

# THE BABYLONIAN GRAPH

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ABSTRACT. The Babylonian graph  $B$  has the positive integers as vertices and connects two integers  $a, b$  if  $a^2 + b^2$  is a square and so define a Pythagorean triple. Triangular sub-graphs correspond to Euler bricks. What are the properties of this graph? Are there tetrahedral sub-graphs corresponding to Euler tesseracts? Is there only one infinite connected component? Are there two Euler bricks in the graph that are disconnected? Do the number of edges or triangles in the sub-graph generated by the first  $n$  vertices grow like of the order  $n$   $W(n)$ , where  $n$  is the product log? We prove here some simple results. In an appendix, we include a handout from a talk on Euler cuboids given in the year 2009.

## 1. BABYLONIAN GRAPHS

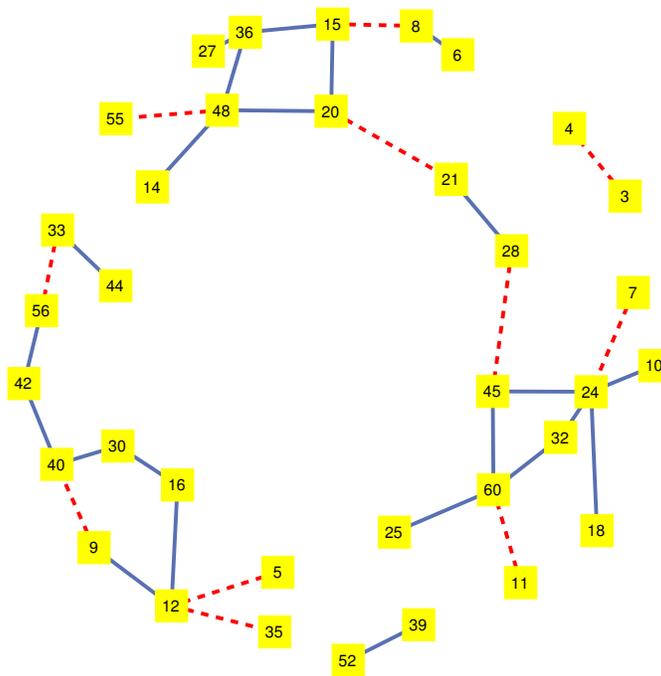


FIGURE 1. The graph  $B_{60}$  without isolated vertices like  $\{1\}$  or  $\{2\}$ . The edges corresponding to primitive Pythagorean triples are dashed.

**1.1.** For every positive integer  $n$ , let **Babylon-n** denote the simple graph  $B_n = (V_n, E_n)$  with vertex set  $V_n = \{1, \dots, n\}$  and edge set  $E_n = \{(a, b), \sqrt{a^2 + b^2} \in \mathbb{N}, a, b \leq n\}$ . We have a nested increasing sequence of graphs  $B(n)$  which starts with  $B_0 = K_1$  and leads to the **Babylonian graph**  $B = (\mathbb{Z}^+, E)$  with  $(a, b)$  as edges if  $(a, b, \sqrt{a^2 + b^2})$  is a **Pythagorean triple**, is infinite. Triangles, complete sub-graphs  $K_3$  in  $B$  correspond to **Euler bricks** [4]

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(chapter XIX, see also [13]. The more extended notes [12] is attached here as an Appendix). A subclass of Euler bricks can be obtained by parametrizations like the **Saunderson parametrization**  $a = u(4v^2 - w^2), b = v(4u^2 - w^2), c = 4uvw$ . Triangles  $(a, b, c)$  for which additionally the sum  $a^2 + b^2 + c^2$  is an integer correspond to **perfect Euler bricks**, an object which has not yet been found and which might not exist. The graphs  $B_n$  are the from  $V_n = \{1, \dots, n\}$  induced sub-graph of  $B$ . Each  $B_n$  is a sub-graph of  $B_{n+1}$  and the limit  $B = (\mathbb{Z}^+, E)$  encodes all Pythagorean triples.

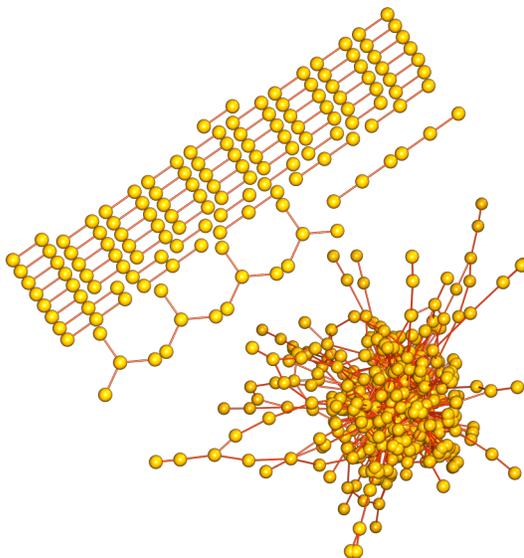


FIGURE 2. The graph  $B_{1000}$  without the 0-dimensional isolated points.

**1.2.** We are interested in the **largest connected component**  $B'_n$  of  $B_n$  and in the connectivity or symmetry properties of  $B$ . Are there  $K_4$  sub-graphs in  $B$ ? What groups act as graph isomorphisms on  $B$ ? While  $B$  has some small components like the single vertex  $\{1\}$  or the single vertex  $\{2\}$  or the isolated  $K_2$  graph  $\{3 \leftarrow 4\}$ , we expect that  $B$  only has one large infinite connectivity component.

**1.3.** The question of existence of perfect Euler bricks appears to be difficult. The popularity of the problem persists. It is now also in a list of problems in [34] discussed beyond the Millenium problems. Euler bricks are triangles in  $B$  and correspond to points  $(x, y, z)$  in  $\mathbb{R}^3$  located on three cylinders  $x^2 + y^2 = a^2, y^2 + z^2 = b^2, z^2 + x^2 = c^2$  with integer radii  $a, b, c$ . In an Euler brick triangle, it is not possible that all three pairs are primitive. The graph induced by the primitive edges has no triangles. Figure illustrates the geometric problem to find integer points  $(x, y, z)$  on the intersection of three perpendicular main axes-centered cylinders with integer radius. The perfect Euler brick problem is to find such points which also have integer distance to the origin. Also the problem of **Euler tesseracts** can be seen geometrically. It is the problem to find the intersection of six perpendicular  $3D$ -cylinders  $x_i^2 + x_j^2 = r_{ij}^2, 1 \leq i < j \leq 4$  with integer radius  $r_{ij}$  in  $\mathbb{R}^4$ . It rephrases to find  $K_4$  sub-graphs of  $B$ . Whether this is possible is not clear.

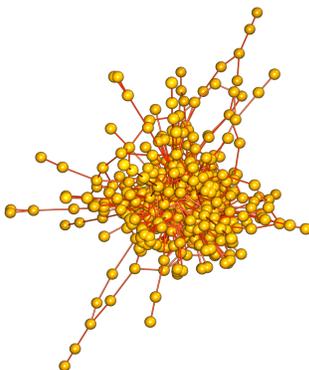


FIGURE 3. The main connected component  $B'_{1000}$  of the Babylonian graph  $B_{1000}$ . Its  $f$ -vector is  $f = (480, 952, 10)$ , its Euler characteristic is  $\chi(B'_{1000}) = 480 - 952 + 10 = -462$  and its Betti vector is  $b = (b_0, b_1) = (1, 463)$ . The graph  $B_{10000}$  itself has  $f$ -vector  $f = (1000, 1034, 10)$ , Betti vector  $b = (b_0, b_1) = (439, 463)$  and  $\chi(B_{10000}) = f_0 - f_1 + f_2 = b_0 - b_1 = -24$ .

**1.4.** There are quite many Diophantine problems using squares. One similar looking problem is the **Mengoli six square problem** of Pietro Mengoli who by the way also would suggest the **Basel problem**, to find the value of  $\sum_{k=1}^{\infty} 1/k^2$ . Mengoli asked for triples  $x \leq y \leq z$  of integers such that the sum and difference of any two are squares. In other words,  $x + y, y + z, x + z, y - x, z - y, z - x$  should all be squares. Euler found the smallest one. The solution found by Ozanam in 1691 is  $(1873432, 2288168, 2399057)$ . [24]

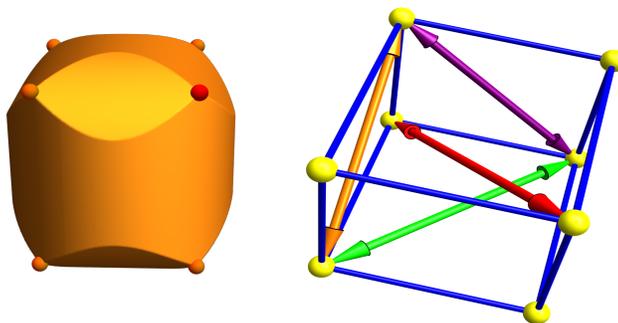


FIGURE 4. The intersection of three cylinders with integer radius defines Euler bricks. If the intersection point  $(x, y, z)$  also has integer distance to the origin, we have a perfect Euler brick. To the right we see an Euler brick.

