TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

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E-320: Teaching Math with a Historical Perspective **O. Knill, 2010-2022**

Lecture 1: Mathematical roots

1.1. The organization of knowledge is a **taxonomy problem**. Early approaches to sort things out were the **canons of rhetoric** with memory, invention, delivery, style, and arrangement or the **liberal arts and sciences** which combined the **trivium**: grammar, logic and rhetoric with the **quadrivium**: arithmetic, geometry, music, and astronomy. Taxonomies are often historically grown and motivated.

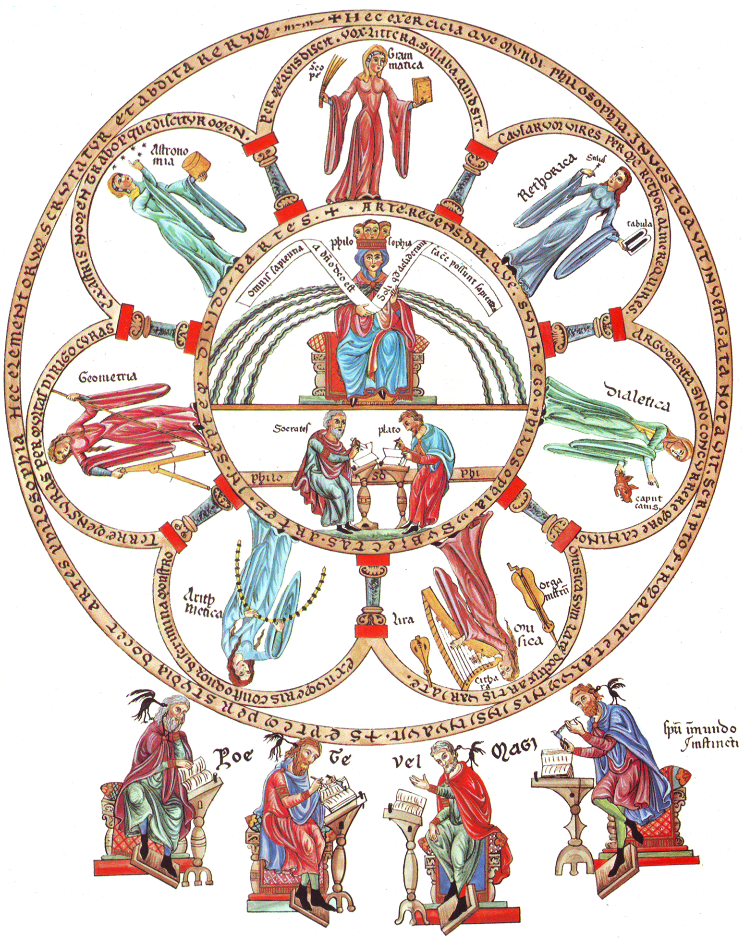

FIGURE 1. "Philosophia at septem artes liberales" (Philosophy within the seven liberal arts and sciences) by Herrad of Landsberg (1125-1195), an 12'th century abbess at the castle of Landsberg. The picture is in the public domain and part of her work "Hortus deliciarum" (Garden of delights).

1.2. A more human-centric approach appears in the **eight ancient roots of mathematics**. It is based on **practical tasks** from daily life. Each of the eight activities is paired with a key area in mathematics:

counting and sorting	**arithmetic**
spacing and distancing	**geometry**
positioning and locating	**topology**
surveying and angulating	**trigonometry**
balancing and weighing	**statics**
moving and hitting	**dynamics**
guessing and judging	**probability**
collecting and ordering	**algorithms**

1.3. Modern mathematics has grown into a rather sophisticated building. Modern classification systems split math into about 100 subfields. To morph the above 8 roots into more modern mathematical areas, we complemented the ancient roots with calculus, numerics and computer science, merge trigonometry with geometry, separate arithmetic into number theory, algebra and arithmetic and turn statics into analysis. Let us call this more modern but still rather arbitrary adaptation the **12 modern roots of Mathematics**:

counting and sorting	**arithmetic**
spacing and distancing	**geometry**
positioning and locating	**topology**
dividing and comparing	**number theory**
balancing and weighing	**analysis**
moving and hitting	**dynamics**
guessing and judging	**probability**
collecting and ordering	**algorithms**
slicing and stacking	**calculus**
operating and memorizing	**computer science**
optimizing and planning	**numerics**
manipulating and solving	**algebra**

1.4. While relating **mathematical areas** with **human activities** is useful, it makes sense to also select specific topics in each of this area. Indeed, any of the subjects has grown itself to a large tree itself. For our course, we will select from each of the 12 topics to build a lecture.

