TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

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E-320: Teaching Math with a Historical Perspective **O. Knill, 2010-2021**

Lecture 10: Analysis

10.1. **Analysis** is a science of measure and optimization. As a rather diverse collection of mathematical fields, it contains **real and complex analysis**, **functional analysis**, **harmonic analysis** and **calculus of variations**. Analysis also has close relations to calculus, geometry, topology, probability theory and dynamical systems.

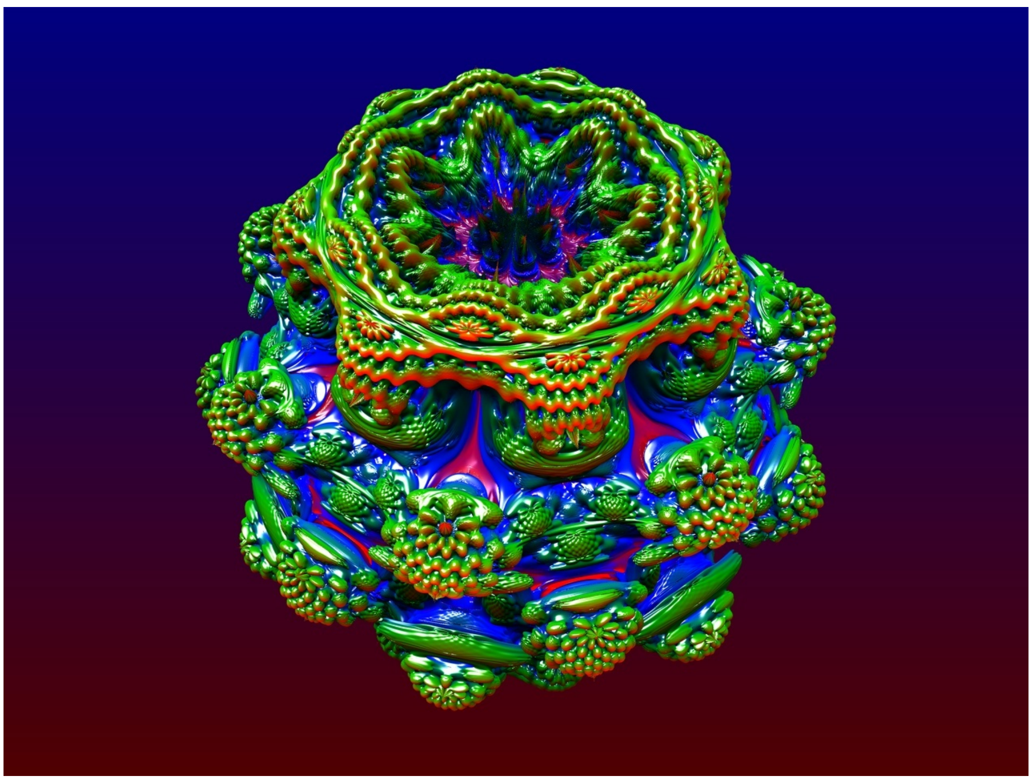


Figure 1. The mysterious Mandelbulb, freshly rendered with the software Mandelbulb 3D.

10.2. We focus here mostly on "the geometry of fractals" which can be seen as part of **dimension theory**. Examples are Julia sets which belong to the subfield of "complex analysis" of "dynamical systems". "Calculus of variations" is illustrated by the Kakeya needle set in "geometric measure theory", "Fourier analysis" appears when looking at functions which have fractal graphs, "spectral theory" as part of functional analysis is represented by the "Hofstadter butterfly". We somehow try to illustrate the vast field of analysis using "pop icons", being aware that it is very much worn out. Being mathematical kitsch should however not disqualify the subject.

10.3. A **fractal** is a set with non-integer dimension. An example is the **Cantor set**, as discovered in 1875 by Henry Smith. Start with the unit interval. Cut the middle third, then cut the middle third from both parts then the middle parts of the four parts etc. The limiting set is the Cantor set. The mathematical theory of fractals belongs to **measure theory** and can also be viewed of a playground for real analysis or topology.

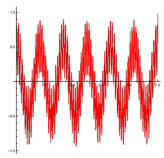
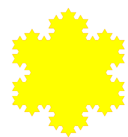
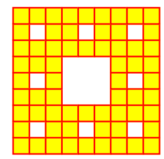
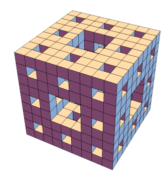
10.4. The term **fractal** had been introduced by Benoit Mandelbrot in 1975, with his book "The fractal geometry of nature". Very important is the notion of **Dimension**. It can be defined in different ways. The simplest is the **box counting definition** which works for most "household fractals": if we need n squares of length r to cover a set, then

$$d = -\log(n)/\log(r)$$

converges to the dimension of the set with $r \to 0$. A curve of length L for example needs L/r squares of length r so that its dimension is 1. A region of area A needs A/r^2 squares of length r to be covered and its dimension is 2. The Cantor set needs to be covered with $n = 2^m$ squares of length $r = 1/3^m$. Its dimension is $-\log(n)/\log(r) = -m\log(2)/(m\log(1/3)) = \log(2)/\log(3)$.

