

This is part 2 (of 3) of the weekly homework. It is due July 22 at the beginning of class.

SUMMARY.

- $\nabla f(x, y) = (0, 0)$  **critical point** or **stationary point** (candidate for max or min).
- $H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$  **Hessian**.  $D = \det(H(x, y)) = f_{xx}f_{yy} - f_{xy}^2$  **determinant**, is called **discriminant**. (It is not necessary that you know determinants or matrices, you can survive with the formula  $D = f_{xx}f_{yy} - f_{xy}^2$ .)
- $D > 0, f_{xx} < 0$  **local maximum**,  $D > 0, f_{xx} > 0$  **local minimum**,  $D < 0$  **saddle point**.

## Homework Problems

- 1) (4 points) Find all the extrema of the function  $f(x, y) = x^3 - 3x + 2y^2 - y^4$  and determine whether they are maxima, minima or saddle points.

**Solution:**

The condition  $\nabla f(x, y) = (f_x, f_y) = (0, 0)$  are the equations

$$3x^2 - 3 = 0 \quad (1)$$

$$4y - 4y^3 = 0 \quad (2)$$

have the solution  $x = \pm 1, y = 0$  or  $y = \pm 1$ .

Point	D	$f_{xx}$	nature
(-1,-1)	48	-6	local maximum
(-1,0)	-24	-6	saddle
(-1,1)	48	-6	local maximum
(1,-1)	-48	6	saddle point
(1,0)	24	6	local minimum
(1,1)	-48	6	saddle point

- 2) (4 points) Find all the extrema of the function  $f(x, y) = x^2 - y^2 + xy$  and determine whether they are maxima, minima or saddle points.

**Solution:**

The condition  $\nabla f(x, y) = (f_x, f_y) = (0, 0)$  are the equations

$$2x + y = 0 \quad (3)$$

$$-2y + x = 0 \quad (4)$$

which have only the solution  $(0, 0)$ . Because  $D = -4 - 1^2 < 0$  this is a saddle point.

- 3) (4 points) Find and classify all the extrema of the function  $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ .

**Solution:**

We have to solve the equations  $\nabla f = ((2x - 2x^3)e^{-x^2 - y^2}, (4y - 4y^3)e^{-x^2 - y^2}) = (0, 0)$  which has the solutions  $x = 0, y = 0, x = 0, y = \pm 1, y = 0, x = \pm 1$ . By evaluating  $D$  and  $f_{xx}$  at

Point	D	$f_{xx}$	nature
(0,0)	8	2	local minimum
(0,1)	$16/e^2$	$-2/e$	local maximum
(0,-1)	$16/e^2$	$-2/e$	local maximum
(1,0)	$-8/e^2$	$-4/e$	saddle point
(-1,0)	$-8/e^2$	$-4/e$	saddle point

each point we find their nature:

- 4) (4 points) Find the critical point of the function  $f(x, y) = x^2 + ky^2 + xy$  for general  $k$ . Describe what happens with that critical point and its nature (maximum, minimum or saddle) as  $k$  changes. Especially, at which values of  $k$  does the nature of this critical point change?

**Solution:**

The equations

$$\begin{aligned} 2x + y &= 0 \\ 2yk + x &= 0 \end{aligned}$$

can be solved by substituting  $y = -2x$  into the second equation:  $-4xk + x = 0$  has for all  $k$  the solution  $x = 0$  leading to  $y = 0$ . So,  $(0, 0)$  is a critical point for all  $k$ .

At that critical point the discriminant is  $D = f_{xx}f_{yy} - f_{xy}^2 = 4k - 1$ . For  $k < 1/4$ , we have  $D < 0$  and the critical point is a saddle. For  $k > 1/4$ , we have  $D > 0$  and  $f_{xx} > 0$  so that the critical point is a local minimum. At the parameter value  $k = 1/4$ , the nature of the critical point changed.

