E-320: Teaching Math with a Historical Perspective **O. Knill, 2010-2022**

Lecture 11: Dynamical systems

11.1. **Dynamical systems theory** is the science of time. If time is **continuous**, the evolution is defined by a **differential equation** $\dot{x} = f(x)$. If time is **discrete**, then we look at the **iteration of a map** $x \to T(x)$. Here is the prototype of a differential equation in three dimensions:

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz .
\end{aligned}
$$

the **Lorenz system**. There are three parameters σ, r, b. For $\sigma = 10, r = 28, b = 8/3$, one observes a **strange attractor** with fractal shape.
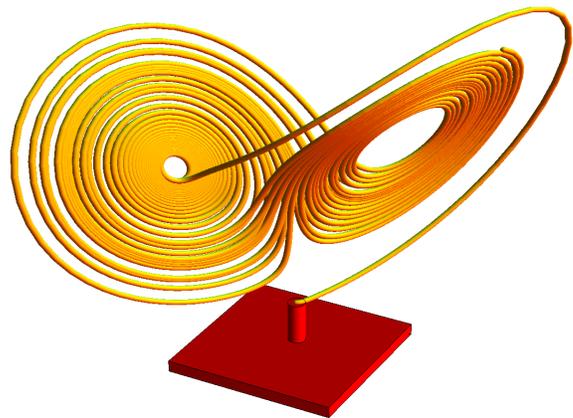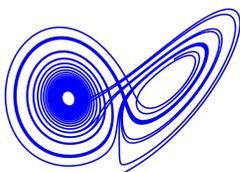
11.2. The goal is to **predict the future** of the system when the present state is known. A **differential equation** is an equation of the form $d/dt\,x(t) = f(x(t))$, where the unknown quantity is a path $x(t)$ in some "phase space". We know the **velocity** $d/dt\,x(t) = \dot{x}(t)$ at all times and the initial configuration $x(0)$), we can to compute the **trajectory** $x(t)$. What happens at a future time? Does $x(t)$ stay in a bounded region or escape to infinity? Which areas of the phase space are visited and how often? Can we reach a certain part of the space when starting at a given point and if yes, when. An example of such a question is to predict, whether an asteroid located at a specific location will hit the earth or not. An other example is to predict the weather of the next week.

11.3. An example of a dynamical systems in one dimension is the differential equation

$$x'(t) = x(t)(2 - x(t)), x(0) = 1 \ .$$

It is called the **logistic system** and describes population growth. This system has the solution $x(t) = 2e^t/(1 + e^{2t})$ as you can see by computing the left and right hand side.

11.4. A **map** is a rule which assigns to a quantity $x(t)$ a new quantity $x(t+1) = T(x(t))$. The state $x(t)$ of the system determines the situation $x(t+1)$ at time $t+1$. An example is is the **Ulam map** $T(x) = 4x(1-x)$ on the interval $[0, 1]$. This is an example, where we have no idea what happens after a few hundred iterates even if we would know the initial position with the accuracy of the Planck scale. We will experiment with that in class.

11.5. Dynamical system theory has applications in all fields of mathematics. We can use dynamical systems for example to find roots of equations. The **Newton map**

$$T(x) = x - f(x)/f'(x)$$

is such a procedure. If we are close enough to the fixed point, applying T again and again will have us converge very fast to the fixed point.

11.6. Dynamical systems also appear in number theory. For large primes p, nonlinear maps like $T(x) = x^2 + c \bmod p$ or $T(x) = a^x \bmod p$ behave rather erratically. And this is good so as the maps can be used for encryption.

11.7. A rather curious system of number theoretical nature is the **Collatz map**

$$T(x) = \frac{x}{2} \text{ (even x)}, 3x + 1 \text{ else } .$$

A system of geometric nature is the **Pedal map** which assigns to a triangle the pedal triangle.

11.8. Lets look a bit at the history of chaos: about 100 years ago, **Henry Poincaré** was able to deal with **chaos** of low dimensional systems. While **statistical mechanics** had formalized the evolution of large systems with probabilistic methods already, the new insight was that simple systems like a **three body problem** or a **billiard map** can produce very complicated motion. It was Poincaré who saw that even for such low dimensional and completely deterministic systems, random motion can emerge.