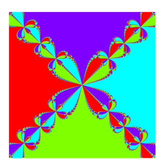
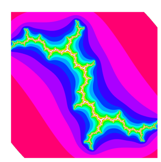
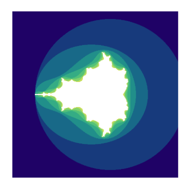
10.5. Examples of fractals (for the first, the dimension is

Weierstrass function	1872
Koch snowflake	1904
Sierpinski carpet	1915
Menger sponge	1926

Complex analysis extends calculus to the complex. It deals with functions $f(z)$ defined in the complex plane. Integration is done along paths. Complex analysis completes the understanding about functions. It also provides more examples of fractals by iterating functions like the **quadratic map** $f(z) = z^2 + c$:

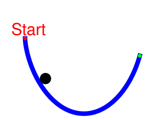
Newton method	1879
Julia sets	1918
Mandelbrot set	1978
Mandelbar set	1989

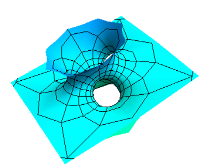
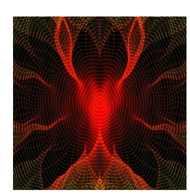





10.6. Particularly famous Julia sets are the **Douady rabbit** and the **dragon**, the **dendrite**, the **airplane**.

10.7. **Calculus of variations** is calculus in infinite dimensions. Taking derivatives is called taking "variations". Historically, it started with the problem to find the curve of fastest fall leading to the **brachistochrone** curve $\vec{r}(t) = (t - \sin(t), 1 - \cos(t))$. In calculus, we find maxima and minima of functions. In calculus of variations, we extremize on much larger spaces. Here are some examples of problems:

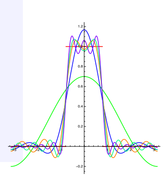
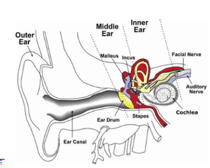
Brachistochrone	1696
Minimal surface	1760
Geodesics	1830
Isoperimetric problem	1838
Kakeya Needle problem	1917

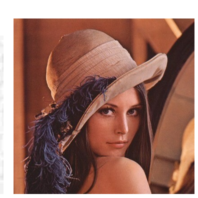



10.8. **Fourier theory** decomposes a function into basic components of various frequencies $f(x) = a_1\sin(x) + a_2\sin(2x) + a_3\sin(3x)...$. The numbers a_i are called Fourier coefficients. Our ear does such a decomposition, when we listen to music. By distinguish different frequencies, our ear produces a Fourier analysis.

Fourier series	1729
Fourier transform (FT)	1811
Discrete FT	Gauss?
Wavelet transform	1930

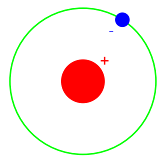
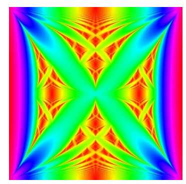
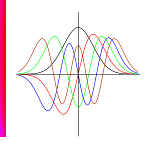



10.9. The Weierstrass function mentioned above is given as a series $\sum_n a^n \cos(\pi b^n x)$ with $0 < a < 1, ab > 1 + 3\pi/2$. The dimension of its graph is believed to be $2 + \log(a)/\log(b)$ which is confirmed for some ranges.

10.10. Spectral theory analyzes linear m.pdf L. The **spectrum** are the real numbers E such that $L - E$ is not invertible. A Hollywood celebrity among all linear m.pdf is the **Matthieu operator** $L(x)_n = x_{n+1} + x_{n-1} + (2 - 2\cos(cn))x_n$: if we draw the spectrum for for each c, we see the **Hofstadter butterfly**. For fixed c the map describes the behavior of an electron in an almost periodic crystal. An other famous system is the **quantum harmonic oscillator**, $L(f) = f''(x) + f(x)$, the **vibrating drum** $L(f) = f_{xx} + f_{yy}$, where f is the amplitude of the drum and $f = 0$ on the boundary of the drum.

