This is part 3 (of 3) of the weekly homework. It is due Monday August 11 at 6 PM in SC102B at the review.

SUMMARY.

- $\mathrm{div}(F)(x,y,z) = F_x + F_y + F_z$ **divergence**. Is a scalar field measuring the how much field is "produced" at a point (x,y,z).
- $\boxed{\int\int\int_E \mathrm{div}(F)\ dV = \int\int_S F \cdot dS}$ **divergence theorem** also called **Gauss theorem**.

Homework Problems

1) (4 points) Verify that the divergence theorem is true for the vector field $F(x,y,z) = (3y, xy, 2yz)$ on the unit cube $[0,1] \times [0,1] \times [0,1]$.

Solution:
The cube consists of 6 faces. The flux through the face $x = 1$ is $\int_0^1 \int_0^1 3y\ dydz = 3/2$.
The flux through the face $x = 0$ is $-3/2$.
The flux through the face $y = 1$ is $\int_0^1 \int_0^1 x\ dxdz = 1/2$.
The flux through the face $y = 0$ is 0.
The flux through the face $z = 1$ is $\int_0^1 \int_0^1 2y\ dxdy = 1$.
The flux through the face $z = 0$ is $\int_0^1 \int_0^1 0\ dxdy = 0$.
The sum of all these fluxes is 3/2.
The divergence of F is $x + 2y$. Integrating this over the unit cube gives $1/2 + 1 = 3/2$.

2) (4 points) Verify that the divergence theorem is true for the vector field $F(x,y,z) = (xy, yz, zx)$ on the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$.

Solution:
The divergence is $(y + z + x)$. Integrated over the cylinder gives $\int_0^1 z\ dz\pi = \pi/2$.
The flux of the vector field through the bottom is 0 because there the vector field has the form (*,*,0)$andthenormalvectoris$(0,0,-1). The flux integral over the top is $\int\int_R x\ dxdy$ where R is the unit disc, which is zero. To compute the flux integral over the boundary of the cylinder, parametrize the cylinder as $\vec{r}(\theta, z) = (\cos(\theta), \sin(\theta), z)$. We have $F(\vec{r}(u,v)) = (\cos(\theta\sin(\theta), \sin(\theta)z, \cos(\theta)z)$ and $r_u \times r_v = (\cos(\theta), \sin(\theta), 0)$. The flux integral is $\int_0^1 \int_0^{2\pi} \cos^2(\theta)\sin(\theta) + \sin^2(\theta)z\ d\theta dz = \pi/2$.

3) (4 points) Use the divergence theorem to calculate the flux of $F(x,y,z) = (x^3, y^3, z^3)$ through the sphere $x^2 + y^2 + z^2 = 1$.

4) (4 points)
a) Verify that $\mathrm{div}(E) = 0$ away from the origin if E is the electric field $E(\vec{x}) = \vec{x}/|\vec{x}|^3$.
b) An electric charge at 0 generates the field E as in a). What is the flux of F through the unit sphere S?
c) What is the flux of E through any other sphere containing the origin $(0,0,0)$ inside?
d) What is the flux of the electric field $F(\vec{x}) = \sum_{i=1}^n E(\vec{x} - \vec{x}_i)$ through S generated by n charges located at points $\vec{x}_i$ inside S?

Solution:
a) The vector field is $(P, Q, R) = (x(x^2+y^2+z^2)^{-1/2}, y(x^2+y^2+z^2)^{-1/2}, z(x^2+y^2+z^2)^{-1/2})$.
$P_x = (x^2 + y^2 + z^2)^{-3/2} - \frac{3}{2}2x^2(x^2 + y^2 + z^2)^{-5/2}$.
$Q_y = (x^2 + y^2 + z^2)^{-3/2} - \frac{3}{2}2y^2(x^2 + y^2 + z^2)^{-5/2}$.
$R_z = (x^2 + y^2 + z^2)^{-3/2} - \frac{3}{2}2z^2(x^2 + y^2 + z^2)^{-5/2}$.
$P_x + Q_y + R_z = 0$.
b) The flux is 4π.
c) The flux is $4\pi n$.

5) (4 points) Find $\int\int_S F \cdot dS$, where $F(x, y, z) = (x, y, z)$ and S is the outwardly oriented surface obtained by removing the cube $[1, 2] \times [1, 2] \times [1, 2]$ from the cube $[0, 2] \times [0, 2] \times [0, 2]$.

Solution:
The divergence is 3. By the divergence theorem, the result is 3 times the area of the solid E which is $(8 - 1)3 = 21$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) We show that Green's theorem in the plane is equivalent to Gauss theorem in the plane:

   a) Let $G(x, y)$ be the vector field orthogonal to $F(x, y)$. Show that $\text{div}(G) = \text{curl}(F)$.

   b) Show that the line integral $\int_C F\ dr$ along a curve C is the same as the flux integral $\int_C G \cdot dn$, where dn is a vector perpendicular to the curve with the same length as $r'(t)dt$.

2) Formulate the divergence theorem in arbitrary dimensions.