This is part 2 (of 2) of the homework for the third week. It is due July 15 at the beginning of class.

SUMMARY.

- **speed** $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.
- **Unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$.
- **Unit normal vector** $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$
- **Binormal vector** $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.
- $\int_a^b |\vec{r}'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$ **arc length**.
- $|\vec{r}\,'(t) \times \vec{r}\,''(t)|/|\vec{r}\,'(t)|^3$ **curvature**.
- Example of graph: $\vec{r}(t) = (t, f(t))$, $\kappa(t) = f''(t)/(1 + f'(t)^2)^{3/2}$.

Homework Problems

1) (4 points) Find the arc length of the curve $\vec{r}(t) = (t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t))$, $0 \le t \le \pi$.

Solution:
The velocity is $\vec{r}\,'(t) = (2t, t\sin(t), t\cos(t))$ and the speed is $|\vec{r}\,'(t)|\sqrt{5t^2} = \sqrt{5}t$. The arc length of the curve is $\int_0^\pi \sqrt{5}t\, dt = \pi^2\sqrt{5}/2$.

2) (4 points) Find the arc length of the curve $\vec{r}(t) = (t^2, 2t, \log(t))$ for $t \in [1, e]$.

Solution:
The volocity is $\vec{r}\,'(t) = (2t, 2, 1/t)$ and the speed is $\sqrt{4t^2 + 4 + 1/t^2} = \sqrt{(2t + 1/t)^2}$. The arc length of the curve is $\int_1^e (2t + 1/t)\, dt = t^2 + \log(t)|_1^e = e^2$.

(Note: log denotes the natural logarithm which satisfies $\log(e) = 1$.)

3) (4 points) Find the curvature of $\vec{r}(t) = (e^t\cos(t), e^t\sin(t), t)$ at the point $(1, 0, 0)$.

Solution:
To use the formula $\kappa(t) = |\vec{r}\ '(t) \times \vec{r}\ ''(t)|/|\vec{r}\ '(t)|^3$, we need to know the veolocity $\vec{r}\ '(t) = (e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t)), 1)$, as well as the acceleration $\vec{r}\ ''(t) = (-2e^t\sin(t), 2e^t\cos(t), 0)$. At the time $t = 0$, we have $\vec{r}\ '(0) = (1, 1, 1)$ and $\vec{r}\ ''(t) = (0, 2, 0)$. Now apply the formula $\kappa(0) = |(1, 1, 1) \times (0, 2, 0)|/\sqrt{3}^3 = |(-2, 0, 2)|/\sqrt{3}^3 = \sqrt{8}/\sqrt{3}^3 s 2\sqrt{6}/9$.

4) (4 points)
(3) Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = (t^2, t^3, 0)$ for $t = 2$.
(1) Do the vectors depend continuously on t for all t?

Solution:
a) $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)| = (1,2,0)/\sqrt{10}$, $\vec{N}(t) = \vec{T}\,'(t)/|\vec{N}\,'(t)| = (-3,1,0)/\sqrt{10}$, $\vec{B}(t) = T(t) \times N(t) = (0,0,1)$.
b) The $\vec{T}$ and $\vec{N}$ vectors do not depend continuously on t, they switch direction at $t = 0$.

5) (4 points)
Let $\vec{r}(t) = (t, t^2)$. Find the equation for the **caustic** $\vec{s}(t) = \vec{r}(t) + \vec{N}(t)/\kappa(t)$ known also as the **evolute** of the curve.

Solution:
The curvature of the graph of $f(x) = x^2$ is $\kappa(t) = f''(t)/(1 + f'(t)^2)^{3/2} = 2/(1+4t^2)^{3/2}$. The normal vector to the curve is $\vec{n}(t) = (-2t, 1)$ which is orthogonal to the velocity vector $\vec{v}(t) = (1, 2t)$. The unit normal vector is $\vec{N}(t) = (-2t,1)/\sqrt{4t^2+1}$. The caustic is $\vec{s}(t) = \vec{r}(t) + \frac{1}{2}(1+4t^2)^{3/2}(-2t,1)/\sqrt{4t^2+1} = \boxed{(4x^3, 1/2 - 3t^2)}$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Find the evolute of the curve $\vec{r}(t) = (t, t^4)$.

2) If $\vec{r}(t) = (-\sin(t), \cos(t))$ is the boundary of a coffee cup and light enters in the direction $(-1, 0)$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. Find a parameterization of this curve.