11.9. While physicists have dealt with chaos earlier by assuming it or artificially feeding it into equations like the **Boltzmann equation**, the occurrence of stochastic motion in simple systems like double penduli, geodesic flows or billiards or restricted three body problems was a surprise. These findings needed half a century to sink in and only with the emergence of computers in the 1960ies, the awakening happened. Icons like Lorentz helped to popularize the findings and we owe them the **"butterfly effect"** picture: a wing of a butterfly can produce a tornado in Texas in a few weeks.

11.10. The reason for this statement is that the complicated equations to simulate the weather reduce under extreme simplifications and truncations to a simple differential equation $\dot{x} = \sigma(y - x), \dot{y} = rx - y - xz, \dot{z} = xy - bz$, the **Lorenz system**. For $\sigma = 10, r = 28, b = 8/3$, Ed Lorenz discovered in 1963 an interesting long time behavior and an aperiodic "attractor". Ruelle-Takens called it a **strange attractor**. It is a **great moment** in mathematics to realize that attractors of simple systems can become fractals on which the motion is chaotic. It suggests that such behavior is abundant. What is chaos? If a dynamical system shows **sensitive dependence on initial conditions**, we talk about **chaos**. We will experiment with the two maps $T(x) = 4x(1-x)$ and $S(x) = 4x - 4x^2$ which starting with the same initial conditions will produce different outcomes after a couple of iterations.

11.11. The sensitive dependence on initial conditions is measured by how fast the derivative dT^n of the n'th iterate grows. The exponential growth rate γ is called the **Lyapunov exponent**. A small error of the size h will be amplified to $he^{\gamma n}$ after n iterates. In the case of the Logistic map with $c = 4$, the Lyapunov exponent is $\log(2)$ and an error of 10^{-16} is amplified to $2^n \cdot 10^{-16}$. For time $n = 53$ already the error is of the order 1. This explains the above experiment with the different maps. The maps $T(x)$ and $S(x)$ round differently on the level 10^{-16}. After 53 iterations, these initial fluctuation errors have grown to a macroscopic size.

11.12. Here is a famous open problem which has resisted many attempts to solve it: Show that the map

$$T(x, y) = (c \sin(2\pi x) + 2x - y, x)$$

with $T^n(x, y) = (f_n(x, y), g_n(x, y))$ has sensitive dependence on initial conditions on a set of positive area. More precisely, verify that for $c > 2$ and all n $\frac{1}{n} \int_0^1 \int_0^1 \log |\partial_x f_n(x, y)| \, dxdy \geq \log(\frac{c}{2})$. I have tried over a decade to prove this using methods from quantum mechanics, calculus of variations and complex analytic methods. The problem is open.

11.13. The left hand side converges to the average of the Lyapunov exponents which is in this case also the **entropy** of the map. For some systems, one can compute the entropy. The logistic map with $c = 4$ for example, which is also called the **Ulam map**, has entropy $\log(2)$. The **cat map**

$$T(x, y) = (2x + y, x + y) \bmod 1$$

has positive entropy $\log |(\sqrt{5} + 3)/2|$. This is the logarithm of the larger eigenvalue of the matrix implementing T.

11.14. While questions about simple maps look artificial at first, the mechanisms prevail in other systems: in astronomy, when studying planetary motion or electrons in the van Allen belt, in mechanics when studying coupled pendulum or nonlinear oscillators, in fluid dynamics when studying vortex motion or turbulence, in geometry, when studying the evolution of light on a surface, the change of weather or tsunamis in the ocean.

11.15. Dynamical systems theory historically started with the problem to understand the **motion of planets**. Newton realized that this is governed by a differential equation, the **n-body problem**

$$x_j''(t) = \sum_{i=1}^n \frac{c_{ij}(x_i - x_j)}{|x_i - x_j|^3} \, ,$$

where c_{ij} depends on the masses and the gravitational constant. If one body is the sun and no interaction of the planets is assumed and using the common center of gravity as the origin, this reduces to the **Kepler problem** $x''(t) = -Cx/|x|^3$, where planets move on **ellipses**, the radius vector sweeps equal area in each time and the period squared is proportional to the semi-major axes cubed. A great moment in astronomy was when Kepler derived these laws empirically. An other great moment in mathematics is Newton's theoretically derivation from the differential equations.