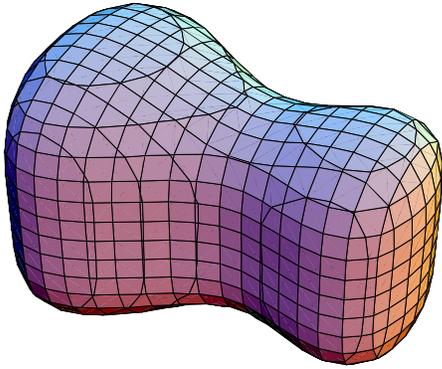
- 5) (4 points)
- a) Consider a level surface  $g(x, y, z) = 0$ . Using the chain rule, verify that such a surface has an extremal  $z$  value at a point  $(x_0, y_0, z_0)$  for which  $g_x(x, y, z) = g_y(x, y, z) = 0$ . Find a geometric interpretation of this result in terms of the gradient of  $g$ .
- b) Consider the quartic surface  $g(x, y, z) = x^4 + x^3 - 4x^2 + 5y^2 + 2z^4 - z^3 - z^2/4 - 3 = 0$ . It looks a bit like a Cello case. Using the result in a), find the maximal  $z$  value.

**Solution:**

a) Taking the partial derivative of the equation  $g(x, y, z(x, y))$  with respect to  $x$  gives  $g_x(x, y, z(x, y)) + g_z(x, y, z(x, y))z_x(x, y) = 0$  so that  $z_x(x, y) = -g_x(x, y, z(x, y))/g_z(x, y, z(x, y))$ . Similarly, we obtain  $z_y(x, y) = -g_y(x, y, z(x, y))/g_z(x, y, z(x, y))$ . In order that  $z(x, y)$  is maximal, we must have  $z_x(x, y) = z_y(x, y) = 0$ . This is the case if and only if  $g_x(x, y, z) = 0$  and  $g_y(x, y, z) = 0$ . Geometrically, this means that the gradient vector  $\nabla g = (0, 0, g_z)$  is vertical. In terms of the tangent space, this means that at a maximal value, **the tangent plane is horizontal**.

b) The equation  $g_x(x, y, z) = 0$  is  $4x^3 + 3x^2 - 8x = 0$  which means  $x = 0$  or  $x = (-3 \pm \sqrt{137})/8$ . The equation  $g_y(x, y, z) = 0$  implies  $y = 0$ . We have now for each of the solutions  $(x, y)$  find the corresponding  $z$  value:

For  $(x, y) = (0, 0)$ , we get  $2z^4 - z^3 - z^2/4 - 3 = 0$  which has a maximal root around 1.3. For  $(x_{\pm}, y) = (-3 \pm \sqrt{137})/8, 0$ . The real solutions  $z$  to  $g(x_{\pm}, 0, z) = 0$  are around  $z = 1.7$  and  $z = 1.4$ .



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## Remarks

(You don't need to read these remarks to do the problems.)

To Problem 4). The constant  $k$  is here called a **parameter**. Parameter values of  $k$ , where the nature of the critical point changes, are called **bifurcation parameters**. They play an important role in physics. If more than one parameter is involved there is a whole theory called **catastrophe theory** which deals with bifurcations. Bifurcations are points, where qualitative changes in the system happen. These can often be catastrophes. For example, for two parameters the most common type of catastrophes is called a "cusp" catastrophe which can be illustrated with a **catastrophe machine**. (Try it out at <http://www.ams.org/new-in-math/cover/cusp4.html>).



Catastrophe theory has been applied to a number of different phenomena, such as the stability of ships at sea and their capsizing, bridge collapse, fight-or-flight behavior of animals or even prison riots.

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## Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Find and classify all critical points of  $\log(200 + \log(100 + \log(200 + 3x^2 + xy + 2x + y^2 + y + 4)))$ .
- 2) How would a classification of critical points and formulate the criterium for local maxima or local minima for functions  $f(x, y, z)$  of three variables?
- 3) Can you prove the "Island theorem" on the handout?