Lecture 1: The Mathematics of Mazes

0 In the first lecture, we discuss a topic with historical and artistic connections. It is the **mathematics of mazes**. Since 2010, when this course was run the first time, we have chosen for the first lecture a topic which is somehow connected to all of mathematics but also relates to research interests of myself at the time. This year, we look at the mathematics of mazes. [1]

1 Informally, a **maze** is a network G of **paths** in a background network K. The obstacles are **hedges** or **walls** G'. We also are given two points A, B in K. The problem is to get from A to B on G without crossing G'. A **labyrinth** is a maze in which G is not branched. The first figure shows an example of a maze. The background graph is a **grid path**. Take a square ruled paper and you have a grid graph! What we have first drawn out is G. The complement G' can be identified by line segments which can not be crossed. Try to find a way from A to B within G. The solution is shown in the second picture. The third picture shows that the complement G' of a maze. Instead of using the complementary edges G in K, we draw the walls perpendicular to them, forming so the obstacles. It is an amazing phenomenon that if you tie together all exits the of G' to a new node, you again get a maze.

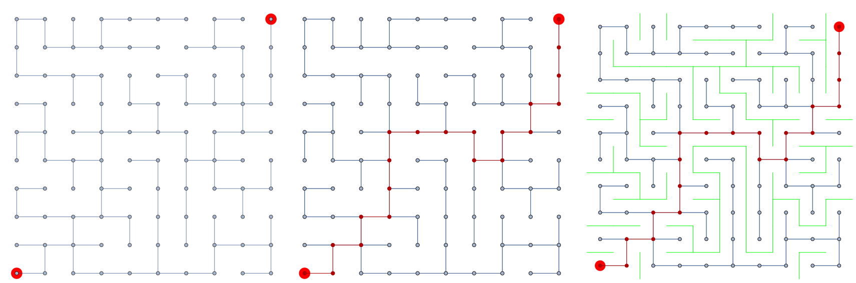

Figure 1: A 10×10 maze is a spanning tree G in a graph K. Here K is a 10×10 grid graph. The solution is a path from A to B. The third picture shows the dual G' consisting of hedges. If all exit paths are tied to a point, this is a again a tree.

2 Before getting to the **mathematics** of mazes, let us look a bit about the **history** and **culture** of mazes. The first "textbook maze" we know is a **Pylos clay tablet** created 1200 BC. Larger mazes have appeared earlier even, as they were used as **architectural tools**, for landscaping or decoration. In pure mathematics, mazes relate to rather serious topics. One principle is the **Jordan curve theorem** which tells that a closed curve in the plane has an interior and exterior. This theorem assures that if the complement of a planar maze contains a non-trivial closed loop, then the dual maze is not connected. Directly related to mazes is also the **Euler polyhedron formula** $V - E + F = 2$, relating the number of vertices, edges and faces of a **polyhedron**. I myself encountered mazes when looking at a quantity called **analytic torsion** which is a rational number defined by an arbitrary **network**.

[1] This document was last updated January 25th 2022.

3 The **lost labyrinth of Egypt** of **Hawara** is located about 80 km south of Cairo. It was built around 1800 BC by pharaoh **Amenemhet III**. The construction was completed by his daughter **Neferuptah**, whose sister **Sobekneferu** was one of the few women that ruled Egypt. According to **Herodots** (who lived 484-425 BC) the **Labyrinth of Minos** featured 3000 rooms in a funeral complex below the pyramid. Based on such descriptions, the Jesuit scholar **Athanasius Kircher** created in 1670 a copper plate picture of the now destroyed labyrinth. About 200 years ago, Egyptologist **Flinders Petrie** discovered the foundations and in 2008, the **Mathara expedition** found more evidence. Just because so much is still unknown, **Egypt's lost labyrinth** has remained an attractive mystery.

4 Labyrinths appear all over the world: examples are the **Labyrinth of the Chartres Cathedral** in France which dates back to 1205, the **11 circuit labyrinth** in San Francisco, the **Damme Priory** in Germany, **the Edge** in South Africa, the **Dunure Castle** in Scotland from the 13th century or the **Old Qing Dynasty Summer Palace** in Beijing or the Gardens shopping mall in Dubai. At Harvard, there is a maze at the **divinity school** located at 45 Museum street.

