This is part 2 (of 3) of the weekly homework. It is due on Monday, August 11 at 6 PM in SC102B at the review.

SUMMARY.

- $\text{curl}(P,Q,R) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ the **curl** of $F = (P,Q,R)$.
- The curl of a gradient field vanishes: $\text{curl}\nabla f = (0,0,0)$.
- $D : (u,v) \mapsto r(u,v) = (x(u,v), y(u,v), z(u,v))$ **surface** $S = r(D)$.
- $\int\int_S F \cdot dS = \int_D F(r(u,v)) \cdot (r_u \times r_v)\, dA$ **flux integral**.
- $\boxed{\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr}$ **Stokes theorem**, C: boundary of S, oriented so that "surface is to your left" if your head points in the normal direction.

Homework Problems

1) (4 points) Evaluate the flux integral $\int\int_S (0,0,yz)\, dS$, where S is the surface with parametric equation $x = uv, y = u+v, z = u-v$ on $R : u^2+v^2 \leq 1$.

Solution:
$\vec{r}_u = (v,1,1), \vec{r}_v = (u,1,-1)$ so that $\vec{r}_u \times \vec{r}_v = (-2, u+v, -u+v)$. The flux integral is $\int\int_R (0,0,u^2-v^2) \cdot (-2,u+v,-u+v)\ dudv = \int\int_R v^2u - u^3 - v^3 + u^2v\ dudv$ which is best evaluated using polar coordinates: $\int_0^1 \int_0^{2\pi} r^4(\sin^2(\theta)\cos(\theta) - \cos^3(\theta) - \sin^3(\theta) + \cos^2(\theta)\sin(\theta))\ d\theta dr = 0$.

2) (4 points) Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot dS$ for $F(x,y,z) = (xy, yz, zx)$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[1,0] \times [0,1]$ and has upward orientation.

Solution:
$\text{curl}(F) = (-y,-z,-x)$. The parametrization $\vec{r}(u,v) = (u,v,4-u^2-v^2)$ gives $r_u \times r_v = (2u,2v,1)$ and $\text{curl}(F)(r(u,v)) = (-v, u^2+v^2-4, -u)$. The flux integral is $\int_0^1 \int_0^1 (-2uv + 2v(u^2+v^2-4) - u)\ dvdu = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$.

3) (4 points) Evaluate the same flux integral as in the previous question but using Stokes theorem.

Solution:
The boundary C consists of 4 curves
$C_1 : \vec{r}(t) = (t, 0, 4t^2), \vec{r}'(t) = (1,0,-2t)$.
$C_2 : \vec{r}(t) = (1, t, 3-^2), \vec{r}'(t) = (0,1,-2t)$.
$C_3 : \vec{r}(t) = (t, 1, 4-t^2-1), \vec{r}'(t) = (1,-,-2t)$.
$C_4 : \vec{r}(t) = (0, t, 4-t^2), \vec{r}'(t) = (0,1,-2t)$.
$I : \int_{C_1} (0,0,4t-t^3) \cdot (1,0,-2t)\ dt = -8/3 + 2/5$.
$II : \int_{C_2} (t, 3t-t^3, 3-t^2) \cdot (0,1,-2t)\ dt = 1/4 - 3/2$.
$III : \int_{C_3} (t, 3-t^2, 3t-t^3) \cdot (1,0,-2t)\ dt = -11/10$.
$IV : \int_{C_4} (t,0,0) \cdot (0,1,-2t) = 0$
$I + II - III - IV = (-8/3 + 2/5) + (-5/4) + 11/10 + 0 = -25/6$. (The line integrals along C_3, C_4 were taken negative because the curves are traced backwards.)

4) (4 points) Use Stokes theorem to evaluate $\int_C F \cdot dr$, where $F(x,y,z) = (x^2y, x^3/3, xy)$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

Solution:
$r_u \times r_v = (-2u^2\cos(v), 2u^2\sin(v), u)$. $F(r(u,v)) = (u^3\cos(v)\sin(v), u^3\cos^3(v)/3, u^2\cos(v)\sin(v))$ $F(r(u,v)) \cdot (r_u \times r_v) = u^3\cos(v)\sin(v)(3 - 6u^2\cos(v) + 2u^2\cos^2(v))$ which gives zero when integrated over $[0, 2\pi]$.

5) (4 points) If S is the surface $x^6 + y^6 + z^6 = 1$ and assume F is a smooth vector field. Show that $\int\int_S \text{curl}(F) \cdot dS = 0$.

Solution:
The flux of curl(F) through a closed surface is zero by Stokes theorem and the fact that the surface does not have a boundary. One can see this also by cutting the surface in two pieces and apply Stokes to both pieces.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Solve Nashs problem distributed as an "in-class-exercice".

2) Use Stokes theorem to show that $\int_C (f\nabla g + g\nabla f) \cdot dr = 0$ for any closed curve C in space and any two functions f, g.
(Hint: the identity also follows from the fundamental theorem of line integrals).

3) Try to figure out, how Stokes theorem would look like in higher dimensions: in four dimensions, it is useful in special relativity.

Start: In dimension d, the curl is a field $\text{curl}(F)_{ij} = \partial_{x_j} F_i - \partial_{x_i} F_j$ with $\binom{d}{2}$ components. In 4 dimensions, it has 6 components. In d dimensions, a surface element in the $i - j$ plane is written as dS_{ij}. The flux integral of the curl of F through S is defined as $\int\int \text{curl}(F) \cdot dS$, where the dot product is $\sum_{i<j} \text{curl}(F)_{ij} dS_{ij}$. If S is given by a map r from a planar domain D to $\mathbb{R}^d$, $U = \partial_u X$ and $V = \partial_v X$ are tangent vectors to that plane and $dS_{ij}(u,v) = (U_iV_j - U_jV_i)\ dudv$.