EXPERIMENTS ON ROOTS OF THE MÖBIUS FUNCTION

OLIVER KNILL

Abstract. What is the nature of roots of the function $p \to \mu(f(p))$ on primes p, where μ is the Möbius function and f is a function on integers built by polynomials or exponentials? No roots of μ on Mersenne numbers $f(p) = 2^p - 1$ nor any roots for Fermat number $f(p) = 2^p + 1$ with prime $p > 3$ are known but roots are expected for larger p [2]. Roots seem to appear quite regularly on the even sparser Woodall numbers $n = p2^p - 1$ or Cullen numbers $n = p2^p + 1$ or other exponential sequences like $n = 2^p + 3$ or $n = 2^p - 3$. As for polynomial functions, we are not aware of any nonlinear polynomial f yet for which $p \to \mu(f(p))$ is root-free on the set of primes P.

1. Introduction

1.1. Given a monotone function $f : \mathbb{N} \to \mathbb{N}$, the question whether the image of f hits the set P of primes is classical: for linear functions $f(x) = mx + b$ Dirichlet's theorem on arithmetic progressions assuring $f(\mathbb{N}) \cap P$ is infinite Already for quadratic functions like $f(x) = x^2+1$, we do not know whether $f(\mathbb{N}) \cap P$ is infinite. This is the famous Landau problem. Statistically, we see primes with the regularity predicted by probabilistic considerations. [1] The number of primes of the form $x^2 + 1$ compare the number of primes of the form $p = 4k + 3$. Until 9493060000000 there are currently 160659700695 primes of the form $x = 4k + 3$ and 220556565794 integers x for which $x^2 + 1$ is prime. The ratio 1.37282 has been predicted by Hardy-Littlewood on statistical grounds. It compares the density of Gaussian primes which are reals with the density of Gaussian primes having one imaginary unit.

1.2. A question of similar spirit is to look for numbers with only square free factors. These are roots of the Möbius μ function. Unlike primes

Date: Feb 11, 2022.
2020 *Mathematics Subject Classification.* 11P32, 11R52, 11A41.
Key words and phrases. Möbius function, square free integers .

[1] I compute primes of the form $x^2 + 1$ essentially nonstop since 2016

which thin out, the roots of μ have a well known constant Lyusternik-Schnirelmann density $(1-1/\zeta(2)) = 1-6/\pi^2$. We expect therefore to have more chance to hit a root of μ along a non-linear sequence. We see this happening on polynomials. Sometimes, there are much less than expected and this is because the sequence hits some squares frequently. But we have not encountered that we actually never hit a root along a polynomial sequence. However, we have examples like $f(x) = 7x^2+66$, where the first 90 values $f(p_k)$ are square free. So, while improbable, it is not statistically impossible to see for an exponential $f(x) = 2^x+1$ that the first 100 are all square free even so we might hit a root later on. There is a principle which describes this: the strong law of small numbers [1]. Coincidences for small numbers which might look odd at first will disappear and destroy any initial guess.

1.3. Since the roots of the Möbius μ function has density $1-6/\pi^2 = 0.392073....$ we tend to expect that $\mu(f(p))$ has some roots in the say first 100 primes. There are cases where we see lots of roots like the **middle binomial coefficient function** $f(p) = B(2p,p)$ or then apparently no roots like the Fermat numbers $f(p) = 2^p-1$. On a sequence like $f(p) = p^2+1$ we hit roots quite regularly. We would expect the chance not to hit a root in the first 90 cases to be $0.39...^{90} \sim 2.510^{-37}$. Still, we see no roots for $f(x) = 7x^2+66$ at first. But $p \to \mu(f(p))$ will have roots. Not so frequent but the density appears to be positive.

2. The Möbius root problem

2.1. A positive integer n is a **Möbius root** if is a root of the **Möbius function** μ. The function μ is 0 if a prime factor appears at least twice in the prime factorization and otherwise is 1 or -1, depending on whether there is an even or odd number of prime factors in n.

2.2. The **radical of an integer** is the product of the distinct prime numbers dividing n. We can readily reformulate being a root of μ using the radical function. Möbius roots are the points which are non-fixed points of the radical function rad. Indeed, they satisfy $\text{rad}(n) < n$.

2.3. A general problem is to investigate the **nature of the roots** of the Möbius function on the set $f(P)$, where P is the set of prime numbers and f is a function like a polynomial or exponential function. When making experiments, we see in general roots along polynomials f. $p \to \mu(p^3+p-1)$ for example appears to be zero pretty regularly, suggesting that there are infinitely many roots. A trivial case is $f(p) = p^k$ for $k > 1$, where $\mu(f(p)) = 0$ for all p. An other case is the middle Binomial coefficient $f(p) = \textbf{Binomial}(2p, p)$ in which case all $\mu(f(n)) = 0$ for all $n > 4$.

```
Do[If[MoebiusMu[Prime[k]^3-1]==0,
  Print[k,"  Root found"],Print[k]],
{k,1000}]
```

2.4. For exponential functions $f(P)$, experiments lead to the impression that either there is one or a few small primes q such that q^2 appears regularly along the sequence $k \to f(p(k))$ with the k'th prime $p(k)$ or then that $f(P)$ only has finitely many roots. This is a bit dangerous in the case of exponential functions, as where we can test only small cases.

2.5. Let us put this down with two concrete questions:

Question: Is there a nonlinear polynomial f such that μ has no roots on $f(P)$?

We assumed non-linear to exclude the trivial case $f(p) = p$ which has no roots or $f(p) = p - 1$ which has as roots all primes of the form $4k+3$. There are some cases like $f(p) = g(p)^2$ for which all primes are roots of $\mu(f(p))$.

2.6.

Question: Is there an exponential function f such that μ has no roots on $f(P)$?