5 Why are mazes so attractive? It is not only the task of **solving puzzles** within architecture or landscaping, labyrinths were used for **contemplation** and **meditation**. In a religious setting, it a symbol for the human journey. The metaphor is that you can not get from A to B directly. Reaching a goal often happens with detours and passing through dead ends. Mazes also appear in **pop culture**. Early literature cameos are in Greek mythology with **Theseus killing the Minotaur**. You might remember also the **tri-wizard maze** in Harry Potter and the Goblet of Fire or the iconic last scene in Stanley Kubrics **Shining**. A more recent movie is the dystopian science fiction movie trilogy **Maze runner**.

6 The next figure shows a more challenging maze. Mathematically G was generated by producing a random spanning tree in a grid graph K. Only **one line of code** in a modern computer algebra system (here Mathematica) is needed to draw such a random maze:

```
n=30;K=GridGraph[{n,n}];G = Graph[FindSpanningTree[Graph[K,
    EdgeWeight->Table[Random[],{Length[EdgeList[K]]}]]]]
```

This allows us to randomly generate many mazes. By the way, we can exactly compute how many mazes there are of a given geometry. The **matrix tree theorem** gives us in the case of 10×10 mazes already 5694319004079097795957215725765328371712000 different mazes! For 15×15 mazes already there are more than 10^{100} different mazes which can be realized. Also only **one line code** is needed to draw the solution from node 1 attached to $(1,1)$ to node 1600 attached at $(40,40)$. The second part of figure 1 was generated as such.

```
HighlightGraph[G,PathGraph[FindShortestPath[G,1,n^2]]]
```

7 Now try to solve the following maze in Figure 2. You might want to print it (or even better) load it onto an drawing app on a tablet or computer and search for the path. You can also solve the classical mazes shown in Figure 3, Figure 4 (we did that in class) or the central maze in Figure 5.

9 Mathematically, it is convenient to look at the maze as a finite structure. We define a maze G as a **spanning tree** embedded in a planar graph $K = (V, E)$. The **vertices** V are locations and E are the **edges** of the graph. For the mazes in Figure 1 or 2, the background graph is a rectangular grid like a square ruled paper. Since K is drawn on a plane it is called a **planar**

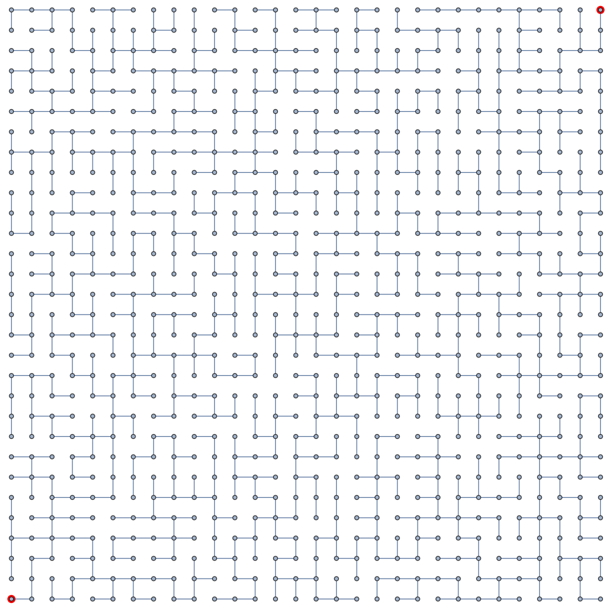

Figure 2: Solve this 30×30 maze by finding a path from A to B. It is now already more challenging to find a path.

