The residue theorem and its applications

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This text contains some notes to a three hour lecture in complex analysis given at Caltech. The lectures start from scratch and contain an essentially self-contained proof of the Jordan normal form theorem, I had learned from Eugene Trubowitz as an undergraduate at ETH Zürich in the third semester of the standard calculus education at that school. My text also includes two proofs of the fundamental theorem of algebra using complex analysis and examples, which examples showing how residue calculus can help to calculate some definite integrals. Except for the proof of the normal form theorem, the material is contained in standard text books on complex analysis. The notes assume familiarity with partial derivatives and line integrals. I use Trubowitz approach to use Greens theorem to prove Cauchy's theorem. *[When I had been an undergraduate, such a direct multivariable link was not in my complex analysis text books (Ahlfors for example does not mention Greens theorem in his book).]* *For the Jordan form section, some linear algebra knowledge is required.*

1 The residue theorem

Definition Let $D \subset \mathbf{C}$ be open (every point in D has a small disc around it which still is in D). Denote by $C^1(D)$ the differentiable functions $D \to \mathbf{C}$. This means that for $f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$ the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

are continuous, real-valued functions on D.

Definition. Let $\gamma : (a,b) \to D$ be a differentiable curve in D. Define the **complex line integral**

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t)) \cdot \dot\gamma(t)\ dt$$

If $z = x + iy$, and $f = u + iv$, we have

$$\int_\gamma f(z)\ dz = \int_a^b (u\dot x - v\dot y) + i(u\dot y + v\dot x)\ dt$$

The integral for piecewise differentiable curves γ is obtained by adding the integrals of the pieces. We always assume from now on that all curves γ are piecewise differentiable.

Example. $D = \{|z| < r\}$ with $r > 1$, $\gamma : [0, k \cdot 2\pi] \to D, \gamma(t) = (\cos(t), \sin(t))$, $f(z) = z^n$ with $k \in \mathbf{N}, \mathbf{n} \in \mathbf{Z}$.

$$\int_\gamma z^n\ dz = \int_0^{k2\pi} e^{int} e^{it} i\ dt = \int_0^{k2\pi} e^{i(n+1)t} i\ dt$$

If $n \neq -1$, we get

$$\int_\gamma z^n\ dz = \frac{1}{n+1} e^{i(n+2)t}|_0^{k2\pi} = 0$$

If $n = -1$, we have $\int_\gamma z^n\ dz = \int_0^{k2\pi} i\ dt = k \cdot 2\pi i$.

We recall from vector calculus the **Green formula** for a vector field (u,v) in $\mathbf{R^2}$

$$\int_\gamma (u\dot x + v\dot y)\ dt = \int_D (v_x - u_y)\ dx \wedge dy\ ,$$

where D is the open set enclosed by closed curve $\gamma = \delta D$ and where $dx \wedge dy = dxdy$ is the **volume form** in the plane. Write $dz = dx + idy$, $d\overline{z} = dx - idy$ and $dz \wedge d\overline{z} = 2idx \wedge dy$. Define for $f \in C^1(D)$

Theorem 1.1 (Complex Green Formula)

> $f \in C^1(D)$, $D \subset \mathbf{C}$, $\gamma = \delta D$.
> $$\int_\gamma f(z)dz = \int_D \frac{\partial f}{\partial \overline{z}} \, dz \wedge d\overline{z} \; .$$

Proof. Green's theorem applied twice (to the real part with the vector field $(u, -v)$ and to the imaginary part with the vector field (v, u)) shows that

$$\int_\gamma f(z) \, dz = \int_a^b (u\dot{x} - v\dot{y}) + i \cdot (u\dot{y} + v\dot{x}) \, dt$$

coincides with

$$\begin{aligned} \int_D \frac{\partial f}{\partial \overline{z}} \, dz \wedge d\overline{z} &= \int_D \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x) \, 2i dx \wedge dy \\ &= \int_D (-u_y - v_x) dx \wedge dy + i \cdot \int_D (u_x - v_y) dx \wedge dy \end{aligned}$$

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We check that

$$\frac{\partial f}{\partial \overline{z}} = 0 \quad \Leftrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \; .$$

The right hand side are called the **Cauchy-Riemann differential equations**.

Definition. Denote by $C^\omega(D)$ the set of functions in $C^1(D)$ for which $\frac{\partial f}{\partial \overline{z}} = 0$ for all $z \in D$. Functions $f \in C^\omega(D)$ are called **analytic** or **holomorphic** in D.