## 2. FOUR QUESTIONS

**2.1.** Here are some natural questions. They can also be formulated as conjectures. Despite having this posted since February 2022, it is probably safer to still keep it as questions and not upgrade it to conjectures. One reason is that one or the other question could turn out to be obvious. When studying a problem for the first time, it is possible to miss something obvious. It can be that one or the other question are already answered in an other context or is a special case of a general theorem. We had looked a couple of years ago at the literature

of Euler bricks while preparing for a math circle talk. That handout in a treasure hunting theme [33] is here attached in an appendix.

**2.2.** First of all, we believe that there is only one largest connected component  $B'_n$  for all  $n$  and that it will end up to be a **single infinite component** in the Babylonian graph  $B$ . In principle, it is not yet excluded that there are several infinite disconnected components of  $B$ . Question  $\textcircled{A}$  asks whether such an “eternal maximal component” exists.

**Question:**  $\textcircled{A}$  Does  $B$  have only one infinite connected component?

**2.3.** Question  $\textcircled{B}$  is about the existence of **Hyper Euler bricks** or **Euler tesseracts**. It addresses the question about the **maximal dimension** of  $B$ . While unlikely, it would in principle be possible that the maximal dimension is infinity, meaning that there are complete sub-graphs  $K_n$  of  $B$  for any integer  $n$ . Already the question of three dimensional complete sub-graphs  $K_4$  is unclear:

**Question:**  $\textcircled{B}$  Is there a  $K_4$  sub-graph in  $B$ ?

**Euler tesseracts** are hyper cubes  $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4, 0 \leq x_1 \leq x, 0 \leq x_2 \leq y, 0 \leq x_3 \leq z, 0 \leq x_4 \leq w\}$  with integer  $x, y, z, w$  side length for which all the 6 2D-face diagonals have integer length. One of the ways to show that a system of Diophantine equations has no solution is to look at the system modulo a prime  $p$ . If there is no solution modulo  $p$ , then there is no solution at all. In the **tesseract problem** we have to solve the system of Diophantine equations

$$x^2 + y^2 = a^2, y^2 + z^2 = b^2, z^2 + w^2 = c^2, w^2 + x^2 = d^2, x^2 + z^2 = e^2, y^2 + w^2 = f^2$$

for the 10 integer variables  $x, y, z, w, a, b, c, d, e, f \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ . We have not started to look for solutions yet, but the strategy is similar than when looking for perfect Euler bricks: take a parametrization  $(x, y, z)$  of Euler bricks like the Saunderson parametrization, then we have a function  $F_{x,y,z}(w) = d(\sqrt{x^2 + w^2}) + d(\sqrt{y^2 + w^2}) + d(\sqrt{z^2 + w^2})$  where  $d(t)$  is the distance to the nearest integer. Now, for large  $x, y, z$  and  $w$ , the dynamics  $F_{x,y,z}(w) \rightarrow F_{x,y,z}(w + 1)$  is by linear approximation close to a translation  $t \rightarrow t + \alpha$  which then by a continued fraction expansion allows to find  $w$  for which  $F_{x,y,z}(w)$  is very small and since the possible distances are quantized, once we are close enough, we hit a solution. This is how we have searched for perfect Euler bricks. A more sophisticated search using a multi-dimensional approach, leading to multi-variable Chinese remainder type problems [11].

**2.4.** Question  $\textcircled{C}$  asks about the maximal dimension of the non-major connected components. Example sub-graphs  $\{1\}, \{2\}, \{3, 4\}$  remain separated also in  $B$ . We have not yet seen any example, where a non-major connected component has maximal dimension larger than 1:

**Question:**  $\textcircled{C}$  Are there two connected components with triangles?

In other words, we look for pairs of Euler bricks, so that there is no connection from one brick to the other brick in  $B$ . We have also not seen any in any  $B_n$ . It could be possible that there exists a  $B_n$  with two disconnected  $K_3$  sub-graphs which however get reunited in a larger  $B_m$  for  $m > n$ .

**2.5.** The fourth and last major question ④ is a particular **growth rate question**. Let  $W(x)$  denote the **product log = Lambert W-function** which is defined as the inverse of the function  $y = x \log(x)$ . This function naturally occurs in the **prime number theorem** which tells that the  $n$ 'th prime  $p_n$  is of the order  $p_n \sim n \log(n)$  meaning  $W(p_n) \sim n$ . Now the **number of primitive edges** in  $B_n$  grow like  $C_1 n$ , where  $C_1$  is a concrete number expressible as an area of the parameters lattice points  $(u, v)$  in  $[0, \sqrt{n}] \times [0, \sqrt{n}]$  such that  $\Phi(u, v) = (u^2 - v^2, 2uv) \in [0, n] \times [0, n]$ . Having a growth  $C_1 n$  of primitive edges, we expect  $C_1 n W(n)$  to be the growth of all edges.

**Question:** ④ Does  $f_1(B_n)/(nW(n))$  converge?

**2.6.** There are many other quantities one could look at. We can look at the **f-vector**  $(f_0, f_1, f_2, \dots) = (n, f_1(B_n), f_2(B_n), \dots)$  or the Betti numbers  $b_k(B_n) = \dim(\ker(L_k(B_n)))$  where  $L_k(B_n)$  is the **k-form Laplacian**. Of interest are the number of connected components  $b_0(B_n)$ , the number  $b_1(B_n)$ , a genus, which measures of the number of one dimensional "holes" in  $B_n$ , the maximal vertex degree, the distribution of the vertex degrees, the growth of the Euler characteristic  $\chi(B_n) = f_0(B_n) - f_1(B_n) + f_2(B_n) - \dots = b_0(B_n) - b_1(B_n) + b_2(B_n) - \dots$ , the inductive dimension of  $B_n$  or  $B'_n$  with the ultimate goal to give lower and upper bounds. We looked first numerically at the diameter change up to  $n = 10000$ . The diameter goes to infinity because  $\text{Diam}(B_1(5000)) = 18$ ,  $\text{Diam}(B_1(10000)) = 29$ . A logarithmic growth which is justified by scaled graphs  $x_1, x_2, \dots, x_n = Mx_1, Mx_2, Mx_3, \dots, Mx_n$  etc.

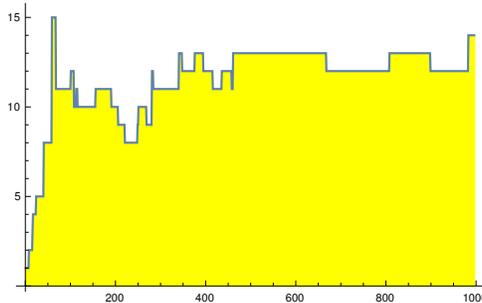


FIGURE 5. The diameter of  $B_n$  up to  $n = 1000$ .

**2.7.** Whenever we have a graph  $G$ , the **graph complement**  $\overline{G}$  is of interest. The operation of taking graph complements is an **involution** on the class of all graphs. Even for very simple graphs, the graph complement can be interesting. See [17] for cyclic or linear graphs, where graph complements are either contractible or homotopic to spheres or wedge sums of spheres. What are the properties of the graph complement of  $B$ ?

### 3. LOW HANGING FRUITS

**3.1.** One can wonder first whether there are **isolated vertices** that are not connected to anything else. These are zero dimensional connected components of the graph. There are exactly two vertices with this property. We know that  $1 + b^2, 4 + b^2$  are never a square.

**Remark:** ① There are exactly two isolated single vertices 1, 2 in  $B$ .

**Proof:** Already the **primitive Babylonian graph**  $B_p$  which connects only points  $a, b$  if  $a, b, \sqrt{a^2 + b^2}$  is a primitive Pythagorean triple has no isolated points except 1, 2: the reason

is that all odd numbers larger than 1 are of the form  $u^2 - v^2$  and all even numbers larger than 2 are of the form  $2uv$  for some positive distinct  $u, v$ .

**3.2.** There are other isolated connected components like  $\{3, 4\}$  belonging to the primitive triple  $3^2 + 4^2 = 5^2$ . This can not be connected to anything else. If  $9 + b^2 = c^2$ , then  $c^2 - b^2 = 9$  which is only possible for  $b = 4, c = 5$ . The relation  $16 + b^2 = c^2$  is only possible for  $b^2 = 9$ .

**3.3.** One can wonder about the asymptotic distribution of the vertex degrees. When looking at the sequence  $B_n$  of graphs, there is an increasing part  $C_n \subset B_n$  for which the vertex degrees do no more change when increasing  $n$ . The reason is that the monotone sequence  $\deg(B_n)(x)$  converges:

**Remark:** ②  $\deg_B(x)$  is finite for all  $x \in V$ .

**Proof:** every edge is a Pythagorean triple which is a multiple of a primitive Pythagorean triple and so of the form

$$(a, b) = (2pqc, (p^2 - q^2)c),$$

where  $p, q, c$  are integers. If we fix an integer like  $b$ , there are only finitely many solutions  $(p^2 - q^2)c = b$  because both  $c$  and  $p^2 - q^2$  have to be recruited from factors of  $b$  which is finite. For a fixed factor  $r$  of  $b$  the Diophantine equation  $p^2 - q^2 = r$  has only finitely many solutions because both  $p, q$  have to be smaller than  $\sqrt{r}$ .

