

This is part 2 (of 3) of the weekly homework. It is due July 29 at the beginning of class.

SUMMARY. $dA = dxdy = r dr d\theta$ area element.

- $\int \int_R f(x, y) dxdy = \int_{\alpha}^{\beta} \int_a^b f(r \cos(\theta), r \sin(\theta)) r dr d\theta$
integral in polar coordinates.
- Reminder: A curve $\vec{r}(t) = (f(t) \cos(t), f(t) \sin(t))$ can in polar coordinates (r, θ) be given as $r(\theta) = f(\theta)$.

Homework Problems

- 1) (4 points) Integrate $f(x, y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

Solution:

The integral in Cartesian coordinates goes less smooth because we have to compute a 1D integral with partial integration: $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 dy dx = \int_{-1}^1 2x^2 \sqrt{1-x^2} dx = \pi/4$.

The integral in polar coordinates is easier to get: using the substitution $x = \cos(u)$ we obtain

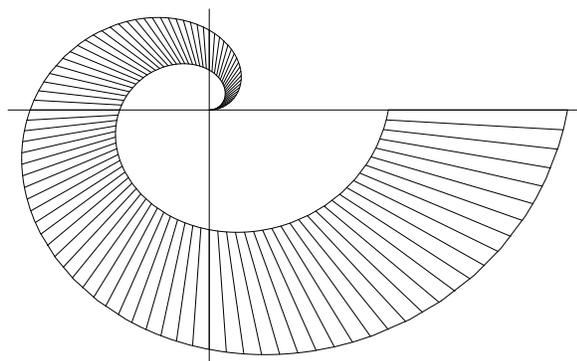
$$\int_0^{2\pi} \int_0^1 r^2 \cos(\theta)^2 r dr d\theta = (1/4)\pi.$$

- 2) (4 points) Find $\int \int_R (x^2 + y^2)^{10} dA$, where R is the part of the unit disc $\{x^2 + y^2 \leq 1\}$ for which $y > x$.

Solution:

Easy in polar coordinates: $\int_0^1 \int_{\pi/4}^{5\pi/4} r^{21} d\theta dr = \pi/22$. This integral would be quite terrible to solve in Cartesian coordinates.

- 3) (4 points) What is the area of the region which is bounded by three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x -axes.



Solution:

$$\int_0^{2\pi} \int_{\theta}^{2\theta} r \, d\theta = \int_0^{2\pi} (2\theta)^2/2 - (\theta)^2/2 \, d\theta = (3/2) \int_0^{2\pi} \theta^2 \, d\theta = (3/2)(2\pi)^3/3 = 4\pi^3.$$

- 4) (4 points) Evaluate the following iterated integral by converting it into polar coordinates

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} \, dydx .$$

Solution:

$$\int_0^3 \int_0^{\pi/2} r^2 \, d\theta dr = (\pi/2)27/3 = 9\pi/2.$$

- 5) (4 points) Find the average value of $f(x, y) = x^2 + y^2$ on the annulus $1 \leq |(x, y)| \leq 2$.

Solution:

The integral of f over the annulus is $2\pi \int_1^2 \int_1^2 r^3 \, drd\theta = 2\pi(16-1)/4$. The area is $2\pi \int_1^2 r \, dr = 2\pi(4-1)/2$. The average is $(15/4)/(3/2) = 5/2$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) Calculate $\iint_{\mathbf{R}^2} e^{-x^2-y^2} dx dy$ and use this to calculate the integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Hint. The function $f(x) = e^{-x^2}$ is known to have no anti-derivative which can be expressed with "known functions" like exp, log, sin etc. You can nevertheless find a closed solution for the definite integral $\int_{-\infty}^{\infty} e^{-x^2} dx$.

- 2) Find the area of the region shaded in the picture. The region is bounded by the polar curves $r(\theta) = 2\theta$ with $\theta \in [0, 6\pi]$ and $r(\theta) = 3\theta$ with $\theta \in [0, 4\pi]$.