Hydrogen atom	1914
Hofstadter butterfly	1976
Harmonic oscillator	1900
Vibrating drum	1680


10.11. All these examples in analysis look unrelated at first. Fractal geometry ties many of them together: spectra are often fractals, minimal configurations have fractal nature, like in solid state physics or in **diffusion limited aggregation** or in other critical phenomena like **percolation** phenomena, **cracks** in solids or the formation of **lighting bolts**

10.12. In Hamiltonian mechanics, minimal energy configurations are often fractals like **Mather theory**. And solutions to minimizing problems lead to fractals in a natural way like when you have the task to turn around a needle on a table by 180 degrees and minimize the area swept out by the needle. The minimal turn leads to a Kakaya set, which is a fractal.

10.13. Finally, lets mention some unsolved problems in analysis. The firs problem is also a problem in number theory: does the **Riemann zeta function** $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ have all nontrivial roots on the axis $\text{Re}(s) = 1/2$? This question is called the **Riemann hypothesis** and is the most important open problem in mathematics. It is an example of a question in **analytic number theory** which also illustrates how analysis has entered into number theory. Some mathematicians think that spectral theory might solve it.

10.14. Also the Mandelbrot set M is not understood yet: the "holy grail" in the field of complex dynamics is the problem whether it M is locally connected. About the Hofstadter butterfly one knows that it has measure zero. What is its dimension?

10.15. An other open question in spectral theory is the "can one hear the sound of a drum" problem which asks whether there are two convex drums which are not congruent but which have the same spectrum.
In the area of calculus of variations, just one problem: how long is the shortest curve in space such that its convex hull (the union of all possible connections between two points on the curve) contains the unit ball.

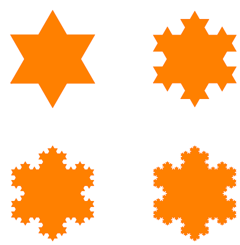
10.16. Here is a shorter summary As Analysis reaches a lot of different areas in mathematics, it is also harder to define. Analysis often extends calculus to areas where traditional calculus does no more apply, like to infinite dimensions or to functions which are not continuous. Sometimes also, it deals with rather strange objects, like fractal geometries or generalized functions. Our goal is to understand **fractals** as they make an appearance in many parts of analysis: spectral theory, complex analysis, harmonic analysis, calculus of variations or functional analysis. Because these fields need some time to learn and explain, the analysis of fractals looks like a nice entry point as it can be seen and the need for a new mathematics is evident. Our story will be mostly pictorial. There is one single formula, we want to understand:

$$\dim(X) = \frac{-\log(n)}{\log(r)} .$$

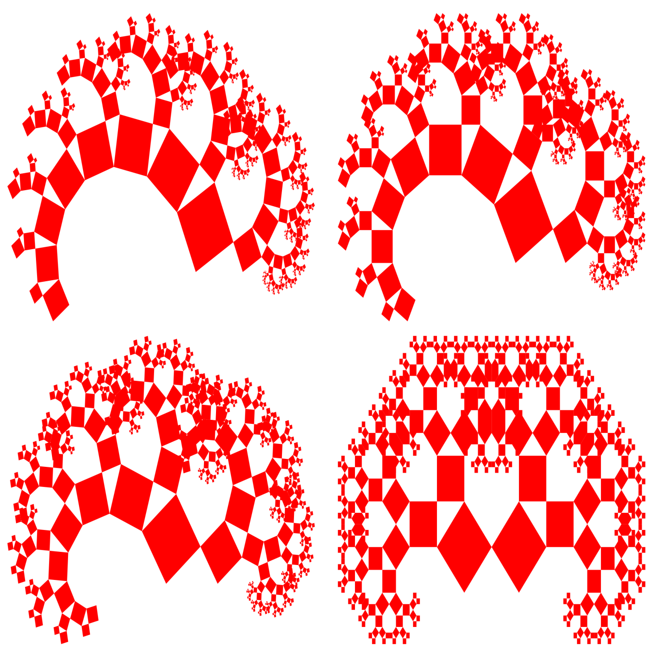
It tells that if we want to find the dimension of an object, we cover it with boxes of size $r > 0$ and count how many boxes we need. Assume this number is n. Dimension is what happens if r goes to zero. The prototype of a fractal is the **Cantor set** which was discovered in 1875 by **Henry Smith**. Start with the unit interval. Cut the middle third, then cut the middle third from both parts then the middle parts of the four parts etc. What is left in the end is the Cantor set for which the dimension is $\log(2)\ \log(3)$.