**3.4.** One can also wonder how many infinite connected components there are  $B$ . We do not know yet. We can get infinite connected components for a primitive  $(a, b) = (2uv, u^2 - v^2)$  if it is connected to a multiple of itself. This can indeed happen and proves

**Remark:** ③ The diameter of  $B$  is infinite.

**Proof:** it is enough to construct a concrete path in  $B$  going to infinity: There is a connection from  $n = 5$  to  $n = 30$  given by  $5 \rightarrow 12 \rightarrow 16 \rightarrow 30$ . This scales. The path  $30 \rightarrow 40 \rightarrow 96 \rightarrow 180$  for example extends the other path so that we have a connection from 5 to 180. We can continue like that and get an infinite path in  $B$ .

**3.5.** We could call a connected component in which a scaling  $(a, b) \rightarrow m(a, b)$  exists a **component with scale symmetry**. We have just seen that such components are infinite. Are there components which are not scale invariant?

**3.6.** Since  $a = u^2 - v^2, b = 2uv$  parametrizes all **primitive triples**, there is a constant  $C$  such that there are asymptotically  $C * n$  primitive edges in the graph  $B_n$ . This limit can be computed explicitly as we can draw out the region in the parameter domain leading to triples  $\leq n$ . But then we have also non-primitive ones which come from scaled smaller primitive ones.

**3.7.** Here is an other simple observation about **leafs** in  $B$ . These are vertices  $x$  for which the unit sphere  $S(x)$  (the sub-graph generated by all vertices attached to  $x$ ) contains only one point.

**Remark:** ④ If  $p$  is an odd prime, then  $p$  is a leaf in  $B$ .

**Proof:** We must have  $p = u^2 - v^2$  so that  $p$  belongs to the primitive Pythagorean triple  $2uv, u^2 - v^2$ . Now,  $u^2 - v^2 = p$  implies with  $v = u - k$  that  $p = u^2 - (u - k)^2 = 2uk - k^2$ . Since  $p$  is prime,  $k = 1$  meaning that we have only one choice to solve  $u^2 - v^2 = p$  for  $u, v$ . The Pythagorean triple is not fixed.

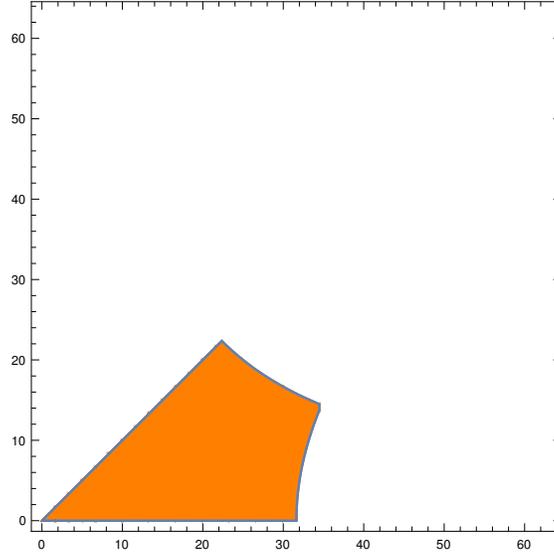


FIGURE 6. The  $uv$ -parameter region  $-n \leq u^2 - v^2 \leq n, 0 \leq 2uv \leq n, u \geq 0, v \geq 0$ . For each lattice point  $(u, v)$  in that region, we get a Pythagorean triple  $u^2 - v^2, 2uv, u^2 + v^2$  on  $[0, 2\sqrt{n}] \times [0, 2\sqrt{n}]$  and so an edge in  $B_n$ .

**3.8.** By definition, the number of vertices  $f_0(B_n) = n$ . Let  $W(x)$  denote the inverse of the function  $x \rightarrow xe^x$ . It is called the **Lambert W function**. Motivated from the prime number theorem telling that that  $n$  grows like  $P(n)W(P(n))$ , where  $P(n)$  is the  $n$ 'th prime number, it is likely that  $\lim_{n \rightarrow \infty} f_1(B_n)/(nW(n)) = C_1$  and  $\lim_{n \rightarrow \infty} f_2(B_n)/(nW(n)) = C_2$  exist:

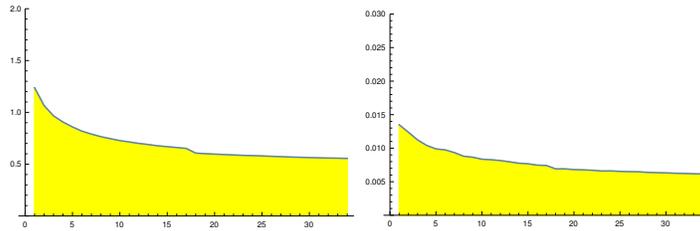


FIGURE 7. The number of edges and triangles for the main component.  $f_1(B'_{n*1000})/(n*W(n)*1000)$  and  $f_2(B'_{n*1000})/(n*W(n)*1000)$  for  $n = 1, \dots, 25$ .

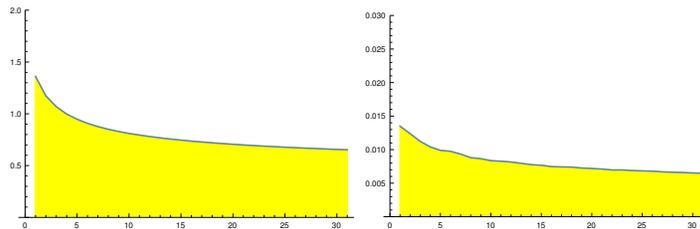


FIGURE 8. Number of edges and triangles for the full graph with all components:  $f_1(B_{n*1000})/(n * W(n) * 1000)$  and  $f_2(B_{n*W(n)*1000})/(n * 1000)$  for  $n = 1, \dots, 20$ .

**3.9.** This would lead to a result that  $C = \lim_{n \rightarrow \infty} \chi(B_n)/(nW(n)) = C_0 - C_1 + C_2 - C_3 + \dots$  exists.

**3.10.** Here is a result which is somehow interesting. A graph can be defined to be planar if it does not contain a homeomorphic copy of  $K_5$  or  $K_{3,3}$ . (There is also the traditional definition of planar using a topological embedding on a 2-sphere, but the just given one is equivalent by **Kuratowski's theorem**. The combinatorial definition has the advantage that it is purely graph theoretical and does not refer to topology of any Euclidean space.

**Remark:** ⑤  $B_n$  is planar if and only if  $n \leq 95$

*Proof.* Since  $B_{95}$  is planar all  $B_n$  with  $n \leq 95$  are planar. Since  $B_{96}$  is non planar, all  $B_n$  with  $n \geq 96$  are non-planar.  $\square$

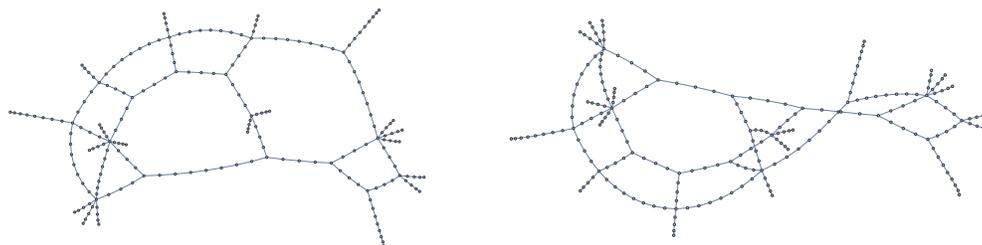


FIGURE 9. The graph  $B_{95}$  (looking like a piglet) is planar, the graph  $B_{96}$  (looking like a chicken) is not. We show here the second Barycentric refinements of the main connected components. The non-planar property of  $B_{96}$  is not well visible. We have to look closely at the crossing of edges. There is a crossing which is not a node.

#### 4. BABYLONIAN TRIPLETS AND PYTHAGORAS

**4.1.** Pythagorean triples appeared in Babylonians tablets. The most famous is Plimpton 322 [31, 21]. An other one is Si.427. We could call the examples of Pythagorean triples which appear in some Babylonian text a **Babylonian triplet**. The major known clay documents which list Babylonian triples appear all during the time 1900-1600 BC. Pythagorean triples also appear in ancient Egyptian mathematics like on the **Berlin Papyrus 6619**. It contains the non-primitive triple 6, 8, 10 and which is a document dated at about a similar time than the clay tablets.

**4.2.** It has been speculated that experimental exploration of Pythagorean triples also had practical **engineering value** because the construction of right angles has architectural or irrigation area measurement applications. The interpretation that some of these clay tablets were school tablets indicate that the topic of Pythagorean triples must also have been of **educational value**. The Pythagorean triples also paved the way for the **Pythagorean theorem**, the statement that  $a^2 + b^2 = c^2$  holds for the sides of a triangle if and only if the triangle has a right angle. In the remarkable tablet YBC 7289, an isosceles right angle triangle appears with a rather astounding approximation of  $\sqrt{2}$ . This was one of the first examples for the Pythagorean theorem with non-integer sides but it is also just an example.