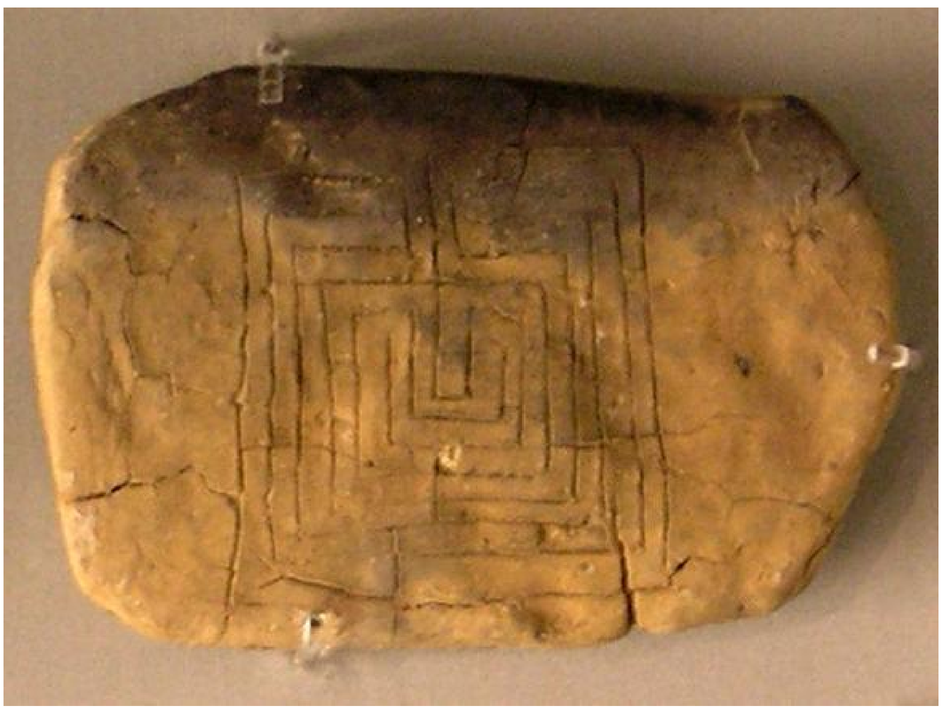

Figure 3: The clay tablet from King Nestor's palace of Pylos representing a classical labyrinth. The Mycenaean palace of Pylos was destroyed by fire around 1200 BC, which accidentally baked the tablet.

Figure 4: The Labyrinth in the Notre-Dames of Chartres was built in 1215. With a diameter of 12.85 meters, it is the largest church's labyrinth constructed in the middle ages. This is a labyrinth, not a maze. Its complement however is a maze.

graph. You can create the maze G by tracing out line segments in K making sure that you reach every crossing V but never produce a closed path. The mazes in Figures 1,2,3 are of this form.

10 It is a crucial assumption to have G a **tree**, meaning that we do not allow maze runners to run in circles. For the purpose of the **maze theorem** discussed below, this is needed. First, we proved the following in class by induction:

Tree lemma: The number vertices minus the number of edges in a tree is 1.

To see this, we know it is true for a tree with two vertices and one edge. Now, whenever you add an other edge, we also add an other vertex. You can build a formal induction proof yourself. You can also see why it is wrong when we allow circles as then we have an opportunity to add an edge without adding an other vertex.

10 To see a planar maze as part of a sphere, just take the outside of the maze and consider it with an additional point. All the paths going to the outside of the maze are attached to this. Now G is a spanning tree in a graph K embedded in a sphere. The graph K divides the sphere into **faces**. In the case of your ruled paper, the faces are all squares. The dual graph K' has as vertices the faces of K. Two faces are connected if they intersect in an edge of K.

11 An important principle which has been seen already in antiquity is the **duality principle**. The dual of K is an other graph K' in which the facets are the vertices and two are connected if they intersect in an edge of K. One has seen that first for polyhedra. If K is the graph of the **cube**, then K' is the graph of the **octahedron**. If K is the graph of the **icosahedron**, then K' is the graph of the **dodecahedron**. Every polyhedron K has a dual polyhedron K' in which the faces F of K are the vertices and where two are connected if they intersected in an edge E.