10.17. Here are again pictures of the more famous fractals: First the Cantor set

The **Koch snowflake** is an example of a fractal with dimension located strictly between 1 and 2. It was first described by the Swedish mathematician **Helge von Koch** (1870-1924) who described it in 1904. It is a simple model for a **snowflake**. There is a simplified version which just is defined over an interval. It is called the **Koch curve**.

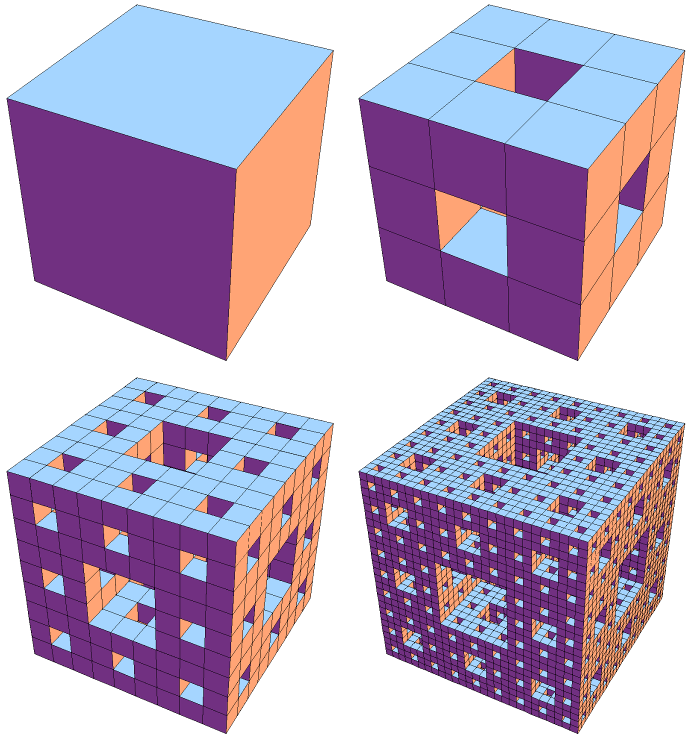


The tree of Pythagoras:



The **tree of Pythagoras** is an example of a fractal with dimension between 1 and 2. We have seen it in our first lecture. The tree of pythagoras inspired antenna designs for small devices.

The **Sierpinski carpet** is a fractal in the plane. Its dimension is $\log(8)/\log(2)$. It was described by **Waclav Sierpinski** in 1916.



The **Menger sponge** is a fractal in space. Its dimension is between 2 and 3. It was first described by Karl Menger (1902-1985). Its dimension is $\log(20)/\log(3)$ which is about 2.7.

10.18. In order to understand the Mandelbrot set we need to look at complex numbers $z = a+ib$ and define complex multiplication

$$(a+ib)(u+iv) = au - bv + (av+bu)i \ .$$

Now look at the function $f(z) = z^2 + c$, where c is a fixed complex number. Start with $z = i$ for example, we get $f(z) = i + c$ and $f^2(z) = f(f(z)) = (i+c)^2 + c$ etc. The **Mandelbrot set** is the set of complex numbers $c = a+ib$ for which $f^n(0)$ stays bounded. The **filled in Julia set** J_c of c is the set of z such that $f^n(z)$ stays bounded. The **Julia set** is the boundary of the filled in Julia set.

For example, for $c = 0$, the map is $f_0(z) = z^2$. Since $|z^n| = |z|^n$ we see that the disc $\{|z| \leq 1\}$ is the filled in Julia set for $c = 0$ and the unit circle $\{|z| = 1\}$ is the Julia set.

10.19. A three dimensional version of the Mandelbrot set is called the **Mandelbulb**. It uses spherical coordinates which have been introduced by Euler.

10.20. The **Hofstadter butterfly** is an example of a fractal which appears in spectral theory. It was first described in 1976 and was popularized in Hofstadters book "Goedel-Escher-Bach".

Work problems

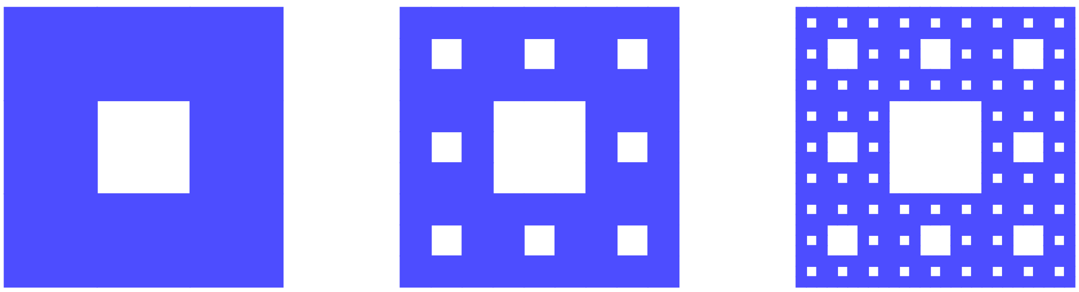
10.21. 1) We want to compute the dimension of various objects in the plane. If we need n squares of side length r to cover an object X. the dimension is defined as