**4.3.** Despite many speculations in that direction, there is **no evidence** that the Babylonians were aware of the Pythagorean theorem. We can **speculate** that they started to **suspect a general rule**. We see in the literature and even encyclopedias formulations like “*may suggest that the ancient Egyptians knew the Pythagorean theorem*”. Still, also for such a

claim, we lack any historical sources. We have no document on which any such conjecture is formulated. Formulations like “may suggest” are a bit reckless as they disregard the difficulty in mathematics to coming up with conjectures and general rules and then to prove them. There are countless many examples, where mathematical rules have been conjectured by looking at small examples and where later, the rule turned out to be false. Many examples are listed in [7]. Proto-Pythagorean themes have also appeared also in Chinese documents, including a proof of the Pythagorean theorem in the 3-4-5 triangle case which indicates that a general statement was in the air. As mathematicians, we know however that stating a fact like  $3^2 + 4^2 = 5^2$  and visualizing it in a picture is not the same than claiming that  $a^2 + b^2 = c^2$  holds in a triangle if one of the angles is a right angle triangle.

**4.4.** Jacob Bronowski took in his book [2] the old fashioned point of view about the discovery of Pythagoras. He of course was aware about the uncertainty of the sources. But it is the so far best guess that Pythagoras was the first who proved the theorem. Because of the lack of original documents of Pythagoras, we might never know who actually proved the Pythagorean theorem the first time. Bronowski tells: *Pythagoras had thus proved a general theorem: not just for the 3:4:5 triangle of Egypt, or any Babylonian triangle, but for every triangle that contains a right angle. He had proved that the square on the longest side or hypotenuse is equal to the square on one of the other two sides plus the square on the other if, and only if, the angle they contain is a right angle. For instance, the sides 3:4:5 compose a right-angled triangle. And the same is true of the sides of triangles found by the Babylonians, whether simple as 8:15:17, or forbidding as 3367:3456:4825, which leave no doubt that they were good at arithmetic. To this day, the theorem of Pythagoras remains the most important single theorem in the whole of mathematics. That seems a bold and extraordinary thing to say, yet it is not extravagant; because what Pythagoras established is a fundamental characterization of the space in which we move, and it is the first time that is translated into numbers. And the exact fit of the numbers describes the exact laws that bind the universe.*

**4.5.** The ability to make good conjectures and to get a **notion of proof** needed to evolve over time. Already the realization that there is a difference between **Examples, Conjectures, Hypothesis, Model** and **Theorems** is a cultural achievement. We see the process when watching students learning. When we learn mathematics we first confuse the process of proving a result in general or to just support a phenomenon by giving **anecdotal data evidence**. The ability of **asking questions** like **why** is what “makes us curious” [19] as it starts a scientific process. Because the object **Babylonian graph** seems historically not have appeared in the literature, we are in a realm, where we can also observe our own first steps and track misconceptions or mistakes.

**4.6.** Even wrong conjectures can be helpful as they illustrate our status of understanding for a subject. We also are interested in the Babylonian graph because we are here in a “data collection” and “forming conjecture” phase. Historians will have to find out whether the Babylonian graph has been mentioned earlier in the literature. In a hundred or thousand years, we might know a lot about this graph. Today in 2022, we seem to be in the “data collection” and “forming conjectures” phase. In a thousand years, there might be powerful theorems which answer all questions. It can of course also be that there is no interest in the object at all building up and that the topic will remain an obscure example.

**4.7.** The origins of the Pythagorean theorem itself is in mystery. Who was the first who conjectured it? This is already a major step going much beyond just listing examples. The next step is a giant one as it is way beyond conjecture. Who was the first who proved the

Pythagorean theorem? This is difficult as no authentic documents of Pythagoras himself are known. See [36, 35, 9, 29, 28, 22, 10]. Still, if we look at the text evidence, we have to give the Greek mathematicians (and especially Euclid from whom we have dated documents) the credit to formalize what a “theorem” and what a “proof” is and distinguish a “general statement” (which is always true) from a “statistical statement” (which is true in most cases or with a few exceptions only [the nonsensical “exceptions prove the rule” is even used in colloquial language]) and especially to distinguish from “anecdotal evidence” (which even in our modern times is often mistaken as “proof” by a mathematically untrained person, or then as a crude but effective tool for advertisement or propaganda.) The Babylonian triplets were **anecdotal evidence** for the theorem for Pythagoras, not more. It was still far from a conjecture about a general relation and even further away from a Pythagorean theorem which is a statement coming with a proof.

**4.8.** There have been a few headlines in the last couple of years claiming that the Babylonians invented trigonometry. There is no indication that Babylonians invented trigonometry. This statement depends on what “trigonometry” means. While trigonometry uses ratios of triangles for the definition of the trigonometric functions, looking at ratios of sides of triangles should not yet count as trigonometry. No school curriculum considers that nomenclature when talking about trigonometry. Looking at ratios of right angle triangles is **proto trigonometry at best**. To cite [31] about research theories in history: *“In general, we can say that the successful theory [in the history of mathematics] should not only be mathematically valid but historically, archaeologically, and linguistically sensitive too.”* [21] for example has produced the **2021 controversy**: the paper has been picked up by media. Math historian Victor Blasjö formulated it nicely: ‘it tricked news outlets into printing nonsense headlines’.

## 5. EXPERIMENTAL EXPLORATIONS

**5.1.** Experimental explorations of a mathematical structure often have predated theorems considerably. Experiments can lead to **examples which suggest a theorem**. But there can be a long journey from experiments to theorems. It took a thousand years to get from **Pythagorean triple explorations** to the **Pythagorean theorem**. Otto Neugebauer already speculated that the parametrizations  $a = u^2 - v^2, b = 2uv, c = u^2 + v^2$  could have been known thousands of years ago ([25] page 39). Bronowski for example gives the example of a pair (3367, 3456) [2] which is the parametrization obtained with  $u = 64, v = 27$ . The largest number in Plimpton 322 is  $(a, b) = (12709, 13500)$  which is obtained with  $u = 125, v = 54$ . Obtaining such large numbers without a parametrization is harder but it is not impossible. It needs a bit of patience and some luck. Such examples make it likely that the Euclid parametrization was known and used but it is still just a guess.

**5.2.** The fact that in the given examples on Plimpton 322, only a few “random” parameter values  $(u, v)$  appear and not a systematic list, ordered according to  $(u, v)$  speak against the knowledge of such a parametrization but it would be conceivable that some structure was seen like that one number is even and trying  $b = 2uv$ , where  $u, v$  are factors. All primitive triples can be obtained as such and also non-primitives like  $6^2 + 8^2 = 10^2$  have been considered. Finding out what really happened is something for Sherlock Holmes [3], where we see the statement: *We can begin by asking if numbers of the form  $a^2 - b^2$  and  $a^2 + b^2$  have any special properties. In doing so, we run the risk of looking at ancient Babylonia from the twentieth century, rather than trying to adopt the autochthonous viewpoint.*

**5.3.** Creighton Buck further writes: “*There is no independent information showing that these facts were known to the Babylonians at the time we conjecture that this tablet was inscribed.*” Indeed, there is no known statement of a result  $a^2 + b^2 = c^2$  for right angle triangles on clay tablets. Three thousand years ago, it had not been excluded that some **super large right triangle** with side length  $a, b, c$  would satisfy  $a^2 + b^2 \neq c^2$ . In most tablets, we only see integer side triangles. There is the triple  $(1, 1, \sqrt{2})$  in YBC 7289 which contains a non-integer side length. We can only imagine how puzzling this must have been.

**5.4.** In the context of finding historical clues, we also can gain insight by looking at what children do. Looking at early learners is like pointing a telescope to the past. The early steps in mathematics resemble the first steps of the pioneers developing the topic. This prompted a historian to claim [6] ”**A student should be taught a subject pretty much in the order in which the subject developed over the ages.**” It is a good rule of thumb but of course not universal. Many secrets from geometry can be appreciated much faster for example when using algebra. The fact that mathematics has evolved for many thousands of years and in an accelerated way requires a modern student also to pass to the modern topics faster and taking shortcuts and bypass times of stagnation.

**5.5.** In the context of pedagogy, there is an anecdote of the teacher letting students construct right angle triangles using paper and ask them measure  $a^2 + b^2 - c^2$ . One student group reported in their presentation that they found a remarkable rule:  $a^2 + b^2 - c^2$  was always small but never zero! This **anti-Pythagorean theorem** is **academically honest** because every measurement comes with errors. The students reported **what they measured** and did not report what they **wanted to see**. It is the most common sin in science to fall into **wishful thinking**. It is a powerful source of motivation, but it is dangerous. We know that error measurements have a continuous distribution so that without prejudice, it is correct that in experiments,  $a^2 + b^2 - c^2 \neq 0$  with probability 1. The students doing the measurements of course had not been exposed to statistics and data science. A more sophisticated approach would be to build a statistical model for the **possible errors**, to make a **hypothesis** and determine the **p-value**, the evidence against a null hypothesis. A good scientist tries to make the p-value as small as possible and so give **data evidence** for the Pythagorean theorem. This theorem would then be a **mathematical model**. The scientist then decides whether the measurements support the model. This is still far from proving the theorem. To prove the theorem one has to place the statement into a particular frame work, like planar Euclidean geometry. This requires to make some **idealizations** and **assumptions**.

