TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

OLIVER KNILL

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Lecture 3: Geometry

3.1. Geometry is a science of **shape, size and symmetry**. It is one of the oldest mathematical disciplines. It appears in some of the earliest documents of human kind like the Rhind papyrus from 1600 BC. As shapes have aesthetic value, geometry also relates to art and design. While arithmetic focuses on numerical structures, geometry builds, relates and describes metric structures. It is far from limited to shapes that we can physically realize in our world; geometry can also describe objects of large, fractional or infinite dimension. In geometry, not even the sky is the limit.

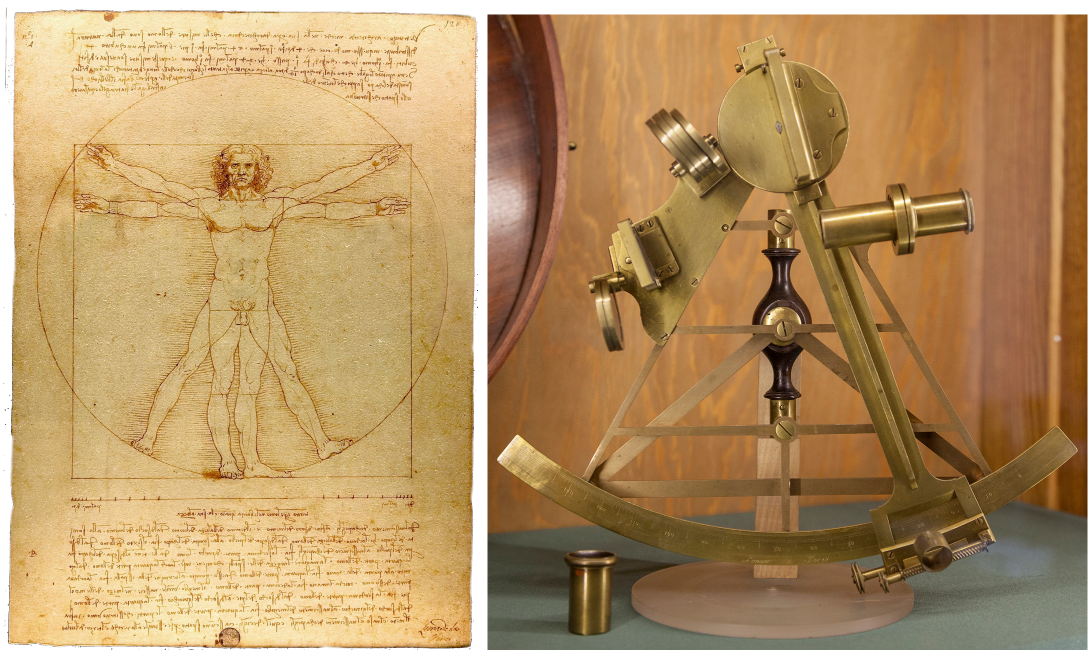


FIGURE 1. The Vitruvian man (1490) by Leonardo da Vinci not only features basic geometric features like angles, proportion, squares and circle, it also relates to art. The sextant (Collegium Matopolski Instytut Kultury) is a symbol for measurements on earth like to determine the position on earth.

3.2. Already the first geometric drawings have relations with arithmetic: the multiplication of two numbers as an **area** of a **shape** that is invariant under a physical **symmetry** rotating the rectangle from a $n \cdot m$ to $m \cdot n$ and so justify a naive commutativity assumption for multiplication. Non-commutative or quantum geometries later generalized geometries even further. A guide is Felix Klein's **Erlanger program** used symmetries to distinguish geometries. Symmetries links geometry also with algebra an other core pillar of mathematics.

3.3. One of the earliest connections between number systems and geometry came through identities like the **Pythagorean triples** $3^2 + 4^2 = 5^2$ which were interpreted and drawn geometrically. It is related to the concept of **right angle** which is the most symmetric angle besides 0. **Symmetry** manifests itself in quantities which are **invariant**. Invariants are most central aspects of geometry and often numerical like Euler characteristic. Generalizing the realm of symmetries leads then to other related fields like topology.

3.4. In this lecture, we first look at a few results related to the first discoveries in geometry. We will start with a few smaller miracles happening in triangles as well as a couple of gems: **Pythagoras**, **Thales**,**Hippocrates**, **Feuerbach**, **Pappus** or **Morley**, **Butterfly**. They all illustrate the importance of symmetry and also extremely approachable as we can realize and see even build the objects.