Corollary 1.2 (Theorem of Cauchy)

> $f \in C^\omega(D)$, $D \subset \mathbf{C}$, $\gamma = \delta D$.
> $$\int_\gamma f(z) \, dz = 0 \; .$$

Proof.

$$\int_\gamma f(z) \, dz = \int_D \frac{\partial f}{\partial \overline{z}} dz \wedge d\overline{z} = 0$$

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Corollary 1.3 (Cauchy's Integral formula)

> *$f \in C^\omega(D)$, D simply connected, $\gamma = \delta D$. For any $a \in D$*
> $$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w) \, dw}{w - a} \; .$$

Proof. Define for small enough $\epsilon > 0$ the set $D_\epsilon = D \setminus \{|z - a| \leq \epsilon\}$ and the curve $\gamma_\epsilon : t \mapsto z + \epsilon e^{it}$. (Because D is open, the set D_ϵ is contained in D for small enough ϵ). Because $\gamma \cup -\gamma_\epsilon = \delta D_\epsilon$ we get by Cauchy's theorem 1.2

$$\int_\gamma \frac{f(w)}{w - a} \, dw - \int_{\gamma_\epsilon} \frac{f(w)}{w - a} \, dw = 0 \; .$$

We compute

$$\int_{\gamma_\epsilon} \frac{f(w)}{w - a} \, dw = \int_0^{2\pi} \frac{f(z + \epsilon \cdot e^{it})}{a + \epsilon \cdot e^{it} - a} \frac{d}{dt}(a + \epsilon e^{it}) \, dt = i \cdot \int_0^{2\pi} f(a + \epsilon \cdot e^{it}) \, dt$$

The right hand side converges for $\epsilon \to 0$ to $2\pi i f(a)$ because $f \in C^1(D)$ implies

$$|f(a + \epsilon e^{it}) - f(a)| \leq C \cdot \epsilon$$

Corollary 1.4 (Generalized Cauchy Integral formulas)

Assume $f \in C^\omega(D)$ and $D \subset \mathbf{C}$ simply connected, and $\delta D = \gamma$. For all $n \in \mathbf{N}$ one has $f^{(n)}(z) \in C^\omega(D)$ and for any $z \notin \gamma$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)\, dz}{(w-z)^{n+1}} \; .$$

Proof. Just differentiate Cauchy's integral formula n times. ∎

It follows that $f \in C^\omega(D)$ is arbitrary often differentiable.

<u>Definition</u> Let $f \in C^\omega(D \setminus \{a\})$ and $a \in D$ with simply connected $D \subset \mathbf{C}$ with boundary γ. Define the **residue** of f at a as

$$\mathrm{Res}(f,a) := \frac{1}{2\pi i} \int_\gamma f(z)\, dz \; .$$

By Cauchy's theorem, the value does not depend on D.

<u>Example</u>. $f(z) = (z-a)^{-1}$ and $D = \{|z-a| < 1\}$. Our calculation in the example at the beginning of the section gives $\mathrm{Res}(f,a) = 1$.

A generalization of Cauchy's theorem is the following residue theorem:

Corollary 1.5 (The residue theorem)

$f \in C^\omega(D \setminus \{z_i\}_{i=1}^n)$, D open containing $\{z_i\}$ with boundary $\delta D = \gamma$.

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{i=1}^n \mathrm{Res}(f, z_i) \; .$$

Proof. Take ϵ so small that $D_i = \{|z - z_i| \leq \epsilon\}$ are all disjoint and contained in D. Applying Cauchy's theorem to the domain $D \setminus \bigcup_{1=1}^n D_i$ leads to the above formula. ∎

2 Calculation of definite integrals

The residue theorem has applications in functional analysis, linear algebra, analytic number theory, quantum field theory, algebraic geometry, Abelian integrals or dynamical systems.

In this section we want to see how the residue theorem can be used to computing definite real integrals.