**5.6.** Also the process of **building a model** or placing a statement in a **particular axiomatic frame work** is an achievement of Greek mathematics which should not be underestimated. I myself was not taught about **methods of science** in mathematics but in a philosophy classes. First in high school and later in college in a lecture series of Paul Feyerabend. Let me mention the high school part: I have been lucky to have a year of philosophy in the Schaffhausen highschool with Markus Werner (1944-2016) who was also a successful writer who won a dozen prestigious prizes like the Herman Hesse literature prize. He started one of the lessons with “What is the color red?” which led to interesting discussions about what color is, and whether it is something we can understand. What happens if we mix colors when drawing with a yellow and blue crayons what happens if we illuminate an object with blue and yellow lights simultaneously. An other of these philosophy lesson started with “What is the sum of the angles in a triangle?”. A student would answer 180 degrees. Werner would ask to prove it. An other student would prove it on the board. The class

would discuss then what kind of assumptions went into the proof. Werner would then draw a triangle on a sphere, where the theorem fails. How could we go wrong? What was wrong with the proof? On a sphere, there are triangles where the sum of the three angles is 270 degrees. There are 8 triangles on a sphere which partition the sphere up like that. In these 90 degree triangles, the Pythagorean result  $a^2 + b^2 = c^2$  fails. Actually, for those special 90-90-90 triangles, one has  $a^2 = b^2 = c^2$ . They are equilateral right angle triangles. Why did the proof which everybody agreed upon fail? What assumptions went into the proof?



FIGURE 10. Markus Werner (1944-2016), a writer and high school teacher.

**5.7.** An even more sophisticated picture appeared since Einstein. Sphere or non-Euclidean geometries are just one of many Riemannian geometries and Riemannian geometry is a more accurate model of our physical space than Euclidean space.<sup>1</sup> The Pythagorean theorem is well known to fail in our **three dimensional physical space**: it is an **idealization** dealing with **flat Euclidean space**. In a linear algebra setting, assuming a linear flat space, the subject can be dealt with quickly: define two vectors to be perpendicular if  $u \cdot v = 0$  and define the length as  $\sqrt{u \cdot u}$ , then check  $|u - v|^2 = (u - v) \cdot (u - v) = u \cdot u + v \cdot v = |u|^2 + |v|^2$ . We make a lot of assumptions although, the result assumes that space is continuous and in particular that there are perpendicular objects. Then we assume that space has an algebraic structure in that we can add and scale.

**5.8.** But we know since more than 100 years now from general relativity that every mass bends space and that right angle triangles only satisfy  $x^2 + y^2 = c^2$  in the **complete absence of matter** or under very special circumstances of the curvature. But even if we assume total absence of matter and ignore the presence of virtual particles (which are confirmed by phenomena like the Casimir effect), we still do not know because we have no access to any **Planck scale features** of space. We have no idea what happens if we take a right angle triangle of side length  $a, b, c$  if  $a, b, c$  are of the order  $10^{-35}$ . Our notions of distance based on measurements using electromagnetic waves do not make sense any more.

**5.9.** On a computer small physical distances are no problem- up to some reasonable scale. We can for example enjoy looking at features of the Mandelbrot set on a scale of say  $10^{-200}$ . It is not difficult for a computer to show us topological features of that mathematical object on such a small scale. But we can also with a computer not explore scales like  $10^{-10^{200}}$ . If the structure of space on the Planck scale would be understood, one can always ask what happens on an even smaller scale. Once the atom was considered the smallest unit, then protons, now we suspect quarks to be part of the smallest ingredients. There was a long way

<sup>1</sup>By the way, Einstein lived 1901/1902 in Schaffhausen for a few months, and lived a few hundred meters from the highschool in Schaffhausen, working as a tutor.

from speculations by philosophers like Democritus to the current standard model of particle physics.

## 6. ABOUT THE QUESTION

**6.1.** We mentioned the Babylonian graph in our first lecture of Math 22 in the spring of 2022. It was aimed as an illustration of the fact that mathematics is not only **eternal**, but also **infinite**. If you solve one problem, ten more problems pop up. Having seen in January, the movie “The eternals” in a movie theater, I called the Babylonian graph problem there the “**Eternals**” **problem**, because it had been communicated to us by the eternal Ajak from the Marvel comics universe. We also used the Babylonian graph as an example in the computer science lecture on May 1st, 2022 Math E 320 to illustrate the process of **experimental mathematics**. A related graph is the graph in which one takes pairs  $a, b$  for which  $a + ib$  is a Gaussian prime. We have played with graphs related to number theory also in [15].

**6.2.** In [16] we looked at the graph  $G_n$  with vertex set  $V = \{1, 2, \dots, n\}$ , where two are connected if their sum is a square. So, the connecting rule is  $a + b = c^2$ , (not  $a^2 + b^2 = c^2$ ). This graph had as a motivation a puzzle posed by Anna Belyakova of the University of Zürich and Dmitriy Nikolenkov of Trogen, a high school in Switzerland: *Write down the numbers 1 – 16 in a row so that the sum of two arbitrary neighbors is a square number.* This means we have to find a Hamiltonian path in  $G_{16}$ . Historically, the use of graph theory is closely tied to puzzles. William Rowan Hamilton came up with the idea of Hamiltonian paths in the context of the **Icosian game**, the problem to find a Hamiltonian cycle on the dodecahedron graph. All questions asked for the Babylonian graph can be asked for this **Balyankova-Nikolenkov graph**.

**6.3.** An other class of natural graphs  $G_n$  appears on square free integers by connecting two such integers if one divides the other [14]. It has the Euler characteristic  $\chi(G_n) = 1 - M(n)$  relates to the **Mertens function**  $M(x) = \sum_{k=1}^n \mu(k)$  with the Möbius function  $\mu$ . The value  $-\mu(k)$  is the **Poincaré-Hopf index** of the vertex  $k$  using the Morse function  $f(k) = k$ . Adding a new integers is part of a **Morse build-up** because every critical point either has index 1 or  $-1$  and “**counting**” is a **Morse theoretical process**, during which more and more ‘handles’ in the form of topological balls are added, building an increasingly complex topological structure. I would see later that this structure has already been studied earlier in [1].

**6.4.** In February 2022, we already looked numerically at the growth of the graph diameter of the main connected component  $B'_n$  of Babylon  $B_n$ . The diameter does not grow monotonically as some parts reconnect. But the experiments suggested already then that the diameter might increase indefinitely. While easy to see, it has not been visible to me at first and I asked it as a question. Only when noticing that there are connections from integers to a multiple of an integer, the infinite diameter of  $B$  became clear and obvious. In retrospect it now would have been ridiculous to formulate an infinite diameter conjecture.

**6.5.** It might well be that one or the other of the questions  $\textcircled{A}, \textcircled{B}, \textcircled{D}, \textcircled{D}$  mentioned here are not difficult to answer. When you look at a new problem the first time, a lot of things which later appear “obvious”, are still obscured. It might also be that some Diophantine problems like the problem of the existence of  $K_4$  or  $K_5$  sub-graphs in  $B$  are difficult. The **perfect Euler brick problem** turned out to be hard and evidence that it is really hard is the fact that it has remained open for so long. It is well possible that the  $K_4$  problem is

## BABYLONIAN GRAPH

easy and that it could be answered by just looking at the problem from the right angle or by having a sufficiently strong computer and patience to find one.

APPENDIX: A TALK ON EULER BRICKS

Math table, February 24, 2009, Oliver Knill  
**Treasure Hunting Perfect Euler bricks**

An Euler brick is a cuboid with integer side dimensions such that the face diagonals are integers. Already in 1740, families of Euler bricks have been found. Euler himself constructed more families. If the space diagonal of an Euler brick is an integer too, an Euler brick is called a perfect Euler brick. Nobody has found one. There might be none. Nevertheless, it is an entertaining sport to go for this treasure hunt for rational cuboids and search - of course with the help of computers. We especially look in this lecture at the Saunderson parametrization and give a short proof of a theorem of Spohn [32] telling that the any of these Euler bricks is not perfect. But there are other parameterizations.

$x^2 + y^2 = a^2, x^2 + z^2 = b^2, y^2 + z^2 = c^2, x^2 + y^2 + z^2 = d$

Paul Halcke      Leonard Euler

Dead men's chest

Sounderson shore

Pari

(132 351 720)

(44 117 240)

(85 132 720)

108 Mr. Peirce's Some Diophantine Impossibilities. Some Diophantine Impossibilities. By H. C. POCKLINGTON, M.A., St. John's College. [Revised October 1912.]

1. The object of this paper is mainly to discuss some equations obtained by creating a quadratic function of  $x^2$  and  $y^2$  to a value, either to prove that they are impossible in integers or rational fractions (§ 2-10), or to completely find their solutions in integers (§ 12). Two theorems on arithmetical progressions of which given terms are to be squares, given in §§ 6, 11, and the impossibility of  $x^2 + y^2 = z^2$  is discussed in § 13.

§ 2. We use the principle of three locuses. The first is that if  $ax^2 + by^2 = z^2$  and  $a$  is prime to  $y$ , then both  $a$  and  $y$  are with factors. The proof is obtained on expressing  $ax^2$  as a product of primes and their powers. For each prime that occurs in  $a$  must occur in  $ax^2$  but not in both, and clearly occurs raised to a power the index of which is some multiple of  $n$ . If  $n=2$  we have an alternate proof in Euler's § 2 and if  $n$  is a power of 2 we can get a result by repeated application of Euler's proposition. We have assumed that  $a$  and  $y$  are positive. If they can be negative we see that  $a = -a', y = -y'$ . By repeated application we see that  $a = \pm a', y = \pm y'$  and  $z = \pm z'$ .

