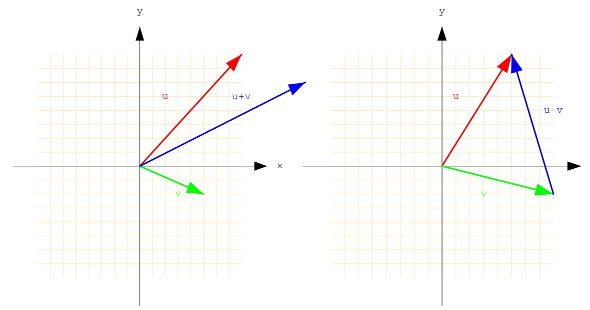
This is part 2 (of 3) of the homework. It is due July 1 at the beginning of class.

SUMMARY.
Vectors $\vec{v} = (v_1, v_2, v_3) = PQ$.
Points are special vectors $\vec{v} = \vec{OP}$, with $O = (0, 0, 0)$.
$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$ **dot product**
$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\alpha)$ where α is the **angle** between vectors.
$\text{proj}_{\vec{w}}(\vec{v}) = \vec{w}(\vec{v} \cdot \vec{w})/|\vec{w}|^2$ **projection** of $\vec{v}$ onto $\vec{w}$.
$\text{comp}_{\vec{w}}(\vec{v}) = |\text{proj}_{\vec{w}}(\vec{v})| = (\vec{v} \cdot \vec{w})/|\vec{w}|$
scalar projection of $\vec{v}$ onto $\vec{w}$.



Homework Problems

1) (4 points) Let $\vec{u} = (1, 2)$ and $\vec{v} = (-2, 1)$.
a) Draw the vectors $\vec{u}, \vec{v}$, $\vec{u} + \vec{v}, \vec{u} - \vec{v}$.
b) What is the relation between the length of $\vec{u} - \vec{v}$ and the lenghts of $\vec{u}$ and $\vec{v}$.
c) Prove Pythagoras: if $\vec{u}, \vec{v}$ are orthogonal $\vec{u} \cdot \vec{v} = 0$, then $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$.

2) (4 points) Denote by I the set of vectors in the plane, which have integer coordinates and have integer length. For example, $\vec{v} = (3, 4)$ is in I.

a) Verify that if s, t are two integers, then $\vec{v} = (x, y)$ is in I, if $x = 2st, y = t^2 - s^2$.
b) Use this information to find orthogonal vectors in I whose sum and difference is in I.
c) Is the sum or difference of two vectors in I always in I?
d) Is the dot product of two vectors in I always an integer?

Solution:
a) $(x, y) = (2st, t^2 - s^2)$ has length $\sqrt{4s^2t^2 + t^4 + s^4 - 2s^2t^2} = (t^2 + s^2)$.
b) If $w = (x, y)$ is in I, take $u = (x, 0)$ and $v = (0, y)$.
c) While the sum and difference of vectors with integer coordinates has integer coordinates, the length of the result does not necessarily have to be an integer. For example $u = (1, 0)$ and $v = (0, 1)$ have sum and difference, which has not an integer as length.
d) Yes, $u = (x_1, y_1), v = (x_2, y_2)$ has the dot product $u \cdot v = x_1 y_1 + x_2 y_2$ which is an integer.

3) (4 points) Colors are encoded by vectors $\vec{v} = (r, g, b)$, where the red, green and blue components are all numbers in the interval $[0, 1]$. Examples are:

(0,0,0)	black	(0,0,1)	blue
(1,1,1)	white	(1,1,0)	yellow
(1/2,1/2,1/2)	gray	(1,0,1)	magneta
(1,0,0)	red	(0,1,1)	cyan
(0,1,0)	green	(1,1/2,0)	orange

a) Determine the angle between the colors yellow and magneta.
b) Find a color which is orthogonal to orange.

c) What does the scaling $\vec{v} \mapsto \vec{v}/2$ do, if $\vec{v}$ represents a color?
d) Vectors on the diagonal $r = g = b$ are called **gray** colors. Find the gray vector which is the vector projection of yellow onto white.

Solution:
a) $\cos(\alpha) = \frac{(1,1,0)\cdot(1,0,1)}{(|(1,1,0)||(1,0,1)|)} = 1/2$ gives $\alpha = 60° = \pi/3$.
b) Blue $\vec{b} = (0, 0, 1)$ is orthogonal to orange $\vec{y} = (1, 1/2, 0)$.
c) It darkens the color.
d) The vector projection of yellow $\vec{y} = (1, 1, 0)$ onto white $\vec{w} = (1, 1, 1)$ is $\vec{w}(\vec{y} \cdot \vec{w})/|\vec{w}|^2 = (2/3, 2/3, 2/3)$.

