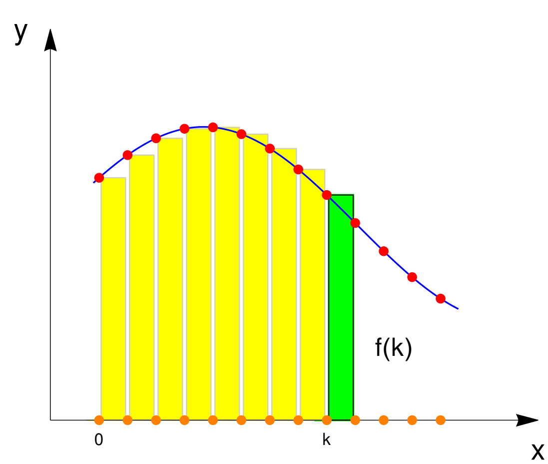
$$DSf(n) = [\sum_{k=0}^{n-1} f(k+1) - \sum_{k=0}^{n-1} f(k)] = f(n) .$$

The process of adding up numbers will lead to the **integral** $\boxed{\int_0^x f(x)\, dx}$. The process of taking differences will lead to the **derivative** $\boxed{\frac{d}{dx} f(x)}$.

$$\boxed{\int_0^x \frac{d}{dt} f(t)\, dt = f(x) - f(0), \qquad \frac{d}{dx} \int_0^x f(t)\, dt = f(x)}$$



Theorem: Sum the differences and get
$SDf(kh) = f(kh) - f(0)$



Theorem: Difference the sum and get
$DSf(kh) = f(kh)$

6.5. If we define the functions $[x]^0 = 1, [x]^1 = n, [x]^2 = x(x-1)/2, [x]^3 = x(x-1)(x-2)/6$ then $D[x] = [1], D[x]^2 = 2[x], D[x]^3 = 3[x]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

The calculus we have just seen, contains the essence of single variable calculus. A major **core idea** is present which will become more powerful and natural if it is used together with the concept of limit.

6.6. Problem: The **Fibonnacci sequence** $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? We take the difference between successive numbers and get a new sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

which is the same sequence again. We can deduce from this recursion that f has the property that $\boxed{Df(x) = f(x-1)}$.

6.7. Problem: Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(x) = Sf(x)$ obtained by summing the first x numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

$$\boxed{\text{We can study differences to understand sums.}}$$

The next problem illustrates this too:

6.8. Problem: Find the next term in the sequence
2 6 12 20 30 42 56 72 90 110 132 . **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132
2	4	6	8	10	12	14	16	18	20	22
2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	0

6.9. Problem: The function $f(n) = 2^n$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \ldots$. It leads to the sequence of numbers

n=	0	1	2	3	4	5	6	7	8	...
f(n)=	1	2	4	8	16	32	64	128	256	...

We can verify that f satisfies the equation $\boxed{Df(x) = f(x)}$ because $Df(x) = 2^{x+1} - 2^x = (2-1)2^x = 2^x$.
This is an important special case of the fact that

$$\boxed{\text{The derivative of the exponential function is the exponential function itself.}}$$

The function 2^x is a special case of the exponential function when the Planck constant is equal to 1. We will see that the relation will hold for any $h > 0$ and also in the limit $h \to 0$, where it becomes the classical exponential function e^x which plays an important role in science.

Calculus has many applications: computing areas, volumes, solving differential equations. It even has applications in arithmetic. Here is an example for illustration. It is a proof that π is irrational. The theorem is due to Johann Heinrich Lambert (1728-1777):

Theorem of Lambert Pi is irrational.

The proof by Ivan Niven is given in a book of Niven-Zuckerman-Montgomery. It originally appeared in 1947 (Ivan Niven, Bull.Amer.Math.Soc. 53 (1947),509). The proof illustrates how calculus can help to get results in arithmetic.

Proof. Assume $\pi = a/b$ with positive integers a and b. For any positive integer n define

$$f(x) = x^n(a - bx)^n/n! \ .$$

We have $f(x) = f(\pi - x)$ and

$$0 \leq f(x) \leq \pi^n a^n/n!(*)$$

for $0 \leq x \leq \pi$. For all $0 \leq j \leq n$, the j-th derivative of f is zero at 0 and π and for $n <= j$, the j-th derivative of f is an integer at 0 and π.