3.5. Much of geometry is based on our ability to measure **length**, the **distance** between two points. Indeed, the etymology of the word geometry comes from measuring distances on the earth. Geometric thinking became more and more important when human extended their horizon. Discovering new worlds required to travel more efficiently, first by ship. Also modern global positioning systems (GPS) are built on simple geometric ideas like that enough differences between distances between various satellites to you determine your positions. It does not stop with space, because light travels with a universal speed, distances are related to time spans.

3.6. Having a notion of distance $d(A, B)$ between any two points A, B, we can look at the next more complicated object, a **triangle** which is a set A, B, C of 3 points. Given a triangle ABC, are there relations between the three possible distances $a = d(B, C), b = d(A, C), c = d(A, B)$? If we fix the scale by $c = 1$, then $a + b \geq 1, a + 1 \geq b, b + 1 \geq a$. For any pair of parameters (a, b) in this region, there is a triangle. A triangle also defines angles leading to **triangulations**, for example using sextants.

3.7. The concept of distance leads to the notion of **spheres**, the set of points with fixed distance r from a point. In the plane, the sphere is called a **circle**. A natural problem is to find the circumference $L = 2\pi$ of a unit circle, or the **area** $A = \pi$ of a unit disc, the **surface area** $S = 4\pi$ of a unit sphere and the **volume** $V = 4 = \pi/3$ of a unit sphere. Measuring the length of segments on the circle leads to **angle** or **curvature**. Because the circumference of the unit circle in the plane is $L = 2\pi$, angle questions are tied to the number π, which Archimedes already has approximated with rational numbers.

3.8. **Volumes** were among the first quantities, Mathematicians wanted to measure and compute. A problem on **Moscow papyrus** dating back to 1850 BC explains the formula $h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h. Archimedes achieved to compute the **volume of the sphere** by seeing that it is the volume the complement of the cone inside the cylinder which has at height z a slice of area $\pi - \pi z^2$. The volume of the half sphere therefore is volume of the complement of the cone inside the cylinder which is $\pi - \pi/3 = 2\pi/3$. Later, using calculus and analysis even stronger tools to compute volumes became available.

3.9. The first geometric explorations were done in flat two-dimensional space. Highlights are **Pythagoras theorem**, **Thales theorem**, **Hippocrates theorem**, and **Pappus theorem**. Discoveries in planimetry have been made later on: an example is the Feuerbach nine-point theorem from the 19th century. Ancient Greek Mathematics is closely related to history. It starts with **Thales** goes over Euclid's era at 500 BC and ends with the threefold destruction of Alexandria 47 BC by the Romans, 392 by the Christians and 640 by the Muslims.

3.10. Geometry was also a place, where, in the hands of Euclid, the **axiomatic method** entered mathematics. More rigorous deductive proofs appeared at the same time also in number theory, especially in the context of prime numbers. The Pythagorean theorem leads naturally to Pythagorean triples, integers satisfying a Diophantine equation $x^2 + y^2 = z^2$. With axioms and proofs, mathematics became more organized and reliable. The first axioms of geometry were the five axioms of Euclid:

1. Any two distinct points A, B determines a line through A and B.
2. A line segment $[A, B]$ can be extended to a straight line containing the segment.
3. A line segment $[A, B]$ determines a circle containing B and center A.
4. All right angles are congruent.
5. If lines L, M intersect with a third so that inner angles add up to $< \pi$, then L, M intersect.

3.11. **Euclid** wondered whether the fifth postulate can be derived from the first four and called theorems derived from the first four "absolute geometry". Only much later, with **Karl-Friedrich Gauss**, **Janos Bolyai** and **Nicolai Lobachevsky** in the 19'th century, one has realized that for a **hyperbolic space** the 5'th axiom does not hold any more. This was just the beginning for many new geometries. Geometry can be generalized to non-flat, or even much more abstract situations.

3.12. Basic examples of **non-Euclidean geometries** are geometry on a sphere leading to **spherical geometry**. Then there is the geometry on the Poincare disc, an example of a **hyperbolic space**. These geometries are still rather limited. **Riemannian geometry**, which is essential for **general relativity theory** generalizes both concepts to a great extent. An example is the geometry on an arbitrary surface. Curvatures of such spaces can be computed by measuring length alone, which is how long light needs to go from one point to the next. Also Riemannian geometries have been generalized to geometries allowing for rather arbitrary metric spaces.

3.13. An important moment in mathematics was the **merge of geometry with algebra**. This step is often attributed to **René Descartes**. Together with algebra, the subject exploded to an enormous building called **algebraic geometry**. It is a geometry that can also can make use of computer algebra systems.