4) (4 points)
a) Find the angle between the diagonal of a cube and one of the diagonal of one of its faces. Assume that the two diagonals go through the same edge of the cube.
Remark. You can leave the answer in the form $\cos(\alpha) = ...$
b) Find the angle between two face diagonals which go through the same edge and are an adjacent faces.

Solution:
a) Put the coordinate system so that

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$$

are the corners of the cube. The main diagonal is the vector $\vec{v} = (1, 1, 1)$, a diagonal in the face is for example $(1, 1, 0)$. Then, $\cos(\alpha) = 2/(\sqrt{3}\sqrt{2})$.
b) We can take the two diagonals $(1, 1, 0), (0, 1, 1)$ and the angle satisfies $\cos(\alpha) = 1/2$ so that $\alpha = \pi/3$.

5) (4 points) Assume $\vec{v} = (-4, 2, 2)$ and $\vec{w} = (3, 0, 4)$.

a) Find the length of $\vec{w}$ and the dot product between $\vec{v}$ and $\vec{w}$.
b) Find the vector projection of $\vec{v}$ onto $\vec{w}$.
c) Find the scalar projection of $\vec{v}$ onto $\vec{w}$.
d) Find a vector parallel to $\vec{w}$ of length 1.

Solution:
a) $|\vec{w}| = 5$, $\vec{v} \cdot \vec{w} = -4$.
b) $\vec{w}(\vec{v} \cdot \vec{w})/|\vec{w}|^2 = -4/25(3, 0, 4) = (-12/25, 0, -16/25)$.
c) $(\vec{v} \cdot \vec{w})/|\vec{w}| = -4/5$.
d) $\vec{w}/|\vec{w}| = (3/5, 0, 4/5)$.

Remarks

(You don't need to read these remarks to do the problems.)

To problem 2) A triple (x, y, z) satisfying $z^2 = x^2 + y^2$ is called a **Pythagorean triple**. The construction of Pythagorean triples has been known already by the Babylonians. The construction is vital, because it allowed (using a rope only) to construct precise right angles or measure area. It is an early example, where a mathematical discovery lead to the enhancement of economics. Measuring area for example was crucial for the trading of land.
By the way, the construction of Pythagorean triples shown in this example enumerates all the

triples: Proof: $x^2 + y^2 = z^2$ defines a point $(X, Y) = (x/z, y/z)$ on the unit circle. If the line through $(-1, 0)$ and (X, Y) has slope t, then $X = (1 - t^2)/(1 + t^2)$ and $Y = 2t/(1 + t^2)$. This shows that the slope t of that line is rational if and only if the point (X, Y) on the circle is rational. If $t = s/t$, then $X = (s^2 - t^2)/(s^2 + t^2)$ and $Y = 2st/(s^2 + t^2)$ which gives $x = s^2 - t^2, y = 2st$ and $z = s^2 + t^2$.

To problem 3) In many computer applications, color is encoded in hexadecimal form, where r,g,b are integers from 1 to 255. For example, the word fa887b indicates the color, where the red component is $fa = 11 + 15 \cdot 16 = 251$, green is $88 = 8 \cdot 16 + 8 = 136$, blue $= 7b = 7 \cdot 16 + 12 = 124$. This color corresponds to the vector $(251/255, 136/255, 124/255)$ in the unit cube. You see expressions like "$bgcolor = \#fa887b$" in HTML pages for example.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Verify that for any two vectors $\vec{a}$ and $\vec{b}$, the inequality $|\vec{a} - \vec{b}| \geq ||\vec{a}| - |\vec{b}||$ holds.

2) Given three numbers g_1, g_2, g_3. Define a new **dot product** $(\vec{v}, \vec{w}) = g_1 v_1 w_1 + g_2 v_2 w_2 + g_3 v_3 w_3$. For $g_1 = g_2 = g_3 = 1$, this is the usual dot product.

   Which properties of the usual dot product still hold for this generalization? For which g_1, g_2, g_3 could the dot product still serve to measure a reasonable "length" $|\vec{v}| = \sqrt{(\vec{v}, \vec{v})}$?

3) The coordinates for the corners of a cube in 4D are the 16 points $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the big diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the "middle diagonal" in one of 3D faces connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, 1)$.