The function

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - ... + (-1)^n f^{(2n)}(x)$$

has the property that $F(0)$ and $F(\pi)$ are integers and $F + F'' = f$. Therefore, $(F'(x)\sin(x) - F(x)\cos(x))' = f\sin(x)$. By the fundamental theorem of calculus, $\int_0^\pi f(x)\sin(x)\,dx$ is an integer. Inequality (*) implies however that this integral is between 0 and 1 for large enough n. For such an n we get a contradiction.

Work problems

6.10. We stack disks onto each other building n layers and count the number of discs. The number sequence we get are called **triangular numbers**.

1	3	6	10	15	21	36	45	...

This sequence defines a **function** on the natural numbers. For example, $f(4) = 10$. Can you find $f(200)$? The task to find this number was given to Carl Friedrich Gauss in elementary school. The 7 year old came up quickly with an answer. How?

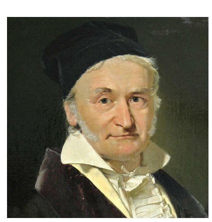


FIGURE 2. Carl-Friedrich Gauss, 1777-1855

6.11. We stack spheres onto each other building n layers and count the number of spheres. The number sequence we get are called **tetrahedral numbers**.

1	4	10	20	35	56	84	120	...

Also this sequence defines a **function**. For example, $g(3) = 10$. But what is $g(100)$? Can we find a formula for $g(n)$? Verify that $g(n) = n(n+1)(n+2)/6$, satisfies $Dg(n) = g(n) - g(n-1) = n(n+1)/2$.

6.12. Problem: Given the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ which satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$?

6.13. Problem: Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, ...$ What sequence is that?

6.14. Problem: Find the next term in the sequence
2 6 12 20 30 42 56 72 90 110 132 .

6.15. Problem: Find the next term in the sequence

$$3, 12, 33, 72, 135, 228, 357, 528, 747, 1020, 1353... .$$

To do so, compute successive derivatives $g = Df$ of f, then $h = Dg$ until you see a pattern.

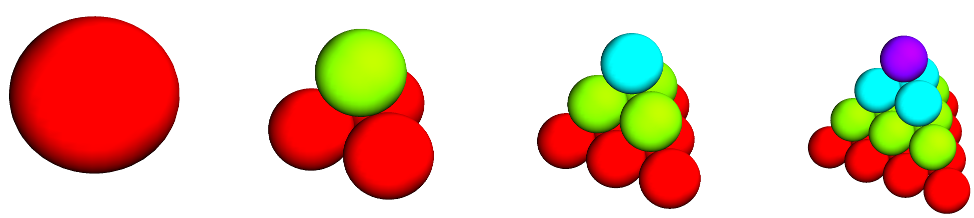


FIGURE 3. Tetrahedral numbers

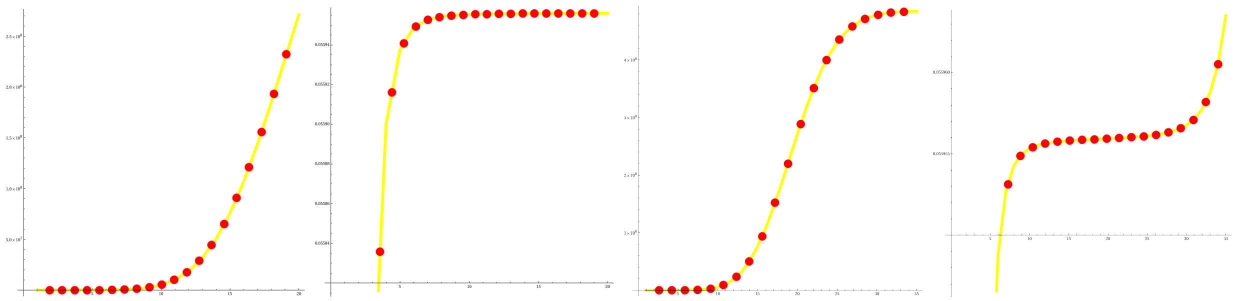
6.16. Henri Poincaré mentions in his "New Methods of Celestial Mechanics" the two sums

