TEACHING MATHEMATICS WITH A HISTORICAL PERSPECTIVE

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E-320: Teaching Math with a Historical Perspective **O. Knill, 2010-2022**

Lecture 9: Topology

9.1. Topology is rubber geometry. It studies properties of geometric objects that do not change under continuous invertible deformations. For a topologist, a coffee cup with a single handle is the same as a doughnut. One can deform one into the other without punching any holes or ripping things apart. Similarly, a plate and a croissant are the same. But a croissant is not equivalent to a doughnut. On a doughnut, there are closed curves which can not be pulled together to a point. For a topologist, the letters O and P are the same but they both are different from the letter B.

9.2. The mathematical setup is beautiful: a **topological space** is a set X with a set O of subsets of X containing both $\emptyset$ and X such that finite intersections and arbitrary unions in O are in O. Sets in O are called **open sets** and O is called a **topology**. The complement of an open set is called **closed**. Examples of topologies are the **trivial topology** $O = \{\emptyset, X\}$, where no open sets besides the empty set and X exist or the **discrete topology** $O = \{A \mid A \subset X\}$, where every subset is open. But these are in general not interesting.

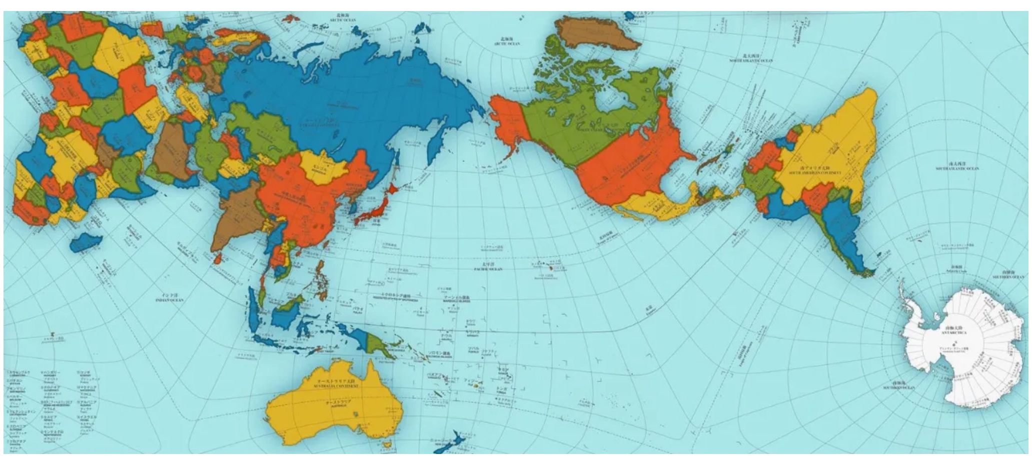


FIGURE 1. To map the spherical earth we need an atlas. An example is the authagraph. It maps the earth onto a tetrahedron then unfolds that to a rectangle and preserves area at about 100 parts of the earth. You can see why it has appeared in Japan (1999) and is not so popular yet in places like Europe.

9.3. For example, let X be the set of points of your paper and $\mathcal{O}$ the set of sets generated by open balls. For the of $\mathcal{O}$ one can use arbitrary unions and finite intersections. Special class of topological spaces are **metric spaces**. These are sets X that are equipped with a **distance function** $d(x,y) = d(y,x) \geq 0$ satisfying the **triangle inequality** $d(x,y) + d(y,z) \geq d(x,z)$ and which has the property that $d(x,y) = 0$ if and only if $x = y$. A set U in a metric space is open, if to every x in U, there is a **ball** $B_r(x) = \{y | d(x,y) < r\}$ of positive radius r contained in U. The set of open sets defines then a topology.

9.4. A substantial part of topology deals with spaces which locally look like our space. An example is the **sphere** for which our earth is an example. At a fixed location on earth the geometry looks like part of a plane but we need an **atlas** to cover all of X. The charts are then glued together with identification maps on the intersection. This gives what one calls a **manifold**. An other example of such a space is the **torus** or the **Klein bottle**. Topological spaces X, Y are "topologically equivalent" if there is an invertible map from X to Y so that this also induces an invertible map on the corresponding topologies.