3.14. Here are some examples of geometries which are determined from the amount of symmetry which is allowed:

Euclidean geometry	Properties invariant under a group of rotations and translations
Affine geometry	Properties invariant under a group of affine transformations
Projective geometry	Properties invariant under a group of projective transformations
Spherical geometry	Properties invariant under a group of rotations
Conformal geometry	Properties invariant under angle preserving transformations
Hyperbolic geometry	Properties invariant under a group of Möbius transformations

3.15. We finally show four pictures about the 4 special points in a triangle. They will be the starting point of our lecture. We will see why in each of these cases, the 3 lines intersect in a common point. It is a manifestation of a **symmetry** present on the space of all triangles. **size** of the distance of intersection points is constant 0 if we move on the space of all triangular **shapes**. It's Geometry!

Beautiful theorems

3.16. The existence of the **centroid** (the intersection of medians) B, the **orthocenter** (the intersection of altitudes) O , the **circumcenter** (the center of the circumcircle) C and the **incenter** (the center of the incircle) I are already not completely obvious. We can show their existence algebraically. The points B, O, C are on a line, the **Euler line**.

3.17. The **Pythagorean theorem** does not need any introduction. Like $E = mc^2$, it is a famous quadratic law: for all right angle triangles of side length a, b, c, the quantity $a^2 + b^2 - c^2$ is zero.

3.18. The **3D Pythagoras theorem** tells that areas of the three sides of the pyramid add up to the base area. It is also called the **Faulhaber extension** or **de Gua theorem**. We can verify it by computing the area of the triangle $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$ which is $|\langle a, -b, 0\rangle \times \langle 0, b, -c\rangle|/2 = |\langle bc, ac, ab\rangle| = (bc^2 + ac^2 + ab^2)/4$.

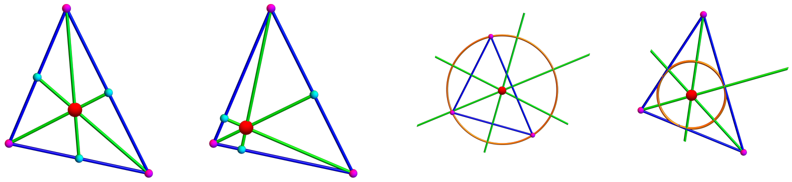


FIGURE 2. The existence of centroid B, orthocenter O, circumcenter C and incenter I are four little miracles for triangles. The points B, O, C are located on a line.

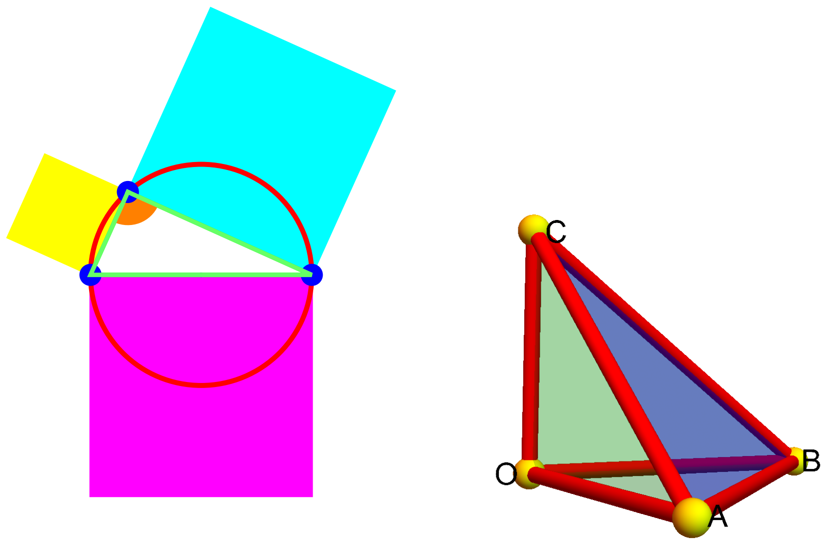


FIGURE 3. The two and three dimensional Pythagoras theorem.

3.19. Thales of Miletus (625 BC -546 BC) showed that if a triangle inscribed in a fixed circle is deformed by moving one of its points on the circle, then the angle at this point does not change.