$$d = \frac{-\log(n)}{\log(r)} \text{ when } r \text{ gets zero.}$$

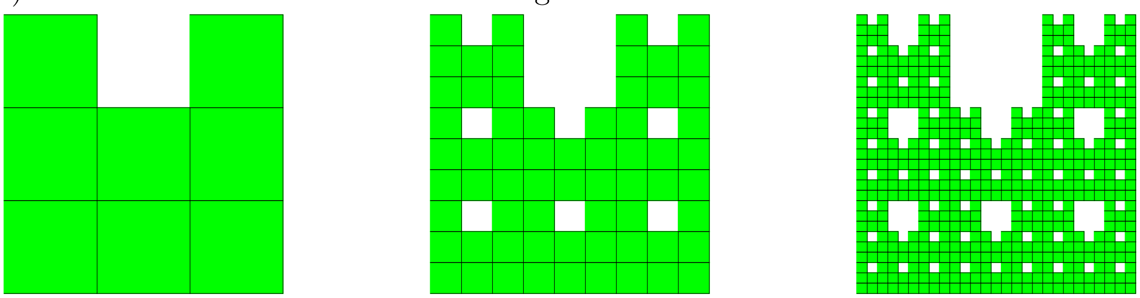
a) Assume a curve is given as the boundary of the unit square. How many squares of length $r = 1/10$ do we need to cover the curve? If we call n this number, what is $-\log(n)/\log(r)$?

b) Assume a region is the unit square. How many squares of length $r = 1/10$ do we need to cover the square? If we call n this number, what is $-\log(n)/\log(r)$?

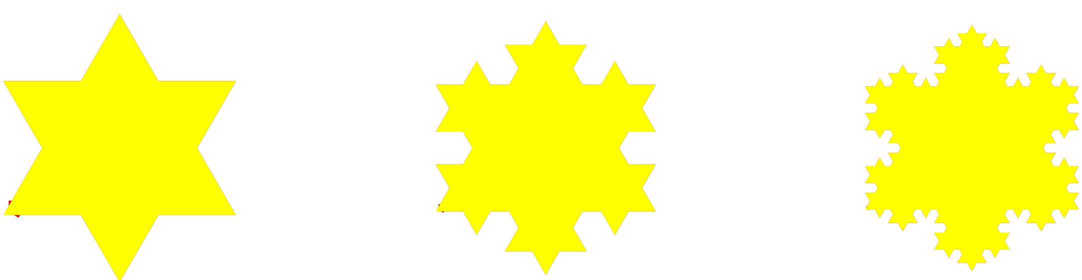
c) The **Sirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k'th step, we need $n = 8^k$ squares of length $r = 1/3^k$ to cover the carpet. What is the dimension?



d) What is the dimension of the following fractal from which we see the first levels of construction?



e) What is the dimension of the Koch snowflake? How large is n, the number of squares we need to covert the flake if the square has size $1/3^k$ assuming that the first triangle has side length 1.



10.22. 2) We want to understand the definition of the Julia sets and the Mandelbrot set. Define

$$T(z) = z^2 + c ,$$

where c is a fixed parameter. The **filled in Julia set** J_c is the set of points z for which the orbit $z, T(z), T(T(z))$... stays bounded. The **Mandelbrot set** M is the set of c for which the point 0 is in the filled in Julia set J_c. It is the set of c such that $0, T(0) = c, T(T(0)) = c^2 + c, T(T(T(0))) = (c^2 + c)^2 + c$... stays bounded. The **Julia set** finally is the boundary of the filled in Julia set.

a) What is the square root of -9?

b) Add $2 + 6i$ with $6 + 8i$.

c) Multiply $2 + 6i$ with $6 + 8i$.

d) What is the length of the complex number $3 + i4$?

e) Assume $c = 2$. Compute the first 3 st.pdf of the orbit of $z = 1$ of the quadratic map.

f) Verify that 0 is inside the Mandelbrot set. Verify that -1 is inside the Mandelbrot set. Verify that i is inside the Mandelbrot set.

g) Verify that 2 is outside the Mandelbrot set. Verify that 1 is outside the Mandelbrot set.

h) Can you verify that $1/4$ is the largest real number in the Mandelbrot set and -2 the smallest real number in the Mandelbrot set?

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