9.5. A basic task is to decide whether two spaces are equivalent in this sense or not. The surface of the coffee cup for example is equivalent in this sense to the surface of a doughnut but it is not equivalent to the surface of a sphere. In an informal sense, two spaces are equivalent, if one can deform one into an other without changing quantities like connectivities, dimension or the collection of closed non-contractible loops in the space. Punching a hole into a paper for example changes the topology of the space. Gluing together left and right of a rectangular paper changes its topology to a cylinder. If the upper and lower parts are identified too, one has the "pacman space" which is a doughnut and called a torus.

9.6. Many properties of geometric spaces X can be understood by replacing them with **finite networks**. The points and connections then form a skeleton of the space. Networks are also called **graphs**. A finite simple graph (V, E) is a finite collection of vertices V paired with a finite set of edges E, where each edge in E connects two different points in V. For example, the set V of cities in the US form a network if one takes as edges are pairs of neighboring cities connected by a street. Two cities x, y are neighboring if there is a direct street from x to y not passing through an other city.

9.7. Graph theory is the birth crib of topology. The **Königsberg bridge problem** was a trigger for the study of graph theory. Also **Polyhedra**, objects already studied by the Greeks has led to graphs. The study of polyhedra is loosely related to the analysis of surfaces. The reason is that one can see polyhedra as discrete versions of surfaces. In computer graphics for example, surfaces are rendered as networks of triangulations.

9.8. The **Euler characteristic** of a convex polyhedron is a remarkable topological invariant. For two dimensional convex polyhedra, it is

$$V - E + F = 2 \, ,$$

where V is the number of vertices, E the number of edges and F the number of **faces**. This formula for the Euler characteristic is also called **Euler's gem**. It comes with a rich history. **René Descartes** seems have stumbled upon it and written it down in a secret notebook. It was Leonard Euler in 1752 was the first to proved the formula for convex polyhedra. There was a long sequence of proofs of refutations which however all boil down to how general one assumes the notion of polyhedron to be. Removing the assumption of convex for example can completely change the story. There are non-convex polyhedra which have not Euler characteristic 2. Kepler has found some. The story has been woven into a dialog written by Lacatos: "Proofs and Refutations". It is a tale of caution to use precise definitions.

9.9. A convex polyhedron is called a **Platonic solid**, if all vertices are on the unit sphere, all edges have the same length and all faces are congruent polygons. A theorem of **Theaetetus** states that there are only five platonic solids: [Proof: Assume the faces are regular n-gons and m of them meet at each vertex. Beside the Euler relation $V + E + F = 2$, a polyhedron also satisfies the relations $nF = 2E$ and $mV = 2E$ which come from counting vertices or edges in different ways. This gives $2E/m - E + 2E/n = 2$ or $1/n + 1/m = 1/E + 1/2$. From $n \geq 3$ and $m \geq 3$ we see that it is impossible that both m and n are larger than 3. There are now nly two possibilities: either $n = 3$ or $m = 3$. In the case $n = 3$ we have $m = 3, 4, 5$ in the case $m = 3$ we have $n = 3, 4, 5$. The five possibilities $(3,3), (3,4), (3,5), (4,3), (5,3)$ represent the five platonic solids.] The pairs (n,m) are called the **Schläfly symbol** of the polyhedron:

Name	V	E	F	V-E+F	Schläfli
tetrahedron	4	6	4	2	$\{3,3\}$
hexahedron	8	12	6	2	$\{4,3\}$
octahedron	6	12	8	2	$\{3,4\}$

Name	V	E	F	V-E+F	Schläfli
dodecahedron	20	30	12	2	$\{5,3\}$
icosahedron	12	30	20	2	$\{3,5\}$