The first example is the **integral-sine**

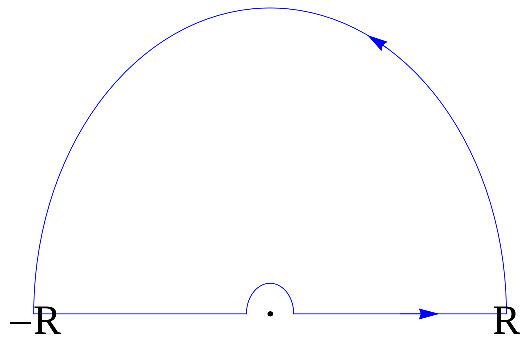
$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt \; ,$$

a function which has applications in electrical engineering. It is used also in the proof of the **prime number theorem** which states that the function $\pi(n) = \{p \leq n \mid p \text{ prime}\}$ satisfies $\pi(n) \sim x/log(x)$ for $x \to \infty$.

$$\boxed{\mathrm{Si}(\infty) = \int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}}$$

Proof. Let $f(z) = \frac{e^{iz}}{z}$ which satisfies $f \in C^\omega(\mathbf{C} \setminus \{\mathbf{0}\})$. For $z = x \in \mathbf{R}$, we have $\mathrm{Im}(f(z)) = \frac{\sin(x)}{x}$. Define for $R > \epsilon > 0$ the open set D enclosed by the curve $\gamma = \bigcup_{i=1}^4 \gamma_i$, where

Figure 1
$\gamma_1 : t \in [\epsilon, R] \mapsto t + 0 \cdot i.$
$\gamma_2 : t \in [0, \pi] \mapsto R \cdot e^{it}.$
$\gamma_3 : t \in [-R, -\epsilon] \mapsto t + 0 \cdot i.$
$\gamma_4 : t \in [\pi, 0] \mapsto \epsilon \cdot e^{it}.$



By Cauchy's theorem

$$0 = \int_\gamma f(z)\, dz = \int_\epsilon^R \frac{e^{ix}}{x} dx + \int_0^\pi \frac{e^{iRe^{it}}}{Re^{it}} iRe^{it} dt + \int_{-R}^{\epsilon} \frac{e^{ix}}{x} dx + \int_\pi^0 \frac{e^{i\epsilon e^{it}}}{\epsilon e^{it}} i\epsilon e^{it} dt \ .$$

The imaginary part of the first and the third integral converge for $\epsilon \to 0$, $R \to \infty$ both to $\mathrm{Si}(\infty)$. The imaginary part of the fourth integral converges to $-\pi$ because

$$\lim_{\epsilon \to 0} \int_0^\pi e^{i\epsilon e^{it}}\ i\ dt \to i\pi \quad .$$

The second integral converges to zero for $R \to \infty$ because $|e^{iRe^{it}}| = |e^{-R\sin(t)}| \leq |e^{-Rt}|$ for $t \in (0, \pi/2]$. ∎

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx = \pi}$$

Proof. Take $f(z) = 1 + z^2$ which has a simple pole $a = i$ in the upper half plane. Define for $R > 0$ the half-disc D with a hole which has as a boundary the curve $\gamma = \bigcup_{i=1}^2 \gamma_i$ with
$\gamma_1 : t \in [-R, R] \mapsto t + 0 \cdot i.$
$\gamma_2 : t \in [0, \pi] \mapsto R \cdot e^{it}.$
f is analytic in $D \setminus \{i\}$ and by the residue theorem

$$\int_\gamma f(z)\ dz = \int_{-R}^R \frac{1}{1+x^2}\ dx = \int_0^\pi \frac{iRe^{it}\ dt}{1 + R^2 \cdot e^{2it}} = 2\pi i \cdot \mathrm{Res}(f(z), i) = 2\pi i \cdot \lim_{z \to i} (z - i) \cdot f(z) = \pi$$

The second integral from the curve γ_2 goes to zero for $R \to \infty$.

Let f, g be two polynomials with $n = \deg(g) \geq 2 + \deg(f)$ such that the poles z_i of $h := f/g$ are not on $\mathbf{R}^+ \cup \{\mathbf{0}\}$. $$\int_0^\infty h(x)\ dx = -\sum_{i=1}^n \mathrm{Res}(h(z) \cdot \log(z), z_i)$$

Proof. Define for $R > r > 0$ the domain D enclosed by the curves $\gamma_R \cup \gamma_- \cup \gamma_r \cup \gamma_+$ with $\gamma_+ : t \in [r, R] \mapsto t + 0 \cdot i.$
$\gamma_- : t \in [R, r] \mapsto t + 0 \cdot i.$
$\gamma_R : t \in [0, 2\pi] \mapsto R \cdot e^{it}.$
$\gamma_r : t \in [0, 2\pi] \mapsto r \cdot e^{-it}.$
and apply the residue theorem for the function $h(z) \cdot \log(z)$:

$$\int_\gamma h(z)\log(z)\ dz \quad = \quad \int_{\gamma_R} h(z)\log(z)\ dz + \int_{\gamma_-} h(z)\log(z)\ dz + \int_{\gamma_+} h(z)\log(z)\ dz + \int_{\gamma_r} h(z)\log(z)\ dz$$