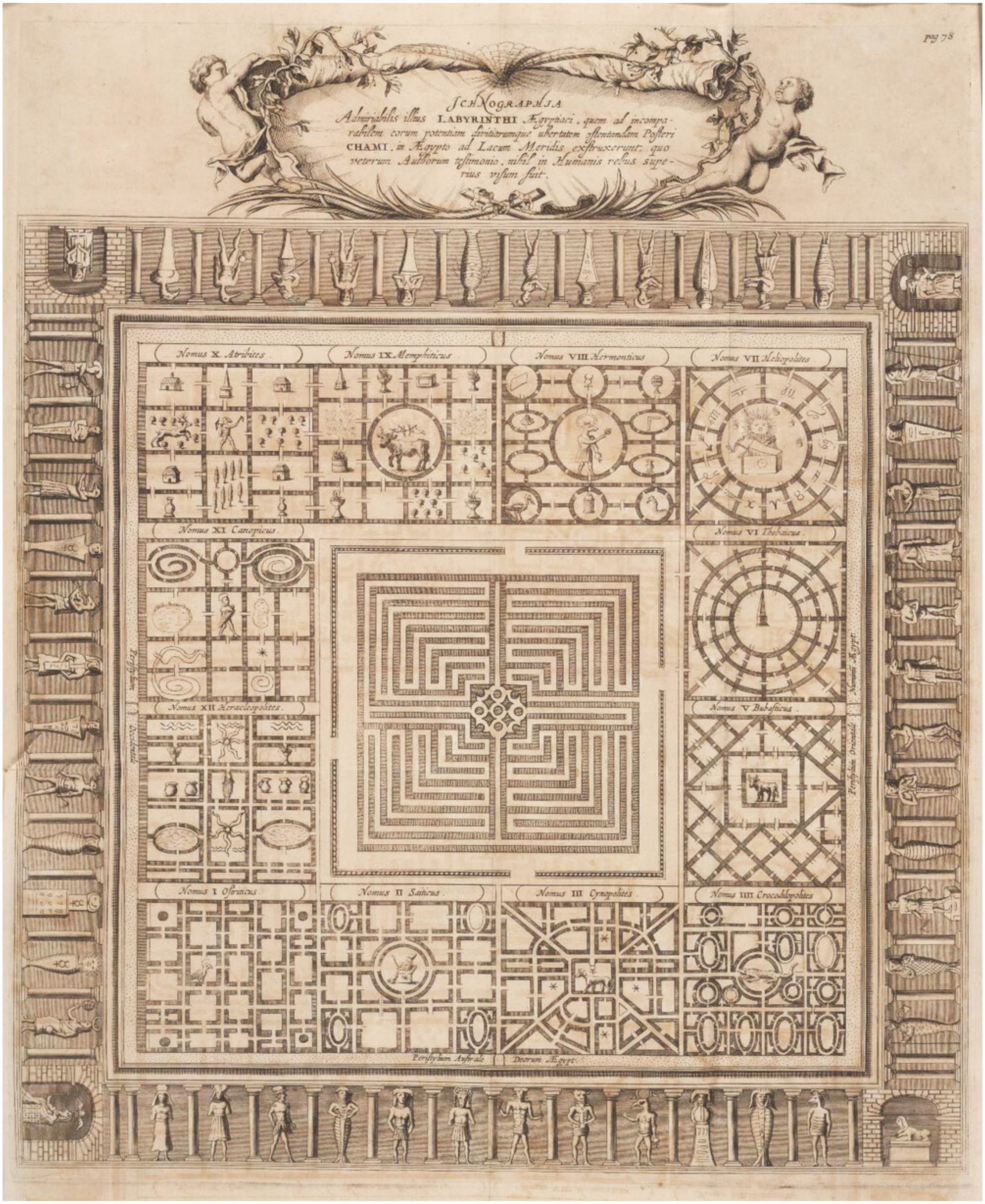

Figure 5: The Jesuit scholar **Athanasius Kircher** created in 1670 a copper plate picture of the lost Egyptian labyrinth. The central square maze is intriguing. We leave it to you to discover why.

10 The following theorem appears implicitly in a book of 1847 by Karl von Staudt who was a student of Gauss. Staudt did not use the language of mazes. But the content was there already.

Duality theorem: if G is a maze in K then G' is a maze in K'.

11 The reason is that G' is again a tree and by construction is spanning because it reaches every face. Why is it a tree? If there was a closed loop on a sphere, then this would divide the sphere into two disjoint regions and the maze G would be disconnected. (This is the Jordan Curve theorem.) But we have assumed that G is a tree which by definition is connected.

Figure 6: A labyrinth G at the Harvard divinity school. You walk on the bright part. The dual of this labyrinth can walked also by staying on the darker stones (the central point does not belong to it). Unlike G, this dual labyrinth G' is a maze and no more a Labyrinth. Try it out.

12 If V, E, F, the number of vertices, edges and faces of a planar K we have the

Euler polyhedron formula $V - E + F = 2$.

