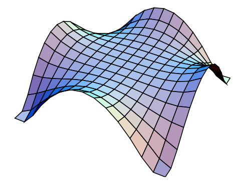
PARTIAL DERIVATIVE. If $f(x,y,z)$ is a function, then $\frac{\partial}{\partial x} f(x,y,z)$ is defined as the derivative of the function $g(x) = f(x,y,z)$, where y and z are fixed. The other derivatives with respect to y and z are defined similarly.

NOTATION. One also writes $f_x(x,y,z) = \frac{\partial}{\partial x} f(x,y,z)$ etc. For iterated derivatives the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

EXAMPLE. $f(x,y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. (A function which satisfies this equation is called **harmonic**.)



CLAIROT THEOREM. If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$. Proof. Compare the two sides:

$dx f_x(x,y) \sim f(x+dx,y) - f(x,y)$
$dydx f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y+dy) - (f(x+dx,y) - f(x,y))$

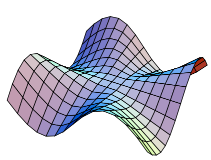
$dy f_y(x,y) \sim f(x,y+dy) - f(x,y)$.
$dxdy f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y) - (f(x,y+dy) - f(x,y))$

CONTINUITY IS NECESSARY. Example: $f(x,y) = (x^3y - xy^3)/(x^2+y^2)$ contradicts Clairot:

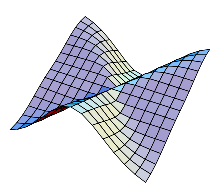
$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,

$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x2$, $f_{y,x}(0,0) = 1$.

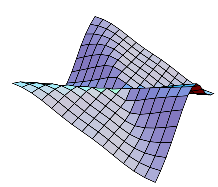
$f(x,y)$



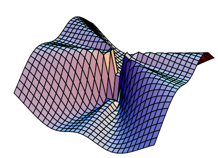
$f_x(x,y)$



$f_y(x,y)$



$f_{xy}(x,y)$



In general, one has not to worry about such examples. But they can occur.

WHY ARE PARTIAL DERIVATIVES IMPORTANT?

- To understand and analyze functions of several variables.
- Find vectors normal to a surface at the point (x,y,z).
- Approximations, linarizations and tangent planes need it.
- Partial differential equations are laws which describe physics.
- Optimize functions.
- Solution to some integration problems using generalizations of fundamental theorem of calculus.

GRADIENT. If $f(x,y,z)$ is a function, then $\nabla f(x,y,z) = \left(\frac{\partial}{\partial x} f(x,y,z), \frac{\partial}{\partial y} f(x,y,z), \frac{\partial}{\partial z} f(x,y,z)\right)$ is called the **gradient** of f. The symbol ∇ is called **Nabla**. It is named after an Egyptian harp, the Hebrew word "nevel"=harp seems to have the same aramaic origin.

1D THE CHAIN RULE REMINDER. If f and g are functions of one variable t, then $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$.

of the function $t \mapsto f(r(t))$ with the derivatives of f and r: $\boxed{d/dt f(r(t)) = \nabla f(r(t)) \cdot r'(t)}$

EXAMPLE. Let $z = \sin(x + 2y)$, where x and y are functions of t: $x = e^t, y = \cos(t)$. What is $\frac{dz}{dt}$? $\nabla f = (\cos(x+2y), 2\cos(x+2y))$ and $\vec{r}\,'(t) = (e^t, -\sin(t))$ so that $\frac{dz}{dt} = \nabla f \cdot r' = \cos(x+2y)e^x - 2\cos(x+2y)\sin(t)$.

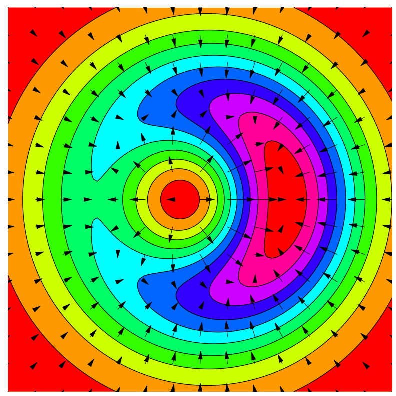
EXAMPLE. $f(x,y) = x^4 + x\sin(xy) = 0$ defines $y = g(x)$. Find $g'(x) =$. **Solution** $f(x, g(x)) = 0$, then $f_x 1 + f_y g'(x) = 0$ so that $g_x(x) = -f_x/f_y = -(4x^3 + \sin(xy) + xy\cos(xy))/(x^2\cos(xy))$.