Because of the degree assumption, $\int_{\gamma_R} h(z)\log(z) \to 0$ for $R \to \infty$. Because h is analytic near 0 and $\log(z)$ goes slower to ∞ than $z \to 0$ we get also $\int_{\gamma_r} h(z)\log(z)\,dz \to 0$ as $r \to 0$. The sum of the last two integrals goes to $-(2\pi i)\int_0^\infty h(x)\,dx$ because

$$\int_{\gamma_+} h(z)\log(z)\,dz = -\int_{\gamma_-} h(z)(\log(z)+2\pi i)\,dz\ .$$

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$$\boxed{\int_0^\pi \frac{d\theta}{a+\cos(\theta)}\,d\theta = \frac{\pi}{\sqrt{a^2-1}}, a > 1}$$

Proof. Put $z = e^{i\theta}$. Then

$$a+\cos(\theta) = a + \frac{z+z^{-1}}{2} = \frac{2az+z^2+1}{2z}\ .$$

Let $\gamma : \theta \mapsto e^{i\theta}$

$$\int_0^\pi \frac{d\theta}{a+\cos(\theta)}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos(\theta)}\,d\theta = \int_\gamma \frac{1}{2i}\frac{2dz}{2az+z^2+1}\ .$$

From the two zeros $-a \pm \sqrt{a^2-1}$ of the polynomial $2az+z^2+1$ the root λ_+ is in the unit disc and λ_- outside the unit disc. From the residue theorem, the integral is

$$2\pi i\frac{1}{i}\mathrm{Res}(\frac{1}{2az+z^2+1}, \lambda_+) = \frac{2\pi}{\lambda_+-\lambda_-} = \frac{\pi}{\sqrt{a^2-1}}\ .$$

3 Jordan normal form for matrices

As an other application of complex analysis, we give an elegant proof of **Jordan's normal form theorem** in linear algebra with the help of the Cauchy-residue calculus.

Let $M(n,\mathbf{R})$ denote the set of real $n \times n$ matrices and by $M(n,\mathbf{C})$ the set $n \times n$ matrices with complex entries. For $A \in M(n,\mathbf{C})$ the **characteristic polynomial** is

$$\det(\lambda - A) = \prod_{i=1}^k (\lambda - \lambda_i)^{\mu_i}\ .$$

We simply write $\lambda - A$ instead of $\lambda I - A$, where I is the identity matrix. The complex numbers λ_i are called the **eigenvalues** of A and μ_i denote their **multiplicities**. Clearly $\sum_{i=1}^k \mu_i = n$.

Two matrices $A, B \in M(n,\mathbf{C})$ are called **similar** if there exists an invertible matrix S such that $A = S^{-1}BS$.

Theorem 3.1 (Jordan normal form theorem)

$A \in M(n,\mathbf{C})$ is similar to a matrix $\begin{bmatrix} [A_1] & 0 & \dots & 0 \\ 0 & [A_2] & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & [A_k] \end{bmatrix}$ *where* $A_i =$ $\begin{bmatrix} \lambda_i & 1 & & & \\ & \cdot & 1 & & \\ & & \cdot & & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}$ *are called* **normal blocks**.

<u>Remark.</u> It follows if all eigenvalues of A are different, then A is diagonalizable.

Denote for $\lambda \neq \lambda_i$ the **resolvent** matrix

$$R(\lambda) = (\lambda - A)^{-1}\ .$$

The function $\lambda \mapsto R(\lambda)$ is analytic in $D = \mathbf{C} \setminus \{\lambda_\mathbf{i}\}$ in the sense that for all i, j, the functions $\lambda \mapsto [R(\lambda)]_{ij}$ are analytic in D. The reason is that there exist polynomials (where $A^{(ij)}$ are the matrices obtained by deleting the i' th row and the j' th column of A), such that

$$[R(\lambda)]_{ij} = \frac{\alpha_{ij}(\lambda)}{\det(\lambda - A)} \; .$$

Lemma 3.2 (The resolvent identity)

$$\boxed{R(\lambda) - R(\lambda') = (\lambda' - \lambda)R(\lambda)R(\lambda')}$$

Proof. From $A(\lambda - A) = (\lambda - A)A$ follows $R(\lambda)A = AR(\lambda)$ and we get by filling in $(\lambda' - A)R(\lambda') = I$ and $(\lambda - A)R(\lambda) = I$

$$R(\lambda) - R(\lambda') = R(\lambda)(\lambda' - A)R(\lambda') - (\lambda - A)R(\lambda)R(\lambda') = (\lambda' - \lambda)R(\lambda)R(\lambda') \; .$$