Arithmetic	numbers and number systems
Geometry	invariance, symmetries, measurement, maps
Number theory	Diophantine equations, factorizations
Algebra	algebraic and discrete structures
Calculus	limits, derivatives, integrals
Set Theory	set theory, foundations and formalisms
Probability	combinatorics, measure theory, statistics
Topology	polyhedra, topological spaces, manifolds
Analysis	extrema, estimates, variation, measure
Numerics	numerical schemes, codes, cryptology
Dynamics	differential equations, iteration of maps
Algorithms	computer science, artificial intelligence

1.5. Like any classification, also this division is rather arbitrary and a matter of personal preferences. The **MCC 2020 AMS classification** distinguishes 63 main areas of mathematics. Many of them are broken off into even finer pieces. Additionally, there are fields which relate with other areas of science, like economics, biology or physics:

00 General
01 History and biography
03 Mathematical logic and foundations
05 Combinatorics
06 Lattices, ordered algebraic structures
08 General algebraic systems
11 Number theory
12 Field theory and polynomials
13 Commutative rings and algebras
14 Algebraic geometry
15 Linear algebra; matrix theory
16 Associative rings and algebras
17 Non-associative rings and algebras
18 Category theory, homological algebra
19 K-theory
20 Group theory and generalizations
22 Topological groups, Lie groups
26 Real functions
28 Measure and integration
30 Functions of a complex variable
31 Potential theory
32 Several complex variables, analytic spaces
33 Special functions
34 Ordinary differential equations
35 Partial differential equations
37 Dynamical systems and ergodic theory
39 Difference and functional equations
40 Sequences, series, summability
41 Approximations and expansions
42 Fourier analysis
43 Abstract harmonic analysis
44 Integral transforms, operational calculus
45 Integral equations
46 Functional analysis
47 Operator theory
49 Calculus of variations, optimization
51 Geometry
52 Convex and discrete geometry
53 Differential geometry
54 General topology
55 Algebraic topology
57 Manifolds and cell complexes
58 Global analysis, analysis on manifolds
60 Probability theory, stochastic processes
62 Statistics
65 Numerical analysis
68 Computer science
70 Mechanics of particles and systems
74 Mechanics of deformable solids
76 Fluid mechanics
78 Optics, electromagnetic theory
80 Classical thermodynamics, heat transfer
81 Quantum theory
82 Statistical mechanics, structure of matter
83 Relativity and gravitational theory
85 Astronomy and astrophysics
86 Geophysics
90 Operations research, Programming
91 Game theory, Economics
92 Biology and other natural sciences
93 Systems theory and control
94 Information, communication, circuits
97 Mathematics education

1.6. One can also try to dissect the body of mathematics alone property lines. A good start is to look at arcs which measure **fancy developments** in mathematics. Michael Atiyah identified in the year 2000 the following **six arcs**:

local	and	global
low	and	high dimension
commutative	and	non-commutative
linear	and	nonlinear
geometry	and	algebra
physics	and	mathematics

1.7. Also this choice is of course highly personal. One could easily add 12 other **polarizing** quantities which help to distinguish or parametrize different parts of mathematical areas. The use of ambivalent pairs can be used to slice through the different areas:

regularity	and	randomness
integrable	and	chaotic
invariants	and	perturbative
experimental	and	deductive
polynomial	and	exponential
applied	and	abstract

discrete	and	continuous
existence	and	construction
finite dim	and	infinite dim
topological	and	differential geometric
practical	and	theoretical
axiomatic	and	example based

1.8. An other possibility to refine the fields of mathematics is to **combine** different areas. Examples are **probabilistic number theory**, **algebraic geometry**, **numerical analysis**, **geometric number theory**, **numerical algebra**, **algebraic topology**, **geometric probability**, **algebraic number theory**, **dynamical probability = stochastic processes**. Almost every pair is has become an actual field.

1.9. Finally, let us try to give a short answer to the question: What is Mathematics?

Mathematics is the science of structure.

1.10. The simplicity of this definition is intended. As soon as we include more topics, we actually exclude other topics. "Structure" is a good word, because mathematics is built by them. Examples are algebraic structures, order structures, topological structures, measure theoretical structures or combinations of such structures. For example, one can let algebraic structures act on topological structures leading, a combination which appears in dynamical systems theory or in geometric group theory. Definitions, theorems and examples form a linguistic structure: the definitions fix the vocabulary, the theorems fix a grammar and examples are the novels written. The structure of all of mathematics is a structure too and today often described using language from category theory.

1.11. The goal is to illustrate some of these structures from a historical point of view. Each week, an other topic will be covered. Every week is a new start. Our focus is historical. By doing so we also learn a great deal about how mathematics is learned. One of the key insights we have in education is that the difficulties of the pioneers developing some new material is often mirrored today, when we learn the material. Understanding how learning works is not only important for teachers it is also important for students of mathematics. And history is not only fun, it provides key insight on how mathematics works and provides us with lessons how we might want to proceed and reevaluate.

1.12. This document has evolved over a decade now. It is the 11th time now, this course is taught here at the Harvard extension school.

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