The answer could be yes and the Mersenne case $f(x) = 2^x - 1$ could be such an example. But we have to caution that for exponential functions, we only can investigate relatively small primes, due to our limitation to factor integers. It leads to a computational challenge however to see how far we can push this.

3. Mersenne and Fermat numbers

3.1. When we evaluated μ on the Mersenne numbers $n = 2^p - 1$ for prime p which when prime are called **Mersenne primes**, we saw first that they are all fixed points of the radical function $\text{rad}(n) = n$ meaning that they are not roots of the Möbius function. The same appears to happen for the Fermat numbers $n = 2^p + 1$ and prime $p > 3$. We checked

Observation: For Mersenne numbers $n = 2^p - 1$ we see that $n = \text{rad}(n)$ for all primes $p \leq p(58)$.

Observation: For Fermat numbers $n = 2^p + 1$ we see that $n = \text{rad}(n)$ for all primes $3 < p \leq p(61)$.

3.2. This could well be a case for the **strong law of small numbers** [1]. Fermat famously conjectured that $F_n = 2^{2^k} + 1$ is prime for all k, a claim which has been refuted already by Euler. But also in the case of F_n, the computational task remains and basic questions like whether there is an other Fermat prime is open. [2] mentions in section A.3 that it has been conjectured that Fermat numbers $2^p + 1$ are square free and also that the question whether $2^p - 1$ is always square free seems to be an "unanswerable question". Guy tells then on page 14: *it is safe to conjecture that the answer is no.* And asks which **repunits** (repeated units) $(10^p - 1)/9$ are square free.

3.3. A brave soul would conjecture that no Mersenne number $2^p - 1$ for prime p is a root of the Möbius function and that 9 is the only root among Fermat numbers $2^p + 1$ for prime p. [2] however expresses the opinion that these statements are almost certainly false. A direct search is difficult as it requires to factor large integers, a computationally difficult task. We can not factor $2^{293} + 1$ already in reasonable time which corresponds to the 62th prime $p = 62$. Of course, the non-ability to factor the number prevents us to see whether it is a root of μ.

3.4. Roots of the Moebius function could be found by checking whether the square of a prime p factors $2^n + 1$. A strategy therefore can be to pick a prime q and then look whether $(2^p - 1)/q^2$ is an integer for a large number of other primes p. This allows to check a lot of different p. But we have only evidence then and not a proof that $n = 2^p - 1$ is not a root of the Möbius function. Also this can be done with a one line experiment:

4. Related experiments

4.1. The situation with roots of μ on sequences $a^p \pm 1$ seems to be a bit on-or off. We either see roots everywhere or then none. For example $3^p + 1$ or $5^p - 1$ are always divisible by 4 for odd p so that they are all roots of μ. But we have not seen any roots for $f(p) = 5^p + 1$ or $f(p) = 9^p + 1$ or $f(p) = 3^p - 1$.

4.2. For other exponential sequences like $a^p + b$ with larger b, we see also mixed behavior. An example is $2^p - 3$, where we see from time to time the factor 25.

4.3. We can also look at functions $ca^p + b$ with constants a, b, c. Also here, we can see mixed behavior like for $3 \cdot 2^p - 1$ but it appears also here that we observe small square factors.

4.4. We can look for roots also along sparser sequences. Numbers of the form $2^p p - 1$ are called **Woodall numbers**. There are many roots of μ on the Woodall numbers and it is not just on-or-off. We see sometimes factors 9, sometimes factors 25.

4.5. Numbers of the form $2^p p + 1$ are called **Cullen numbers**. We see many roots on the Cullen sequence.

5. Background

5.1. The topic came up when preparing for a number theory lecture in a Harvard extension school course. The question appears to be interesting because it is close to the heart of exciting and popular topics in elementary number theory. We throw it out here in order to see whether anybody has seen the question already. In any case, there is quite a bit of context to the subject:

5.2. **Mersenne numbers** $n = 2^p - 1$ for prime p are of immense interest because they lead the pack in the search for largest primes. The search for new Mersenne primes, Mersenne numbers which are prime, is part of the GIMPS distributed computing program.

5.3. The **Riemann hypothesis** RH would follow if μ was sufficiently random on non-roots. As noted by Mertens, if the law of iterated logarithm holds for μ then the Mertens conjecture holds $M(n) = O(n^{1/2+\epsilon})$ for all $\epsilon > 0$ with $M(n) = \sum_{k=1}^n \mu(k)$ which is equivalent to RH.

5.4. Interesting is also the question whether μ is random on $f(\mathbb{N})$ or $f(P)$, where f is a function. That the polynomial case is already tough is illustrated by the Landau conjecture that there are infinitely many primes of the form $n^2 + 1$. Tougher is already to decide whether there are infinitely many primes of the form say $4p^2 + 1$ where p is an other prime. Of course, a positive answer would settle also the Landau problem.

5.5. As for Fermat numbers $n = 2^p + 1$, they can only be prime for p a power of 2 because of Fermat's little theorem which tells that $2^k + 1$ is divisible by $p - 1$ if p is a prime factor of k.

5.6. The radical function appears in the **abc conjecture** claiming that for co-prime a, b, c satisfying $a + b = c$, there exists for every $\epsilon > 0$ a constant K such that $c < K\mathbf{rad}(abc)^{1+\epsilon}$.

5.7. Finally, the question about roots of Moebius shows how large mathematics is and how little we know about the structure on how numbers are built using primes.

References

[1] R. Guy. The strong law of small numbers. *Amer. Math. Monthly*, 95:697–712, 1988.

[2] R. K. Guy. *Unsolved Problems in Number Theory.* Springer, Berlin, 3 edition, 2004.

Department of Mathematics, Harvard University, Cambridge, MA, 02138