9.10. The Greeks proved the classification result geometrically: Euclid showed in the "Elements" that each vertex can have either 3,4 or 5 equilateral triangles attached, 3 squares or 3 regular pentagons. (6 triangles, 4 squares or 4 pentagons would lead to a total angle which is too large because each corner must have at least 3 different edges). **Simon Antoine-Jean L'Huilier** refined in 1813 Euler's formula to situations with holes: $V - E + F = 2 - 2g$, where g is the number of holes. For a doughnut with one hole we have $V - E + F = 0$. . Cauchy first proved that there are exactly 4 non-convex regular **Kepler-Poinsot** polyhedra. Their Euler characteristic can be different.

Name	V	E	F	V-E+F	Schläfli
small stellated dodecahedron	12	30	12	-6	$\{5/2,5\}$
great dodecahedron	12	30	12	-6	$\{5,5/2\}$
great stellated dodecahedron	20	30	12	2	$\{5/2,3\}$
great icosahedron	12	30	20	2	$\{3,5/2\}$

If two different face types are allowed but each vertex still look the same, one obtains 13 **semi-regular polyhedra.** They were first studied by **Archimedes** in 287 BC. Since his work is lost, **Johannes Kepler** is considered the first person since antiquity to describe the whole set of thirteen in his "Harmonices Mundi". The Euler characteristic $\chi = 2 - 2g$ is also useful for surfaces. One can reduce the question to graphs, triangularizations of the surface.

9.11. It turns out that the Euler characteristic completely characterizes smooth compact surfaces if they are orientable. A non-orientable surface, the **Klein bottle** can be obtained by gluing ends of the Möbius strip. Classifying higher dimensional manifolds is more difficult and finding good invariants is part of modern research. Higher analogues of polyhedra are called **polytopes** (Alicia Boole Stott). **Regular polytopes** are the analogue of the platonic solids in higher dimensions. Here they are for the first few dimensions:

dimension	name	Schläfli symbols
2:	Regular polygons	$\{3\},\{4\},\{5\},...$
3:	Platonic solids	$\{3,3\},\{3,4\},\{3,5\},\{4,3\},\{5,3\}$
4:	Regular 4D polytopes	$\{3,3,3\},\{4,3,3\},\{3,3,4\},\{3,4,3\},\{5,3,3\},\{3,3,5\}$
≥ 5:	Regular polytopes	$\{3,3,3,\dots,3\},\{4,3,3,\dots,3\},\{3,3,3,\dots,3,4\}$

9.12. Ludwig Schläfly found in 1852 that there are exactly six convex regular convex 4-polytopes or **polychora**. The expression "choros" is Greek for "space". Schlaefli's polyhedral formula tells that for any **convex polytope** in four dimensions, the relation

$$V - E + F - C = 0$$

holds, where C is the number of 3-dimensional **chambers**. In dimensions 5 and higher, there are only 3 types of polytopes: the higher dimensional analogues of the tetrahedron, octahedron and the cube. A general formula $\sum_{k=0}^{d-1}(-1)^k v_k = 1-(-1)^d$ gives the Euler characteristic of a convex polytop in d dimensions with k-dimensional parts v_k.

Work problems

9.13. 1) The digits 0-9:
a) The numbers $0, 4, 6, 9$ are topologically equivalent.

0 4 6 9

b) The numbers $1, 2, 3, 5, 7$ are topologically equivalent.

1 2 3 5 7

c) The number 8 is not topologically equivalent to any other digit.

8

d) Are there any numbers which are disconnected?

e) Which numbers are simply connected?