We shall also require the following application of the lemma. Let  $x = \frac{p}{q}, y = \frac{r}{s}$  where  $q$  or  $s$  is prime to  $xy$ . Then  $ax^2 + by^2 = z^2$  for  $ax^2 + by^2$  a prime common divisor of  $x$  and  $y$  is a prime divisor of  $z$  and  $ax^2 + by^2 = z^2$  becomes  $a(x')^2 + b(y')^2 = z'^2$  where  $ax' = x/q, y' = y/s, z' = z/(qs)$ . Thus we have given equation  $X^2 + Y^2 = Z^2$  where  $X = ax', Y = by', Z = z'$ . We see that  $X^2 + Y^2 = Z^2$  where  $ax' = X, y' = Y$  and  $z' = Z$  we see that  $X^2 + Y^2 = Z^2$  where  $ax' = X, y' = Y$  and  $z' = Z$ .

Next suppose that the greatest common divisor is 2, i.e. that  $X, Y, Z$  are all even. Then if  $y$  is even we have  $x = \frac{p}{2q}, y = \frac{r}{2s}, z = \frac{z'}{2}$  where  $q$  or  $s$  is prime to  $(x, y)$ . This is before we had  $x = \frac{p}{q}, y = \frac{r}{s}, z = \frac{z'}{qs}$ . If  $z$  is odd,  $X$  must be divisible by 2,  $Y$  must be divisible by 2,  $Z$  must be odd. Hence  $X = 2x', Y = 2y', Z = z'$  where  $ax' = X, y' = Y, z' = Z$  where  $ax' = X, y' = Y, z' = Z$  where  $ax' = X, y' = Y, z' = Z$ .

(160 231 792)

2,3,4,5,8,9,11,16 cove

(88 234 480)

$a = 6t^5 - 20t^3s^2 + 6t^5s$   
 $b = -s^6 + 15t^2s^4 - 4t^4s^2 + 2t^6$   
 $c = 8s^5t - 8st^5$

(140 480 693)

Mathematica

$(\sqrt{2^2 + 117} + 1) \cdot (\sqrt{2^2 + 117} + 1) \cdot 100$

*Euler Brick Island*  
 February 20, 2009, O.Knill

$x = 2mn(3m^2 - n^2)(3n^2 - m^2)$   
 $y = 8mn(m^4 - n^4)$   
 $z = (m^2 - n^2)(m^2 - 4mn + n^2)(m^2 + 4mn + n^2)$

Literature: L.E. Dickson, Theory of Numbers, Volume II, 1966 J. Leech, American Mathematical Monthly, 84 p. 518, 1977 R. Guy, unsolved problems in number theory, 2004 Robert L. Stevenson, Treasure Island

## Introduction: the map of John Flint

An **Euler brick** is a cuboid of integer side dimensions  $a, b, c$  such that the face diagonals are integers. If  $u, v, w$  are integers satisfying  $u^2 + v^2 = w^2$ , then the Saunderson parametrization

$$(a, b, c) = (|u(4v^2 - w^2)|, |v(4u^2 - w^2)|, |4uvw|)$$

leads to an Euler brick.

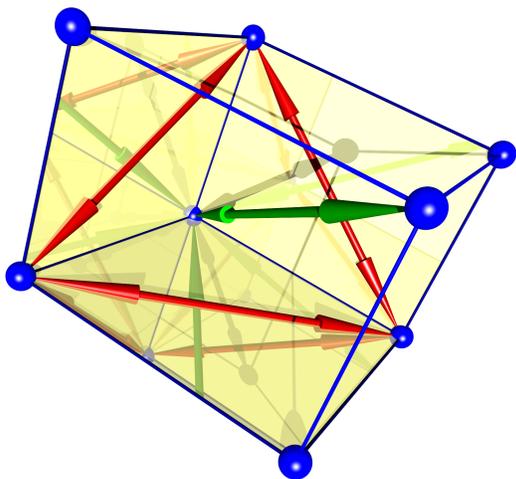


Fig 1. An Euler brick has integer face diagonals. It is perfect if the long diagonal is an integer too.

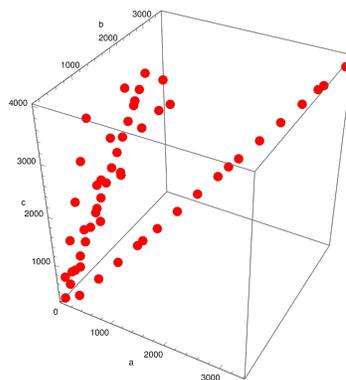


Fig 2. The smallest Euler bricks  $(a, b, c)$  with  $a \leq b \leq c$  plotted in the parameter space.

The cuboid with dimensions  $(a, b, c) = (240, 117, 44)$  is an example of an Euler brick. It is the smallest Euler brick. It has been found in 1719 by Paul Halcke ( - 1731) [4].

If also the space diagonal is an integer, an Euler brick is called a **perfect Euler brick**. In other words, a cuboid has the properties that the vertex coordinates and all distances are integers.

It is an open mathematical problem, whether a perfect Euler bricks exist. Nobody has found one, nor proven that it can not exist. One has to find integers  $(a, b, c)$  such that

$$\sqrt{a^2 + b^2}, \sqrt{a^2 + c^2}, \sqrt{b^2 + c^2}, \sqrt{a^2 + b^2 + c^2}$$

are integers. This is called a system of Diophantine equations. You can verify yourself that that the Saunderson parametrization produces Euler bricks.

If we parametrize the Pythagorean triples with  $u = 2st, v = s^2 - t^2, w = s^2 + t^2$ , we get  $a = 6ts^5 - 20t^3s^3 + 6t^5s, b = -s^6 + 15t^2s^4 - 15t^4s^2 + t^6, c = 8s^5t - 8st^5$ . This defines a parametrized surface

$$r(s, t) = \langle 6ts^5 - 20t^3s^3 + 6t^5s, -s^6 + 15t^2s^4 - 15t^4s^2 + t^6, 8s^5t - 8st^5 \rangle$$

which leads for integer  $s, t$  to Euler bricks.

Indeed, one has then:  $a^2 + b^2 = (s^2 + t^2)^6, a^2 + c^2 = 4(5s^5t - 6s^3t^3 + 5st^5)^2, b^2 + c^2 = (s^6 + 17s^4t^2 - 17s^2t^4 - t^6)^2$ .

A perfect Euler brick would be obtained if  $f(t, s) = a^2 + b^2 + c^2 = s^8 + 68 * s^6 * t^2 - 122 * s^4 * t^4 + 68 * s^2 * t^6 + t^8$  were a square.

**Brute force search: yo-ho-ho and a bottle of rum!**

There are many Euler bricks which is not parametrized as above:

A brute force search for  $1 \leq a, b, c \leq 300$  gives  $a = 44, b = 117, c = 240$  and  $a = 240, b = 252, c = 275$  as the only two Euler bricks in that range. In the range  $1 \leq a < b < c \leq 1000$  there are 10 Euler bricks:

a	b	c
44	117	240
85	132	720
88	234	480
132	351	720
140	480	693
160	231	792
176	468	960
240	252	275
480	504	550
720	756	825

In the  $1 \leq a < b < c \leq 2000$ , there are a 15 more, totalling 25.

a	b	c
170	264	1440
187	1020	1584
220	585	1200
264	702	1440
280	960	1386
308	819	1680
320	462	1584
352	936	1920
480	504	550
720	756	825
960	1008	1100
1008	1100	1155
1200	1260	1375
1440	1512	1650
1680	1764	1925

Searching  $1 \leq a < b < c \leq 4000$ , we get 54 Euler cuboids, in  $1 \leq a < b < c \leq 8000$  there are 120:

44	117	240	528	5796	6325	968	2574	5280	1680	1764	1925
85	132	720	560	1920	2772	980	3360	4851	1755	4576	6732
88	234	480	561	3060	4752	1008	1100	1155	1920	2016	2200
132	351	720	572	1521	3120	1012	2691	5520	2016	2200	2310
140	480	693	595	924	5040	1056	2808	5760	2160	2268	2475
160	231	792	616	15	3360	1100	2925	6000	2400	2520	2750
170	264	1440	640	924	3168	1120	1617	5544	2496	2565	7920
176	468	960	660	1755	3600	1120	3840	5544	2640	2772	3025
187	1020	1584	680	1056	5760	1144	3042	6240	2880	3024	3300
195	748	6336	700	2400	3465	1155	6300	6688	3024	3300	3465
220	585	1200	704	1872	3840	1188	3159	6480	3120	3276	3575
240	252	275	720	756	825	1200	1260	1375	3360	3528	3850
255	396	2160	748	1989	4080	1232	3276	6720	3600	3780	4125
264	702	1440	748	4080	6336	1260	4320	6237	3840	4032	4400
280	960	1386	765	1188	6480	1276	3393	6960	4032	4400	4620
308	819	1680	780	2475	2992	1280	1848	6336	4080	4284	4675
320	462	1584	792	2106	4320	1287	2640	7020	4320	4536	4950
340	528	2880	800	1155	3960	1320	3510	7200	4560	4788	5225
352	936	1920	828	2035	3120	1364	3627	7440	4800	5040	5500
374	2040	3168	832	855	2640	1400	4800	6930	5040	5292	5775
396	1053	2160	836	2223	4560	1408	3744	7680	5040	5500	5775
420	1440	2079	840	2880	4158	1440	1512	1650	5280	5544	6050
425	660	3600	850	1320	7200	1440	2079	7128	5520	5796	6325
429	880	2340	858	1760	4680	1452	3861	7920	5760	6048	6600
440	1170	2400	880	2340	4800	1540	5280	7623	6000	6300	6875
480	504	550	924	2457	5040	1560	2295	5984	6048	6600	6930
480	693	2376	935	1452	7920	1560	4950	5984	6240	6552	7150
484	1287	2640	935	5100	7920	1600	2310	7920	6480	6804	7425
510	792	4320	960	1008	1100	1656	4070	6240	6720	7056	7700
528	1404	2880	960	1386	4752	1664	1710	5280	6960	7308	7975

The number of Euler bricks appears to grow with respect to the box size because if  $(a, b, c)$  is an Euler brick, then  $(ka, kb, kc)$  is an Euler brick too. It would be interesting to know how primitive Euler bricks are distributed.