Work problems

1) We experiment with simple transformations can produce chaotic outcome. Make sure your calculator is in the "Rad" mode. Remember that 2π radians is equal to 360 degrees. You can check whether your calculator is in Radian mode, by computing $\cos(\pi)$ and get the result -1. Make sure your calculator is in rad mode. Use a scientific calculator. In the iphone calculator for example, turn the device to get to the scientific mode.
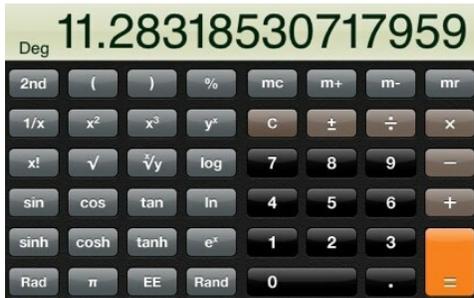
The Scientific Calculator built in by default in the Iphone/Ipod/Ipad appears when you turn the device.

a) Take a calculator, and pushing repetitively the button cos. What do you observe?

b) Now repeat pushing the sin button. What do you observe?

c) Now push x^2 repetitively.

d) Now push $\sqrt{x}$ repetitively.

e) What do you see if you push the buttons sin, then type $1/x$ and repeat this process again and again?

f) Experiment with the button tan. Also here, change from tan and cot (in the cot case, you might have to hit tan and $1/x$ buttons after each other).

g) Look for other "chaotic" key combinations? Experiment also with Deg and Rad changes and try especially the log functions.