Proof: (von Staudt) Since $V(G) = E(G)+1$ and $F(G') = V(G') = E(G')+1$ and $E(G)+E(G') = E(K)$, we have we have $V - 1 + F - 1 = E$. Rearranging gives the formula.

13 By the duality theorem, we also know that the number of spanning trees in G is the same than the number of spanning trees in G'. For example, if G is an icosahedron, then G' is a dodecahedron. Indeed, we can count them. There are 5184000 mazes in both polyhedra.

Amazing Theorem: Number of mazes in K = Number of mazes in K'.

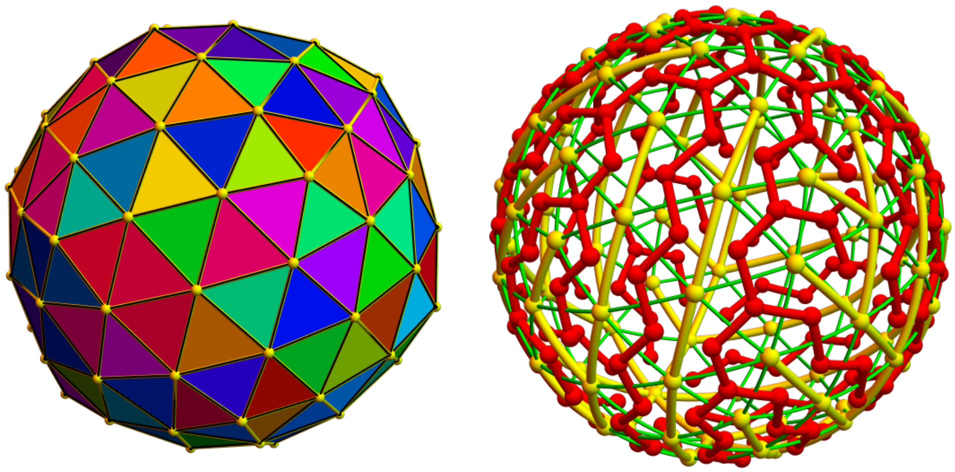

Figure 7: K is a refined Icosahedron. G (yellow) is a random maze. G' (red) is the dual maze in the dual network K'. There are 3099429905094565335814618948040597127493990800000000000000000000 possible mazes on K. This is also the number of mazes in K'.

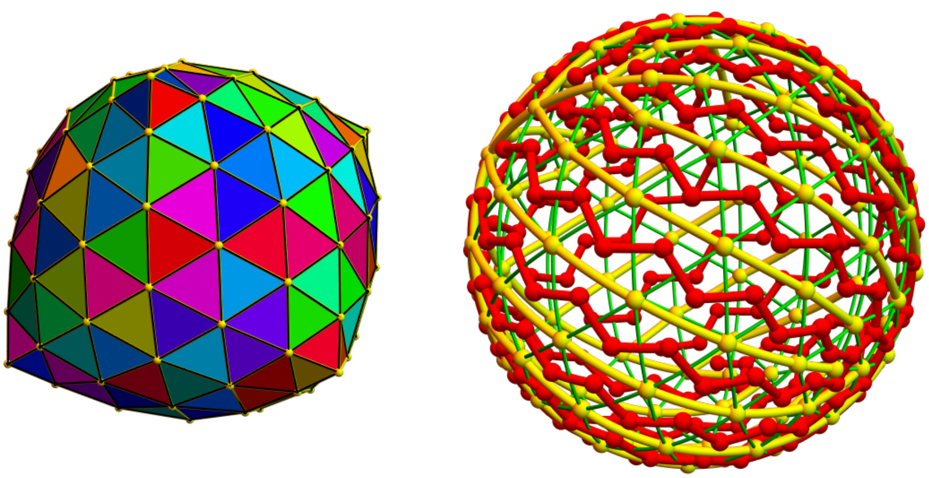

Figure 8: K is a refined Octahedron. G (yellow) is a random maze. G' (red) is the dual maze in the dual network K'. There are 899299297631504199387667566219148533377663103674658512321051081821913088000 possible mazes on K. This is also the number mazes in K'. This is about 10^{63}. **Archimedes** estimated this to be the number of grains of sand in the Aristarchian universe. Modern cosmology estimates the number of particles in the universe to be about 10^{80}. Contemplate about this the next time, you walk a maze.