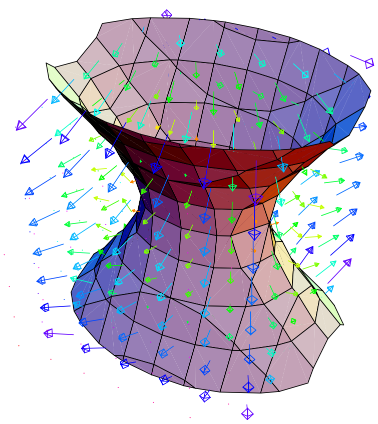
DIRECTIONAL DERIVATIVE. If $\vec{v}$ is a vector, then $\nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$. One writes $\nabla_v f$ or $D_v f$. $\boxed{D_v f(x,y,z) = \nabla f(x,y,z) \cdot v}$ It is usually assume that $\vec{v}$ is a unit vector. Using the chain rule, one can write $\frac{d}{dt} D_{\vec{v}} f = f(x + t\vec{v})$.
EXAMPLES. partial derivatives are directional derivatives where $\vec{v} = (1,0,0), (0,1,0)$ or $(0,0,1)$.

GRADIENTS AND LEVEL SURFACES.

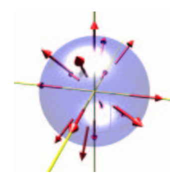
Gradients are orthogonal to level surfaces.

The level surface of the linear approximation is tangent to the surface. Every vector $\vec{x} - \vec{x}_0$ in the tangent surface satisfies $\nabla f \cdot (\vec{x} - \vec{x}_0) = 0$ and is so orthogonal to ∇f.





TANGENT PLANE. Having the direction of the normal immediately gives the tangent plane to level surfaces.
Example: The gradient of $f(x,y,z) = x^2 + y^2 + z^2$ at a point (x,y,z) is $(2x, 2y, 2z)$. At the point $(1,2,2)$ this gradient is $(2,4,4)$. It is normal to the level surface $f(x,y,z) = 9$ which is a sphere of radius 3. The plane $2x+4y+4z = 36$ is the tangent plane to that surface.



STEEPEST DECENT. The directional derivative satisfies $|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|$, because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}||\cos(\phi)| \leq |\nabla f||\vec{v}|$. The direction $\vec{v} = \nabla f$ is the direction, where f increases most, the direction $-\nabla f$ is called **the direction of steepest decent**.

LINEAR APPROXIMATION. $\boxed{\text{1D:}}$ The linear approximation of $f(x)$ at x_0 is $\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$.

$\boxed{\text{2D:}}$ The linear approximation of $f(x,y)$ at (x_0, y_0) is $\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(y - y_0)}$.

$\boxed{\text{3D:}}$ The linear approximation of $f(x,y,z)$ at (x_0, y_0, z_0) is

$\boxed{L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0) + f_z(x_0,y_0,z_0)(z - z_0)}$.

Vector notation in all dimensions: $\boxed{L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)}$

EXAMPLE (2D) Find the linear approximation to $f(x,y,z) = xy + yz + zx$ at the point $(1,1,1)$. **Solution.** $f(1,1,1) = 3$, $\nabla f(x,y,z) = (y+z, x+z, y+x), \nabla f(1,1,1) = (2,2,2)$ so that $L(x,y,z) = f(1,1,1) + (2,2,2) \cdot (x-1, y-1, z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

EXAMPLE (3D). Use the linear approximation to $f(x,y,z) = e^x\sqrt{y}z$ to estimate the value of f at $(0.01, 24.8, 1.02)$. **Solution.** At $(x_0,y_0,z_0) = (0,25,1)$, we have $f(x_0,y_0,z_0) = 5$. The gradient $\nabla f(x,y,z) = (e^x\sqrt{y}z, e^x z/(2\sqrt{y}), e^x\sqrt{y})$ at $(x_0,y_0,z_0) = (0,25,1)$ is $(5, 1/10, 5)$. The linear approximation is $g(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02) = 5.1306$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$.