2) We graphically compute a few iterates of one dimensional maps. This can be done on paper. One produces a so called cobweb.
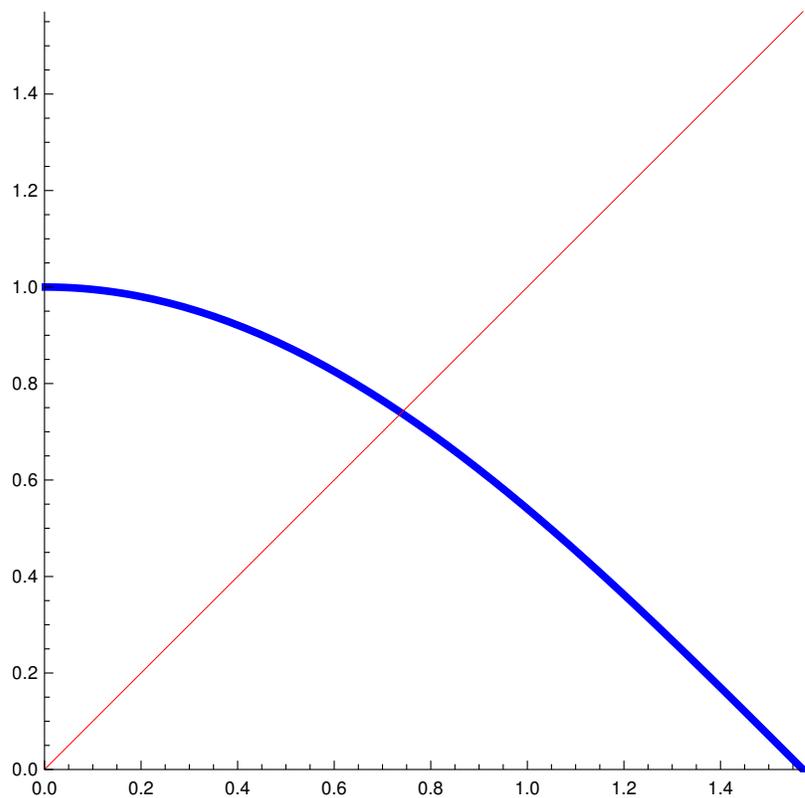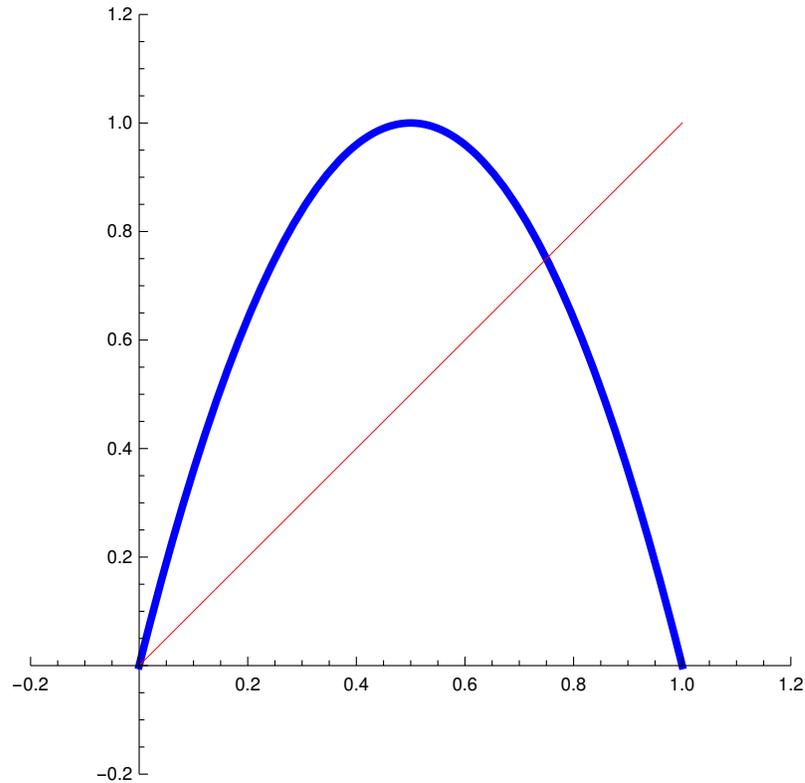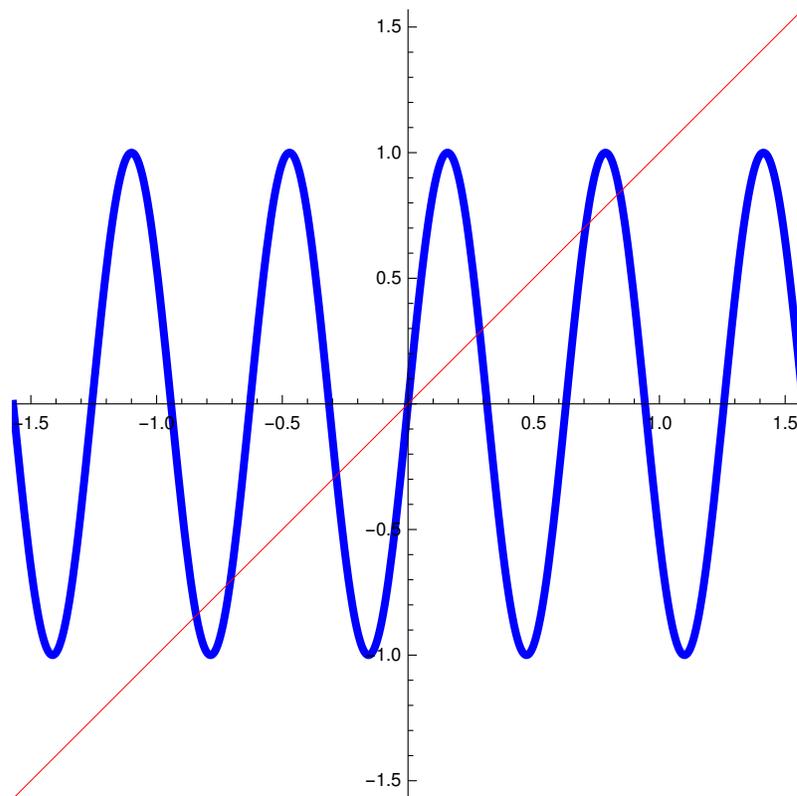
3) We look at a dynamical system of number theoretical nature. In the **Collatz system**, we start with an integer and map it with the following rule:

$$T(x) = \begin{cases} x/2 & x \text{ even} \\ 3x + 1 & x \text{ odd} \end{cases}$$

The question is whether the orbit always ends up with 1. For example: $x = 7$ produces $7, 22, 11, 34, 17$.
a) Start with the initial condition 26:
b) Start with the initial condition 9:
c) Start with the initial condition 2048:
d) What is wrong with the following proof of the Collatz conjecture?