## Modular considerations: pieces of eight! Pieces of eight!

If we take a Diophantine equation and consider it modulo some number  $n$ , then the equation still holds. Turning things around: if a Diophantine equation has no solution modulo  $n$ , then there is no solution in the integers. By checking all possible solutions in the finite space of all possible cases, we can also determine some conditions, which have to hold.

Example:  $5x^4 = 3 + 7y^4$  has no integer solutions because modulo 8, we have no solution because modulo 8 we have  $x^4, y^4 \in \{0, 1\}$ .

To use this idea, let's assume we deal with prime Euler bricks, bricks for which the greatest common divisor of  $a, b, c$  is 1.

**For  $n \in \{2, 3, 5, 11\}$  as well as  $n \in \{2^2, 3^2, 4^2\}$ , there exists at least one side of an Euler brick which is divisible by  $n$ .**

Proof. The case  $n = 2, 4, 16$  follows directly from properties of Pythagorean triples, for  $n = 9$ , use that if two (say  $x, y$ ) are divisible by 3, then  $x^2 - y^2 = a^2 - b^2$  is divisible by 9 and  $a = b \pmod{3}$  showing that also  $z$  has to be divisible by 3 and the cube is not prime.

## Searching using irrational rotation: on a dead man's chest

The problem of solving Diophantine equations has a dynamical system side to it. Take one of the variables  $x$  as time, solve with respect to an other variable say  $y$  then write  $y = (f(x))^{1/n}$  where  $f$  is a polynomial. We can study the dynamical system  $(f(x))^{1/k} \rightarrow f(x+1)^{1/k} \pmod{1}$  and look for  $n$  to reach 0. If there are several parameters, we have a dynamical system with multidimensional time.

For the problem to find  $s, t$  for which

$$\sqrt{s^8 + 68t^2s^6 - 122t^4s^4 + 68t^6s^2 + t^8}$$

is close to an integer, we can change the parameter  $s, t$  along a line and get incredibly close. Unfortunately, we can not hit a lattice point.

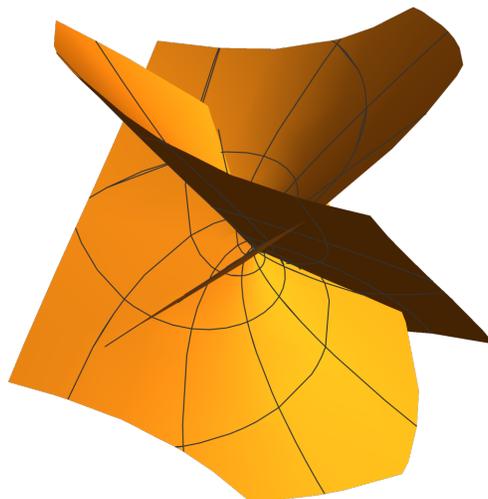


Fig 1. The Saunderson surface: a parametrized surface  $r(s, t) = (a(s, t), b(s, t), c(s, t))$  of Euler bricks.

**The treasure is not there: ney mate, you are marooned**

Spohn is the "Ben Gunn" of the Euler brick treasure island. He has moved the treasure elsewhere. But maybe it does not exist. Anyway, Spohn [32] proved in 1972:

**Theorem (Spohn): There are no perfect Euler bricks on the Saunderson surface of Euler bricks.**

Proof. With  $a = u(4v^2 - w^2)$ ,  $b = v(4u^2 - w^2)$ ,  $c = 4uvw$ ;  $w^2 = u^2 + v^2$ , we check  $a^2 + b^2 + c^2 = w^2(u^4 + 18u^2v^2 + v^4)$ . Pocklington [27] has shown first in 1912 that  $u^4 + 18u^2v^2 + v^4$  can not be square. His argument is more general. We can prove this more easily however:

**Lemma (Pocklington): Unless  $xy = 0$ , the Diophantine equation  $x^4 + 18x^2y^2 + y^4 = z^2$  has no solution.**

Proof:  $x, y$  can not have a common factor, otherwise we could divide it out and include it to  $z$ . Especially, there is no common factor 2. If  $x^4 + 18x^2y^2 + y^4 = (x^2 + y^2)^2 + 4^2x^2y^2 = z^2$  then we have Pythagorean triples which can be parametrized.

a) Assume first the triples are primitive, there is no common divisor among the triple  $(x^2 + y^2)^2, 4^2x^2y^2, z^2$ .

(i) If  $x, y$  are both odd, we must have

$$\begin{aligned} x^2 + y^2 &= 2uv \\ 4xy &= u^2 - v^2 . \end{aligned}$$

The first equation proves that  $x^2 + y^2 = 2 \pmod 4$ . If  $2uv = 2 \pmod 4$ , both  $u, v$  must be odd. The second equation can now not be solved modulo 8. If  $u = 4n \pm 1, v = 4m \pm 1$ , then  $u^2 - v^2$  is divisible by 8. But the left hand side of the equation is congruent to 4 modulo 8.

(ii) If  $x$  is odd and  $y$  is even, the Pythagorean triple representation is

$$\begin{aligned} x^2 + y^2 &= u^2 - v^2 \\ 2xy &= uv . \end{aligned}$$

Because  $y$  is even, the second equation shows that  $uv$  is divisible by 4 and because  $u, v$  have no common divisor, wither  $u$  is divisible by 4 or  $v$  is divisible by 4. If  $u$  is divisible by 4, the first equation can not be solved modulo 4. If  $v$  is divisible by 4, the first equation has no solution modulo 16: the right hand side is 0, 1, 4, 9 modulo 16 while the left hand side is congruent to 5, 13 modulo 16.

b) If there is a common divisor  $p$  among  $(x^2 + y^2)^2$  and  $4^2x^2y^2$  then it has to be 2, because any other factor  $p$  would be a factor of either  $x$  or  $y$  as well as of  $x^2 + y^2$  and so of both  $x$  and  $y$ , which we had excluded at the very beginning. With a common factor 2, we have a Pythagorean triple parametrization

$$\begin{aligned} x^2 + y^2 &= 4uv \\ 4xy &= 2(u^2 - v^2) . \end{aligned}$$

but since  $x, y$  are both odd,  $x^2 + y^2$  is congruent 2 modulo 4 contradicting the first equation.

This finishes the proof of the lemma and so the theorem of Spohn. It is remarkable that the result of Pocklington does not use infinite decent in this case. By the way, the article of Pocklington of 1912 has been checked out many times at Cabot library since this volume almost falls to dust.

Side remark: quartic Diophantine equations of this type form an old topic [23] (section 4). Fermat had shown using infinite descent that  $u^4 + v^4$  is never a square so that  $u^4 + v^4 = z^4$  has no solution. As is well known, he concluded a bit hastily that he has a proof that  $x^p + y^p = z^p$  has no solution for all  $p > 2$  but that the margin is not large enough to hold it.

### Large numbers: shiver my timbers!

There are more parametrizations to be explored. Euler got

$$\begin{aligned} a &= 2mn(3m^2 - n^2)(3n^2 - m^2) \\ b &= 8mn(m^4 - n^4) \\ c &= (m^2 - n^2)(m^2 - 4mn + n^2)(m^2 + 4mn + n^2) \end{aligned}$$

for which  $x^2 + y^2 = 4m^2n^2(5m^4 - 6m^2n^2 + 5n^4)^2$ ,  $x^2 + z^2 = (m^2 + n^2)^6$ ,  $y^2 + z^2 = (m - n)^2(m + n)^2(m^4 + 18m^2n^2 + n^4)^2$ .

In that case, we have  $x^2 + y^2 + z^2 = (m^2 + n^2)^2(m^8 + 68m^6n^2 - 122m^4n^4 + 68m^2n^6 + n^8) = (m^2 + n^2)^2[(m^4 + n^4)^2 + 2m^2n^2(17m^4 - 31m^2n^2 + 17n^4)]$ .

Computer algebra systems like to compute as long as possible in algebraic fields. For example:

```
Expand[(5 + Sqrt[5])^6]
```

produces the result

```
72000+32000 Sqrt[5]
```

This is a much more valuable result than a numerical value like 143554.1753... The evaluation of numerical values in Mathematica is quite mysterious: sometimes, it works quite well:

```
N[Sqrt[2^171 + 1]] - N[Sqrt[2^171 + 1], 100]
```

Sometimes, it does not

```
N[Sqrt[2^117 + 1]] - N[Sqrt[2^117 + 1], 100]
```

which gives in this case a value of -64. Even increasing the accuracy like with

```
$$$MaxExtraPrecision=20000000000;
```

Wolfram research promised to fix this problem.