$\blacksquare$

<u>Definition</u> $C^\omega(D, M(n, \mathbf{C}))$ denotes the set of functions $f : D \mapsto M(n, \mathbf{C})$, such that for all i, j the map $z \mapsto [f(z)]_{ij}$ is in $C^\omega(D)$. Given a curve γ in D we define the complex integral $\int f(z) \, dz$ by

$$[\int f(z) \, dz]_{ij} = \int [f(z)]_{ij} \, dz \; .$$

Define for $\delta < \min_{ij} |\lambda_i - \lambda_j|$ and $\gamma_i : t \mapsto z_i + \delta e^{it}$ the matrices

$$\begin{aligned} P_i &= \frac{1}{2\pi i} \int_{\gamma_i} R(\lambda) \, d\lambda \\ N_i &= \frac{1}{2\pi i} \int_{\gamma_i} (\lambda - \lambda_i) R(\lambda) \, d\lambda \; . \end{aligned}$$

Theorem 3.3 (Jordan decomposition of a matrix)

1) $P_iP_j = \delta_{ij}P_j$,
2) $\sum_{i=1}^k P_i = I$.
3) $N_iP_j = \delta_{ij}N_i = P_jN_i$
4) $N_iN_j = 0, \; i \neq j, \;\; P_i(A - \lambda_i) = N_i, \;\; N_i^{\mu_i} = 0$
5) $A = \sum_{i=1}^k \lambda_i P_i + \sum_{i=1}^k N_i$

Proof.
1) For $i \neq j$ we have using the resolvent identity

$$\begin{aligned} (2\pi i)^2 P_iP_j &= \int_{\gamma_i} R(\lambda) \, d\lambda \int_{\gamma_j} R(\lambda') \, d\lambda' = \int_{\gamma_i} \int_{\gamma_j} R(\lambda)R(\lambda') \, d\lambda d\lambda' \\ &= \int_{\gamma_i} \int_{\gamma_j} \frac{R(\lambda) - R(\lambda')}{\lambda' - \lambda} \, d\lambda \, d\lambda' \\ &= \int_{\gamma_i} R(\lambda) \int_{\gamma_j} \frac{1}{\lambda' - \lambda} \, d\lambda' \, d\lambda + \int_{\gamma_i} R(\lambda') \int_{\gamma_j} \frac{1}{\lambda' - \lambda} \, d\lambda \, d\lambda' = 0 \; . \end{aligned}$$

On the other hand, with $\gamma_i' : t \mapsto \lambda_i + \delta/2 \cdot e^{it}$

$$\begin{aligned} (2\pi i)^2 P_iP_i &= \int_{\gamma_i} \int_{\gamma_i} \frac{R(\lambda) - R(\lambda')}{\lambda' - \lambda} \, d\lambda \, d\lambda' = \int_{\gamma_i} \int_{\gamma_i'} \frac{R(\lambda) - R(\lambda')}{\lambda' - \lambda} \, d\lambda \, d\lambda' \\ &= \int_{\gamma_i} R(\lambda) \int_{\gamma_i'} \frac{1}{\lambda' - \lambda} \, d\lambda' \, d\lambda - \int_{\gamma_i'} R(\lambda') \int_{\gamma_i} \frac{1}{\lambda' - \lambda} \, d\lambda \, d\lambda' \\ &= 2\pi i \cdot \int_{\gamma_i'} R(\lambda') d\lambda' \; , \end{aligned}$$

where we used $\int_{\gamma_i'} \frac{1}{\lambda' - \lambda} \, d\lambda' = 0$ and $\int_{\gamma_i} \frac{1}{\lambda' - \lambda} \, d\lambda = 2\pi i$.