$$S_n = \sum_{n=0}^{\infty} \frac{1000^n}{n!}$$

and

$$S_n = \sum_{n=0}^{\infty} \frac{n!}{1000^n} .$$

Why does the first one have a limit? Can you give its value? You might have to look up the series for the exponential function. Why does the second series not have a limit? Just take some large n and see what the terms are you are summing up. Experimental evidence would rule the first to be divergent and the second to be convergent. The experiments would be too extreme in Poincaré's example. Therefore, we replace 1000 with 20. Lets look at the first 20 values of S_n



6.17. The harmonic series
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$
does not converge. Lets see why.

a) Why is the third and forth therm together larger than 1/2?
$$\frac{1}{3}+\frac{1}{4}\,.$$
b) Why is the sum of the fifth up to eighth term larger than 1/2?
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\,.$$
c) How do we continue the argument?
Experimentally, the series seems to stay bounded. To get to 100, we would need $e^{100} = 10^{43}$ steps. But the universe is only 10^{17} seconds old.

6.18. Here are three of the **Zeno paradoxa**.
1) "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."
2) "That which is in locomotion must arrive at the half-way stage before it arrives at the goal."
3) "If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless." Can you reformulate and resolve these paradoxa?

6.19. How would you summarize calculus in one paragraph? What are the 10 most important results for you in calculus? **Eli Maor** expresses it well in his book "The facts on file: Calculus Handbook, 2003":

"Over the past 25 years or so, the typical college calculus textbook has grown from a modest 350-page book to a huge volume of some 1,200 pages, with thousands of exercises, special topics, interviews with career mathematicians, 10 or more appendixes, and much, much more. But as the old adage goes, more is not always better. The enormous size and sheer volume of these monsters (not to mention their weight!) have made their use a daunting task. Both student and instructor are lost in a sea of information, not knowing which material is important and which can be skipped. As if the study of calculus is not a challenge already, these huge texts make the task even more difficult."

6.20. *Here are some pioneers of single variable calculus and calculus teaching. here are some important mathematicians for single variable calculus. Can you find more?*
Zeno of Elea *490-430 Notion of derivative*
Democritus *460-370 Cone and Pyramid. Atomic structure of matter*
Eudoxus *408-355 BC method of exhaustion*
Archimedes *287-212 BC area of disc, volume of sphere*
Johannes Kepler *1571-1630, velocity and acceleration*
Rene Descartes *1596-1650, tangents, rule of signs*
Bonaventura Cavalieri *1598-1647 Cavalieri principle*

Pierre de Fermat *1601-1665 Maxima, Integral of power function*
John Wallis *1616-1703 integral calculus with* x^a*, infinite series*
Christiaan Huygens *1629-1695 Waves, gravity,*
Blaise Pascal *1623-1662, expectation, Pascal triangle*
Isaac Barrow *1630-1677 Calculating tangents*
James Gregory *1638-1675 Fundamental theorem of calculus*
Robert Hooke *1635-1703 Inverse square law*
Isaac Newton *1643-1727 Fluxions = Derivatives*
Gottfried Leibniz *1646-1716 Modern version of calculus*
Michel Rolle *1652-1719 Critic of calculus, Roles theorem*
Guillaume de L'Hospital *1661-1704 Textbook, Hospitals law*
Johann Bernoulli *1667-1748 First textbook (written with L'Hospital)*
Brook Taylor *1685-1731 Taylor series, Difference calculus*
Leonard Euler *1707-1783 Basel problem, analytic geometry*
Maria Agnesi *1718-1799 Textbook in calculus*
Bernard Bolzano *1781-1848 Rigor, intermediate and extremal value theorem*
Augustin Cauchy *1789-1857, continuity, complex calculus*
Karl Weierstrass *1815-1897 Rigorous foundation of calculus*
Bernhard Riemann *1826-1866 Riemann integral, Zeta functions*
Henri Lebesgue *1875-1941 Modern integration*

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