9.14. 2) The letters:

A B C D E

F G H I J

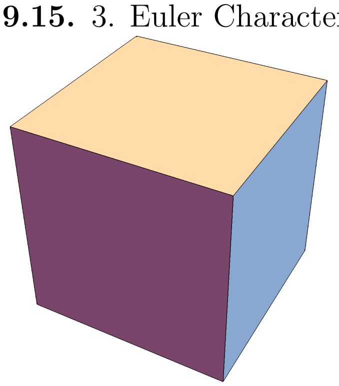
K L M N O

P Q R S T

U V W

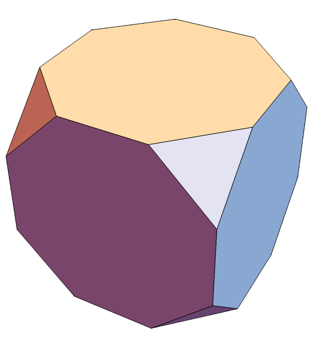
X Y Z

9.15. 3. Euler Characteristic



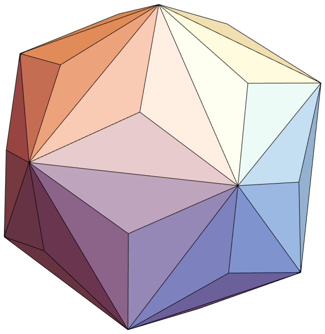
a) Compute the Euler Characteristic $V - E + F$ for the cube

Vertices V	Edges E	Faces F



b) Cutting the corners of the cube produces 8 new faces and renders the old faces octagonal. Count the Euler characteristic of the new object which is now a semi-regular polyhedron.

Vertices V	Edges E	Faces F



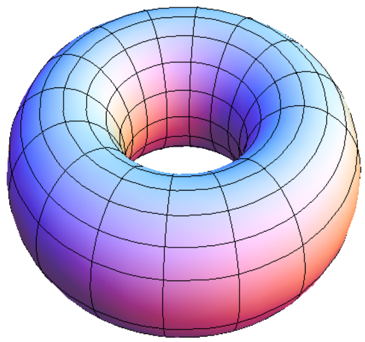
c) Start again with the cube, but now cut each of the faces into 4 faces by drawing the diagonals in the squares. If the midpoints are lifted up a bit so that all triangles become equilateral, the new object is called a **stellation** of the cube. It is an other semi-regular polyhedron.

Vertices V	Edges E	Faces F

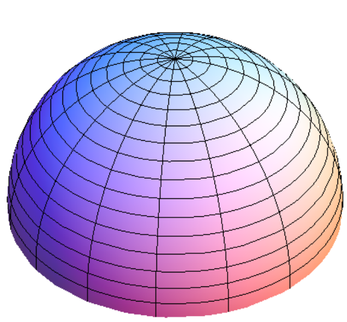
9.16. 3) Which of the following pieces of cloth are topologically equivalent?

9.17. 4) By identifying sides of a square we obtain models of compact surfaces: the **sphere**, the **torus**, the **projective plane** and the **Klein bottle**. We want to explore here the topology of these spaces, especially the simply conntectedness: can one pull any closed rope in this space to a point? Only the sphere is simply connected.

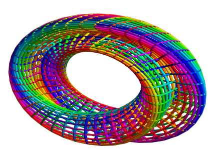
a) Draw some curves on the torus, which can not be pulled together to a point.



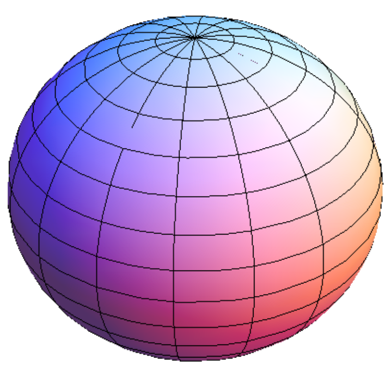
b) Draw a curve on the projective plane, which can not be pulled together to a point.



c) Move a letter R around on the Klein bottle, what happens with the letter as it moves over the boundary to the right and appears to the left?



d) Draw a curve on the sphere. Visualize that you can pull it together to a point.



e) By triangulating a space, we are also able to compute the Euler characteristic of these spaces. The Euler characteristic of the sphere is 2, the Euler characteristic of the torus is 0, the Euler characteristic of the projective plane is 1, the Euler characteristic of the Klein bottle is 0. Can you show this in the examples?

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