2) Using Cauchy's theorem we have for any curve $\gamma_R = \{|\lambda| = R\}$ enclosing all the eigenvalues

$$\begin{aligned} \sum_{i=1}^k P_i &= \frac{1}{2\pi i} \sum_{i=1}^k \int_{\gamma_i} R(\lambda) \, d\lambda = \frac{1}{2\pi i} \int_{\gamma_R} R(\lambda) \, d\lambda \\ &= \frac{1}{2\pi i} \int R(\lambda) - \frac{A}{\lambda} R(\lambda) \, d\lambda + \frac{1}{2\pi i} \int \frac{A}{\lambda} R(\lambda) \, d\lambda \end{aligned}$$

The claim follows from

$$\frac{1}{2\pi i}\int_{\gamma_R}\frac{1}{\lambda}d\lambda = 1$$

and from the fact that for $R\to\infty$

$$\int_{\gamma_R}\frac{A}{\lambda}R(\lambda)\,d\lambda \to 0$$

since $|R(\lambda)_{ij}| \le C\cdot\lambda^{-1}$ for a constant C only dependent on A.

3) For $i\neq j$, we get with the resolvent identity

$$\begin{aligned}(2\pi i)^2 N_i P_j &= \int_{\gamma_i}\int_{\gamma_j}(\lambda-\lambda_i)\cdot R(\lambda)R(\lambda')\,d\lambda\,d\lambda' \\ &= \int_{\gamma_i}\int_{\gamma_j}\frac{\lambda-\lambda_i}{\lambda'-\lambda}(R(\lambda)-R(\lambda'))\,d\lambda\,d\lambda' \\ &= \int_{\gamma_i}\int_{\gamma_j}\frac{\lambda-\lambda_i}{\lambda'-\lambda}R(\lambda)\,d\lambda'\,d\lambda - \int_{\gamma_i}\int_{\gamma_j}\frac{\lambda-\lambda_i}{\lambda'-\lambda}R(\lambda')\,d\lambda\,d\lambda' = 0\,.\end{aligned}$$

Using the curve $\gamma_i' = \{|\lambda-\lambda_i| = \delta/2\}$ we have

$$\begin{aligned}(2\pi i)^2 N_i P_i &= \int_{\gamma_i}\int_{\gamma_i'}(\lambda-\lambda_i)\cdot R(\lambda)R(\lambda')\,d\lambda\,d\lambda' \\ &= \int_{\gamma_i}\int_{\gamma_i'}\frac{\lambda-\lambda_i}{\lambda'-\lambda}(R(\lambda)-R(\lambda'))\,d\lambda\,d\lambda' \\ &= \int_{\gamma_i}(\lambda-\lambda_i)R(\lambda)\,d\lambda\int_{\gamma_i'}\frac{1}{\lambda'-\lambda}\,d\lambda' + \int_{\gamma_i}(\lambda-\lambda_i)R(\lambda')\,d\lambda\int_{\gamma_i'}\frac{1}{\lambda'-\lambda}\,d\lambda' \\ &= (2\pi i)^2 N_i\,.\end{aligned}$$

4) $N_iN_j = 0$ is left as an exercise. (The calculation goes completely analogue to the already done calculations in 1) or in 3)

$$\begin{aligned}(2\pi i)P_i(A-\lambda_i) &= \int_{\gamma_i}R(\lambda)(A-\lambda_i)\,d\lambda = \int_{\gamma_i}R(\lambda)(A-\lambda_i) - I\,d\lambda \\ &= \int_{\gamma_i}R(\lambda)(A-\lambda_i) - R(\lambda)(A-\lambda)\,d\lambda \\ &= \int_{\gamma_i}(\lambda-\lambda_i)R(\lambda)\,d\lambda = (2\pi i)N_i\,.\end{aligned}$$

Using 3) and 1) we get from the just obtained equality

$$\begin{aligned}(2\pi i)N_i^k &= (2\pi i)P_i(A-\lambda_i)^k = \int_{\gamma_i}R(\lambda)(A-\lambda_i)^k\,d\lambda \\ &= \int_{\gamma_i}R(\lambda)(A-\lambda_i)^k - R(\lambda)(A-\lambda_i)^{k-1}(A-\lambda)\,d\lambda \\ &= \int_{\gamma_i}R(\lambda)(A-\lambda_i)^{k-1}(\lambda-\lambda_i)\,d\lambda \\ &= \int_{\gamma_i}R(\lambda)(A-\lambda_i)^{(k-1)}(\lambda-\lambda_i) - R(\lambda)(A-\lambda_i)^{k-2}(A-\lambda)\,d\lambda \\ &= \int_{\gamma_i}R(\lambda)(A-\lambda_i)^{k-2}(\lambda-\lambda_i)^2\,d\lambda\,.\end{aligned}$$