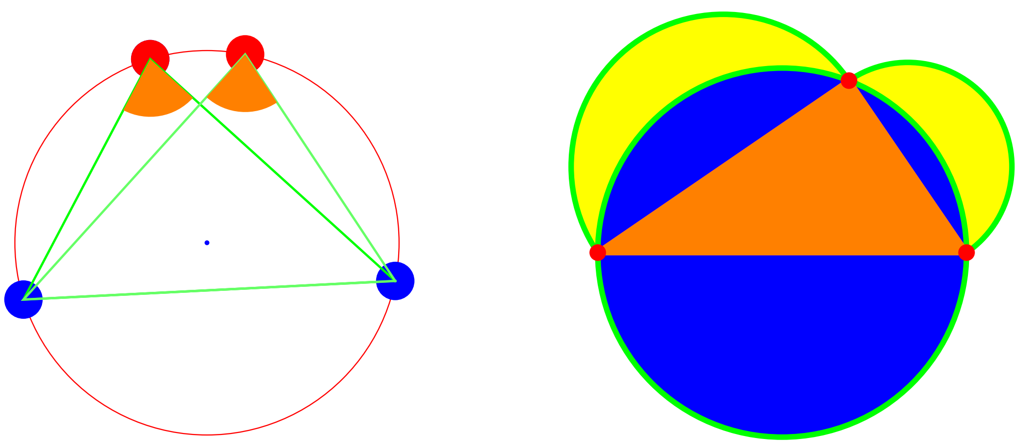
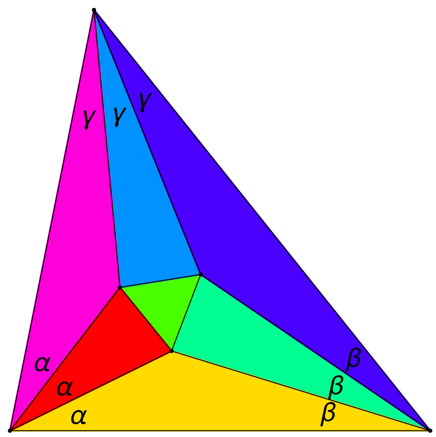


FIGURE 4. Thales theorem assures that the angle does not change when moving the point point on the circle. The Hippocrates theorem equates the areas of the "lunes" (moon shaped figures) with the triangle area.

3.20. The quadrature of the Lune is due to **Hippocrates of Chios** (470 BC - 400 BC). It is the first rigorous quadrature of a curvilinear area. The sum $L + R$ of the area L of the left moon and the area R of the right moon is equal to the area T of the triangle.

3.21. The proof of Morley's miracle we can decompose the triangle with 7 triangles.



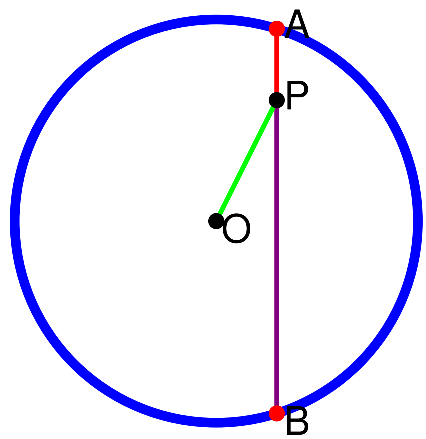


FIGURE 5. Morley's theorem produces an equilateral triangle from angle trisectors of an arbitrary triangle. The Fasskreis theorem assures $1 - |PO|^2 = |PA||PB|$.

3.22. Given a circle of radius 1 and a point P inside the circle. For any line through P which intersects the circle at points A, B we have $1 - |PO|^2 = |PA||PB|$. Proof with Pythagoras. By scaling, translation and rotation we can assume the circle is the unit circle and that the line through the point $P = (a, b)$ is vertical. The intersection points with the circle are then $A = (a, -\sqrt{1-a^2}), B = (a, \sqrt{1-a^2})$. Now

$$|PA||PB| = (\sqrt{1-a^2} - b)(\sqrt{1-a^2} + b) = 1 - a^2 - b^2 = 1 - |PO|^2 .$$

3.23. The **Butterfly theorem** in a triangle is determined by two intersecting line segments in the circle. This defines 5 points, where 4 are on the circle and one point is the intersection point M. The configuration now defines two congruent triangles which intersect in M and which look like the wings of a butterfly. Now draw an additional segment through M. The theorem tells that this segment cuts intervals through the wings which have equal length. It looks like a simple theorem but it is surprisingly hard to prove.

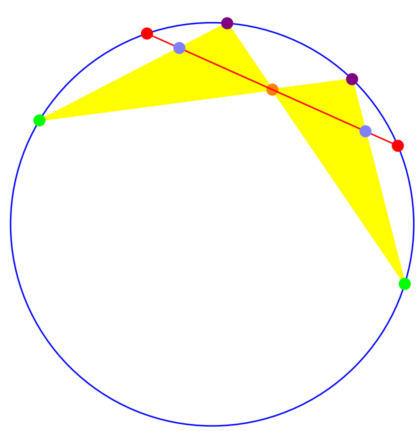


FIGURE 6. The butterfly theorem.