Proof. Consider only the odd numbers in the Collatz sequence. We show that each odd number is in average $3/4$ times smaller than the previous one:

With probability $1/2$ the number $3x + 1$ is divisible by 2 and not 4: this increases x by $3/2$
With probability $1/4$ the number $3x + 1$ is divisible by 4 and not 8: this decreases x by $3/4$
With probability $1/8$ the number $3x + 1$ is divisible by 8 and not 16: this decreases x by $3/8$

To compute the probability, we take logarithms and compute $a = \sum_{n=1}^{\infty} \frac{1}{2^n} \log(3/2^n)$. The average decay rate of the size of a number is the factor $e^a = 3/4$.
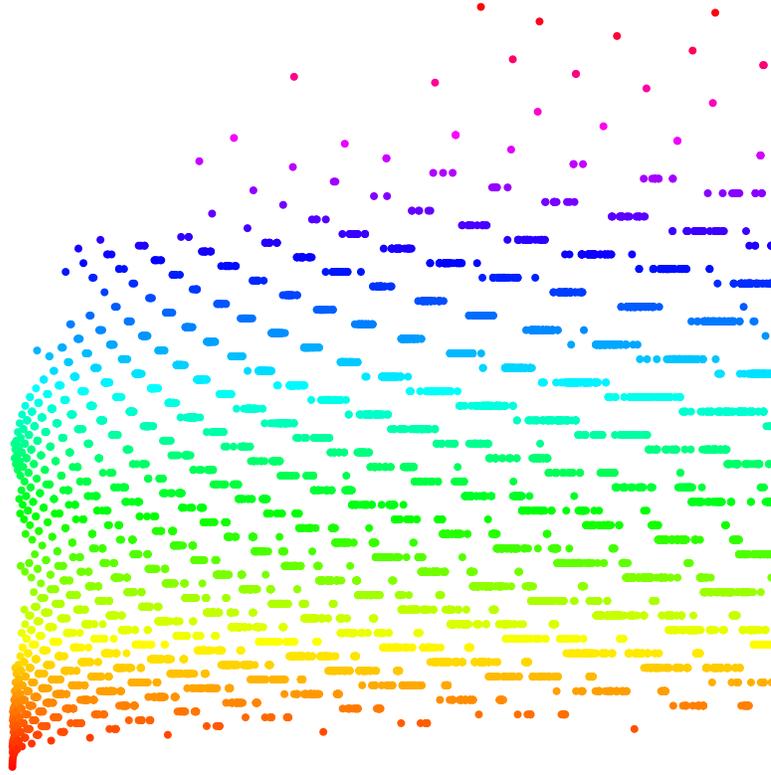
e) The Collatz system certainly can be modified. Can you find one, for which there is a nontrivial loop?

4) We look at a dynamical systems called Cellular automata. These are continuous maps on sequence spaces in which the evolution rule is translational invariant.

neighborhood	new middle cell
111	0
110	0
101	0
100	1
011	0
010	0
001	1
000	0

to an offspring 1, we and $100 = 4, 001 = 1$ in binary, we have $2^4 + 2^1 = 18$.

a) run it:

1	0	1	0	0	0	1	0	0	1	0	1	1	0	1	0

Cellular Automata Offer New Outlook on Life, the Universe, and Everything

What kind of world do we live in? The question has been bandied about for thousands of years by philosophers, theologians, and politicians. More recently, a spectrum of talk show hosts have weighed in on the subject. So far, no one's come up with an answer that everyone can agree on. Mathematicians have considered the same question. But where others worry over the blurred boundaries of Good and Evil, mathematicians ponder a sharper dichotomy: the Continuous versus the Discrete.

Continuous mathematics, exemplified by calculus and differential equations, has long dominated mathematical descriptions of the world. But discrete mathematics is making a bid for primacy. With modern computers, researchers have discovered astonishingly complex behavior in seemingly simple, finite systems. The results have led some theorists to speculate that discrete models, which lend themselves to digital computation, are the "right" way to study nature.

Erica Jen, a mathematician at Los Alamos National Laboratory in Los Alamos, New Mexico, is one of a growing number of researchers who believe that discrete mathematics can mirror many aspects of physical reality fully as well as the more customary continuous theories. Jen has been studying mathematical properties of discrete systems known as cellular automata. These systems, she says, are useful models for many types of complex physical, chemical, or biological systems. They also have an amazing life of their own.