Repeating like this $k-2$ more times, we get

$$(2\pi i)N_i^k = \int_{\gamma_i}(\lambda-\lambda_i)^k R(\lambda)\,d\lambda\,.$$

The claim $N_i^{\mu_i}$ follows from the fact that $(\lambda-\lambda_i)^{\mu_i}R(\lambda)$ is analytic.
5) From $P_i(A-\lambda_i) = N_i$ in 4) and using 2) we have

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Remark. It follows that the matrix A leaves invariant the subspaces $\mathcal{H}_i = P_i\mathcal{H}$ of $\mathcal{H} = \mathbf{C^n}$ and acts on $\mathcal{H}_i$ as

$$v \mapsto \lambda_i v + N_i v \; .$$

There is a basis of $\mathcal{H}_i$, the matrices N_i have 1 in the side diagonal and are 0 everywhere else: by 4), we know that μ_i is the smallest k such that $N_i^k = 0$. It implies that all eigenvalues of N are 0. There exists a vector $v \in \mathcal{H}_i$ such that $\{N_i^k v = v_k\}_{k=0}^{n-1}$ form a basis in $\mathcal{H}_i$. (This is an exercise: any nontrivial relation $\sum_j a_j N_i^j v = 0$ would imply that N_i had an eigenvalue different from 0). In that basis, the transformation N_i is the matrix $N_i = \begin{bmatrix} 0 & 1 & & & \\ & \cdot & 1 & & \\ & & \cdot & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$.

The matrix A is now a direct sum of **Jordan blocks** $A = \oplus_{i=1}^k A_i$, where

$$A_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \cdot & 1 & & \\ & & \cdot & & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} .$$

Exercises: 1) Perform the calculation which had been left out in the above proof: show that

$$N_i N_j = 0, i \neq j \; .$$

2) Show that if a linear transformation N on a μ dimensional space has the property $N^\mu = 0$ and $N^k \neq 0$ for $k < \mu$, then there is a basis in which N is a Jordan block with zeros in the diagonal.

4 The argument principle

Cauchy's integral formula and the residue formula can be expressed more naturally using the notion of the winding number.

Lemma 4.1 *Let γ be closed curve in $\mathbf{C}$ avoiding a point $a \in \mathbf{C}$. There exists $k \in \mathbf{Z}$ such that*

$$\int_\gamma \frac{dz}{z-a} = 2\pi i k \; .$$

Proof. Define

$$h(t) := \int_0^t \frac{z'(s)\, ds}{z(s) - a} \; .$$

Since $h'(t) = z'(t)/(z(t) - a)$, we get

$$\frac{d}{dt} e^{-h(t)}(z(t) - a) = h'(t) e^{-h(t)}(z(t) - a) + e^{-h(t)} z'(t) = 0 \; .$$

and $e^{-h(t)}(z(t) - a) = e^{-h(0)}(z(0) - a)$ is a constant. Therefore $e^{h(t)} = \frac{z(t)-a}{z(0)-a}$ and especially $e^{h(2\pi)} = \frac{z(2\pi)-a}{z(0)-a} = 1$ which means $h(2\pi) = 2\pi i k$. ■

Definition The **index** or **winding number** of a closed curve γ with respect to a point $a \notin \gamma$ is

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in 2\pi i \mathbf{Z} \; .$$

The definition is legitimated by the above lemma.

Cauchy's integral formula and the residue theorem holds more generally for any closed curve γ in a simply con-

If $f \in C^\omega(D)$
$$n(\gamma, z) \cdot f(z) = \int_\gamma \frac{1}{2\pi i} \frac{f(w)\,dw}{w - z} .$$

If $f \in C^\omega(D \setminus \{z_i\}_{i=1}^n)$
$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_{i=1}^n n(\gamma, z_i) \operatorname{Res}(f, z_i) .$$

Theorem 4.2 (Argument principle for analytic functions)

Given $f \in C^\omega(D)$ with D simply connected. Let a_i be the zeros of f in D and γ a curve in D avoiding a_i. Then
$$\sum_i n(\gamma, a_i) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,dz$$

Proof. Write
$$f(z) = (z - a_1)(z - a_2) \dots (z - a_n) g(z)$$
where $g(z)$ has no zeros in D. We compute
$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_1} + \frac{1}{z - a_2} + \dots \frac{1}{z - a_n} + \frac{g'(z)}{g(z)} .$$
Cauchy's theorem gives
$$\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$$
and the formula follows from the definition of the winding number. ∎

Definition If $f \in C^\omega(D \setminus \{z\})$ for a neighborhood D of a, then a is called an **isolated singularity** of f. If there exists $n \in \mathbf{N}$ such that
$$\lim_{n \to \infty} (z - a)^{n+1} \cdot f(z) = 0$$
then a is called a **pole** of f. The smallest n such that the above limit is zero is called the **order** of the pole. If an isolated singularity is not a pole, it is called an **essential singularity**. If $f \in C^\omega(D \setminus \{z_i\})$ and each z_i is a pole then f is called **meromorphic** in D.