By the way, this issue is much better in Pari. How to compute with large accuracy in the open source algebra system Pari/GP? Pari projects algebraic integers correctly, even with millions of digits:

```
\p 1000000
a=sqrt(2^117+1)
```

It can compute up to 161 million significant digits (you have to increase the stack size to do so), like defining

```
parisize = 800M
```

in the .gprc file. It still can produce an overflow depending on your machine. But working a million digits or so is ok.

## History: Captain Flints logbook

In 1719 by **Paul Halcke**, a German accountant, who would also do astronomical computations, found the smallest solution [4]. Nothing earlier seems to be known.

N. Saunderson found in 1740 the parametrization with two parameters mentioned above. Only in 1972, it was established by Spohn that the parametrization does not lead to that these parametrizations do not lead to perfect Euler bricks. Jean Lagrange gave an other argument in 1979 also.

Leonard Euler found in 1770 a second parametrization and in 1772 a third parametrization. After his death, more parametrizations were found in his notes.



## Modern considerations: the black spot

The topic has appeared several times in American Mathematical Monthly articles and was even a topic for a PhD theses in 2000 in Europe and 2004 in China. Because of its simplicity, it is certainly of great educational value. The topic appears for example in a journal run by undergraduates similar to HCMR [26].

Noam Elkies told me:

*“The algebraic surface parametrizing Euler bricks is the intersection in  $P^5$  of the quadrics  $x^2 + y^2 = c^2$ ,  $z^2 + x^2 = b^2$ ,  $y^2 + z^2 = a^2$  which happens to be a  $K3$  surface of maximal rank, so quite closely related to much of my own recent work in number theory. Adding the condition  $x^2 + y^2 + z^2 = d^2$  yields a surface of general type, so it might well have no nontrivial rational points but nobody knows how to prove such a thing.”*

Noam also remarked that Euler’s parametrization would only lead to a finite number of perfect Cuboids as a consequence of Mordell’s theorem. There seems however no reason to be known which would tell whether there are maximally finitely many primitive perfect cuboids.

There are also relations with elliptic curves since a system of quadratic equations often define an elliptic curve. See [20]. The article [18] which mentions also relations with rational points on plane cubic curves.

The problem appeared also in articles for the general public. In 1970 Martin Gardner asked to find solutions for which 6 of the 7 distances in the cuboid are integers. If the large diagonal

is an integer, these are no more Euler bricks, unless we would have a perfect brick.

As for any open problem, it is also interesting to look more fundamental questions. As with many open problems, the problem to find a perfect Euler brick could be undecidable: we would not be able to find a proof that there exists no Euler brick. This is possible only if there is indeed no Euler brick. You can read an amusing story about Goldbach conjecture in “Uncle Petros and the Goldbach conjecture”, where the perspective of such an option blew all motivation of poor uncle Petros to search for the Goldbach grail. [5]

### Treasure problems: scatter and find ‘em!

Many unsolved problems like the Goldbach conjecture, the Riemann hypothesis, the problem to find perfect numbers, or the problem of finding perfect Euler bricks, finding dense sphere packings in higher dimensions, are mathematical tasks which could in principle be solved quickly: by finding an example - if it should exist:

- Writing down an integer which can not be written as a sum of two primes would settle the Goldbach conjecture.
- Finding an integer for which the sum of the proper divisors is the number itself.
- Find a root of the zeta function with  $Re(z) \neq 1/2$ . Just one lucky punch would be needed to solve the problem.

But like treasure hunting, aiming to catch such a treasure is not a good business plan or a way to make a living: the treasure simply does not need to be there. If it does not exist, the most skillful treasure hunter can not be successful.

But it is the search which is interesting, not the prospect of finding anything.

By the way, numerical searches for the grail of a perfect cuboid have been done. Randal Rathbun has found no perfect cuboid with least edge larger than 333750000. The greatest edge is larger than  $10^9$ . See [8]. Treasure hunters all over the world have probably gone even further. See [30] on ArXiv.

**Update, May 20 2022:** Robert Matson (Matson, Robert D. ”Results of a Computer Search for a Perfect Cuboid” (PDF). [unsolvedproblems.org](http://unsolvedproblems.org). Retrieved May 23, 2022.) reports that *there are no perfect cuboids with odd side less than 25 trillion, and no perfect cuboids with minimum side less than 500 billion.*

### REFERENCES

- [1] A. Björner. A cell complex in number theory. *Advances in Appl. Math.*, 46:71–85, 2011.
- [2] J. Bronowski. *The Ascent of Man*. BBC, 1973.
- [3] R.C. Buck. Sherlock Holmes in Babylon. *American Mathematical Monthly*, 87:335–345, 1980.
- [4] L.E. Dickson. *History of the theory of numbers. Vol.II:Diophantine analysis*. Chelsea, New York, 1966.
- [5] A. Doxiadis. *Uncle Petros and Goldbach’s Conjecture*. Bloomsbury, USA, New York, 2000.
- [6] H. Eves. *Great moments in mathematics (I and II)*. The Dolciani Mathematical Expositions. Mathematical Association of America, Washington, D.C., 1981.
- [7] R. Guy. The strong law of small numbers. *Amer. Math. Monthly*, 95:697–712, 1988.
- [8] R. K. Guy. *Unsolved Problems in Number Theory*. Springer, Berlin, 3 edition, 2004.
- [9] C.H. Kahn. *Pythagoras and the Pythagoreans, A brief history*. Hackett Publishing Company, 2001.
- [10] V. Katz. *Mathematics of Egypt, Mesopotamia, China, India and Islam*. Princeton Univ. Press, 2007.
- [11] O. Knill. A multivariable chinese remainder theorem. <https://arxiv.org/abs/1206.5114>, 2005-2012.
- [12] O. Knill. Hunting for Perfect Euler Bricks. *The Harvard College Mathematics Review*, 2, no 2, 2008.
- [13] O. Knill. Treasure Hunting Perfect Euler Bricks. Mathtable talk, February 24, 2009, 2009.

- [14] O. Knill. On primes, graphs and cohomology. <https://arxiv.org/abs/1608.06877>, 2016.
- [15] O. Knill. Some experiments in number theory. <https://arxiv.org/abs/1606.05971>, 2016.
- [16] O. Knill. Exploring creativity through computer algebra. *Tech-Based Teaching: Computational Thinking in the Classroom*, 2018. <https://medium.com/tech-based-teaching/exploring-creativity-through-computer-algebra-c3788d1a06b7>.
- [17] O. Knill. Graph complements of circular graphs. <https://arxiv.org/abs/2101.06873>, 2021.
- [18] J. Leech. The rational cuboid revisited. *American Mathematical Monthly*, 84:518–533, 1977.
- [19] M. Livio. *Why? - What makes us Curious*. Schuster, 2017.
- [20] Allen J. MacLeod. Parametric expressions for a nearly-perfect cuboid. [maths.paisley.ac.uk/allanm/PDFILES/Newsid.pdf](https://maths.paisley.ac.uk/allanm/PDFILES/Newsid.pdf).
- [21] D.F. Mansfield. Plimpton 322: A study of rectangles. *Foundations of Science*, 26:977–1005, 2021.
- [22] E. Maor. *The Pythagorean Theorem: A 4000 year history*. Princeton University Press, 2007.
- [23] L.J. Mordell. *Diophantine Equations*, volume 30 of *Pure and Applied Mathematics*. Academic Press, London and New York, 1969.
- [24] P. Nastasi and A. Scimone. Pietro mengoli and the six-square problem. *Historia Mathematica*, 21:10–27, 1994.
- [25] O. Neugebauer. *The exact Sciences in Antiquity*. Dover Publications, second edition, 1969.
- [26] A. Ortan and V. Quenneville-Bélaïr. Euler’s brick. *The Delta-Epsilon: Mc Gills Undergraduate Mathematics Magazine*, Issue 2006. <http://sums.mcgill.ca/delta-epsilon/issue06.html>.
- [27] H.C. Pocklington. Some Diophantine impossibilities. *Proceedings Cambridge Philosophical Society*, 18:110–118, 1912.
- [28] A.S. Posamentier. *The Pythagorean Theorem: The Story of its Power and Beauty*. Prometheus, 2010.
- [29] E. Kaplan R. Kaplan. *Hidden Harmonies*. Bloomsbury, 2011.
- [30] R. L. Rathbun. The rational cuboid table of Maurice Kraitchik. <http://arxiv.org/abs/math.HO/0111229>, 2001.
- [31] E. Robson. Words and Pictures: New Light on Plimpton 322. *Amer. Math. Monthly*, 109:105–120, 2002.
- [32] W. Spohn. On the integral cuboid. *American Mathematical Monthly*, 79:57–59, 1972.
- [33] R.L Stevenson. *Treasure Island*. Cassell and Co., 1883.
- [34] I. Stewart. *Visions of Infinity*. Basic Books, 2013.
- [35] P. Strathern. *The big idea: Pythagoras and his theorem*. Arrow books, 1997.
- [36] F.J. Swetz and T.I. Kao. *Was Pythagoras Chinese?* Pennsylvania State University Press, 1988.

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