Cellular automata "exhibit an extremely rich and diverse range of pattern formation," Jen says. Among the most interesting are "self-organizing" patterns: highly structured features that seem to emerge spontaneously from a "primordial soup" of random binary
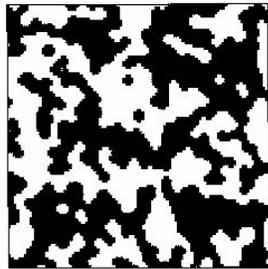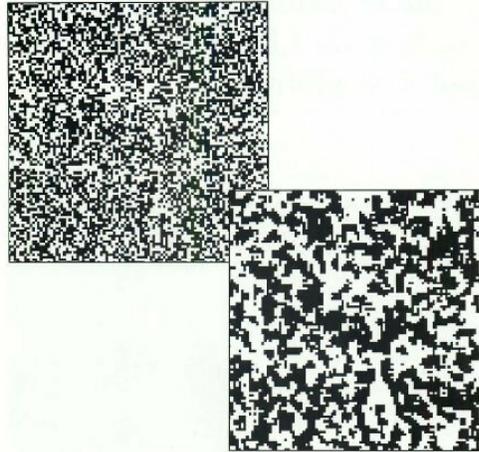
Erica Jen. (Photo courtesy of Erica Jen.)

Figure 2. A 100×100 "majority vote" cellular automaton proceeds from a random initial state (top) to a final state (bottom right, facing page). On each ballot, every cell looks at the cells around it, and changes color value if its current value is in the minority. Most cells have 8 neighbors, but cells on the edges have 5 neighbors, and corner cells only 3. Some features of the final state take shape with the first round of voting (middle).

digits. Jen and her colleagues hope to understand exactly how these patterns arise and precisely what properties they possess. By studying cellular automata with mathematical tools from areas such as abstract algebra and number theory, Jen hopes to bring theoretical rigor to a subject that is often as much art as science.

Loosely speaking, a cellular automaton is a "digitization" of space and time. Instead of varying continuously from point to point and moment to moment, cellular automata consist of discrete "cells" with discrete values that change instantaneously at discrete intervals, much like frames in a movie. The crucial feature, moreover, is a rule that prescribes exactly how each cell's value changes depending on the values of nearby cells.

One possible rule, for example, is a "majority vote": Each cell in a system of black and white squares could be programmed to switch color if the majority of its immediate neighbors are of the opposite color (see Figure 2). Another rule might specify that the value of each cell change to the sum of the values of the cells surrounding it, or, reducing things to black and white again, to the parity of the sum (black could be odd and white even).

The essential features of cellular automata are that they are deterministic and discrete in space, time, and state values; they evolve according to local interaction rules; and these rules apply

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Computer technology was not really up to the job of exploring cellular automata until the 1980s.

synchronously and homogeneously across the system." Jen explains. These features accord well with standard physical assumptions about the uniformity of space and time (the laws of physics are the same everywhere) and the impossibility of instantaneous action at a distance (nothing travels faster than the speed of light). They also lend themselves to modeling complex systems consisting of a large number of simple components that are locally connected. Perhaps most important, these features are tailor-made for digital computation.

Cellular automata were first dreamt of in the early 1950s by John von Neumann and Stanislaw Ulam, as tools for studying biological systems. In the late 1960s, John Conway, then at Cambridge University (now at Princeton), invented rules for a cellular automaton he called the Game of Life, which Martin Gardner popularized in his column for Scientific American. But computer technology was not really up to the job of exploring cellular automata until the 1980s, when color graphics workstations replaced the clattering teletype machines that trailed alphanumeric symbols with a room-sized mainframe in another building.

With today's high-speed machines (fated, no doubt, to seem painfully slow in another few years), researchers can glimpse the complex patterns that often arise from the repeated application of the simple rules that define cellular automata. Fast computers allow experiments with relatively large systems: Automata with thousands of cells can be followed for hundreds of time steps on a personal computer; workstations and supercomputers can track systems with millions of cells for thousands of time steps.

Jen's research focuses on a class of one-dimensional systems called "elementary" cellular automata. Each state of such a system is represented by a row of black and white pixels, corresponding to a string of 1's and 0's, and the update rule uses only the value of a given cell and the values of its two adjoining cells. (To simplify the description, researchers often work with a "wrap-around" model, in which the two ends are joined, so that all cells are treated alike.) The evolution of a one-dimensional automaton is conveniently displayed in a two-dimensional format, each new row below its predecessor. (Researchers also often "colorize" their elementary systems to highlight key features.) The result can be as richly textured as a Navajo weaving.

In the early 1980s, Stephen Wolfram, then at the Institute for Advanced Study in Princeton, roughed out a classification scheme