Theorem 4.3 (Argument principle for meromorphic functions)

Let f be meromorphic in the simply connected set D, a_i the zeros of f, b_i the poles of f in D and γ a closed curve avoiding a_i, b_i.
$$\sum_i^n n(\gamma, a_i) - \sum_j^k n(\gamma, b_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,dz$$

Proof. The function
$$g(z) := f(z) \cdot (z - b_1) \dots (z - b_2) \dots (z - b_k)$$
is analytic in D and has the zeros a_i. Write
$$\frac{1}{2\pi i} \int_\gamma \left(\frac{f'(z)}{f(z)} + \frac{1}{z - b_1} + \frac{1}{z - b_2} \dots + \frac{1}{z - b_k}\right) dz = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)}\,dz$$
The right hand side is by argument principle for analytic maps equal to $\sum_i n(\gamma, a_i)$. The left hand side is
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,dz + \sum_j n(\gamma, b_j) .$$

Theorem 4.4 (Generalized argument principle)

Let f be meromorphic in the simply connected set D, a_i the zeros of f, b_i the poles of f in D and γ a closed curve avoiding a_i, b_i and $g \in C^\omega(D)$.

$$\sum_i^n g(a_i)n(\gamma, a_i) - \sum_j^k g(b_i)n(\gamma, b_j) = \frac{1}{2\pi i}\int_\gamma g(z) \cdot \frac{f'(z)}{f(z)}\, dz$$

Proof. Write again

$$f(z) = h(z)\frac{\prod_{i=1}^n (z-a_i)}{\prod_{j=1}^k (z-b_j)}$$

with analytic h which is nowhere zero and so

$$g(z) \cdot \frac{f'(z)}{f(z)} = \sum_{i=1}^n \frac{g(a_i)}{z-a_i} - \sum_{j=1}^k \frac{g(b_j)}{z-b_j} + g(z)\frac{h'(z)}{h(z)} \ .$$

■

As an application, take $h \in C^\omega(\Delta)$ with $\Delta = \{|z-a| < r\}$ and $D = h(\Delta)$. For $z \in \Delta$ put $\xi = h(z)$. The function $f(w) = h(w) - \xi$ has only one zero in D. Apply the last theorem with $g(w) = w$. We get

$$h^{-1}(\xi) = z = \frac{1}{2\pi i}\int_{\delta(D)} \frac{wh'(w)}{h(w)-\xi}\, dw$$

which is a formula for the inverse of h.

Assume $f \in C^\omega(D \setminus \{z_i\})$ is meromorphic. Denote by $Z_f = Z_f(D)$ the number of zeros of a function in D and with $P_f = P_f(D)$ the number of poles of f in D.

Theorem 4.5 (Rouché's theorem)

Given meromorphic $f, g \in C^\omega(D \setminus \{z_i\})$. Assume for $D_R = \{z \mid |z-a| < R\} \subset D$, $\gamma = \delta D_R$

$$|f(z) - g(z)| < |g(z)|,\ z \in \gamma \ .$$

Then

$$Z_f - P_f = Z_g - P_g \ .$$

Proof. The assumption implies that

$$|\frac{f(z)}{g(z)} - 1| < 1,\ z \in \gamma$$

and $h := f/g$ maps therefore γ into the right half plane. We can define $\log(f/g)$ by requiring that

$$\mathrm{Im}(\log(h(z))) = \arg(h(z)) \in (-\pi/2, \pi/2) \ .$$

The function $\log(h(z))$ is a primitive of $h'/h = \frac{(f/g)'}{f/g}$. We have so

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{f/g}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{g'}{g}\, dz \ .$$

Apply the argument principle. ■

Rouché's theorem leads to an other proof of the fundamental theorem of algebra:

Theorem 4.6 (Fundamental theorem of algebra)

Every polynomial $f(z) = p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ has exactly n roots.

Proof. $\frac{p(z)}{z^n} = 1 + a_1/z + \ldots a_n/z^n$ goes to 1 for $|z| \to \infty$. Consider the function $g(z) = z^n$ and $D = \{|z| < R\}$.