GREEN'S THEOREM — Maths21a, O. Knill

LINE INTEGRALS (RECALL). If $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is a vector field and $C : r(t) = (x(t), y(t)), t \in [a,b]$ is a curve, then

$$\int_C F\ ds = \int_a^b F(x(t),y(t)) \cdot (x'(t), y'(t))\ dt$$

is called the **line integral of** F **along** C. Its helpful to think of the integral as the work of a force field F along C. It is positive if "the force is with you", negative, if you have to "fight against the force", while going along the path C.

THE CURL OF A 2D VECTOR FIELD. The **curl** of a 2D vector field $F(x,y) = (P(x,y), Q(x,y))$ is defined as the scalar field

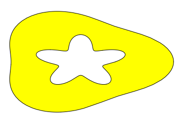
$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y)\ .$$

INTERPRETATION. curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times F = \text{curl}(F)$ for the curl of F because the 2D cross product of (∂_x, ∂_y) with $F = (P,Q)$ is $Q_x - P_y$.

EXAMPLES.
1) $F(x,y) = (-y, x)$. curl$(F)(x,y) = 2$.
2) $F(x,y) = \nabla f$, (conservative field = gradient field = potential) Because $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$, we have curl$(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$.

GREEN'S THEOREM. (1827) If $F(x,y) = (P(x,y), Q(x,y))$ is a vector field and R is a region which has as a boundary a piecewise smooth closed curve C traversed in the direction so that the region R is "to the left". Then

$$\int_C F \cdot ds = \int\int_R \text{curl}(F)\ dxdy$$



Note: for a region with holes, the boundary consists of many curves. They are always oriented so that R **is to the left**.

GEORGE GREEN (1793-1841) was one of the most remarkable physicists of the nineteenth century. He was a self-taught mathematician and miller, whose work has contributed greatly to modern physics. Unfortunately, we don't have a picture of George Green.

SPECIAL CASE. If F is a gradient field, then both sides of Green's theorem are zero:

$\int_C F \cdot ds$ is zero by the fundamental theorem for line integrals.	$\int\int_R \text{curl}(F) \cdot dA$ is zero because curl(F) = curl(grad(f)) = 0.

The fact that curl(grad(f)) = 0 can be checked directly but it can also be seen from $\nabla \times \nabla f$ and the fact that the cross product of two identical vectors is 0. One just has to treat ∇ as a vector.

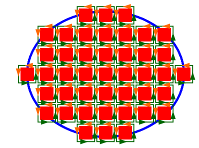
APPLICATION: CALCULATING LINE INTEGRALS. Sometimes, the calculation of line integrals is harder than calculating a double integral. Example: calculate the line integral of $F(x,y) = (x^2 - y^2, 2xy) = (P,Q)$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: curl$(F) = Q_x - P_y = 2y - 2y = -4y$ so that $\int_C F\ dr = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

Remark. One can easily find examples, where the calculation of the line integral is not possible in closed form directly, but where Green allows to do it nevertheless.

WHERE IS THE PROOF? (Quote: General Hein in "Final Fantasy").

To prove Green's theorem, look first at a small square $R = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $F = (P,Q)$ along the boundary is $\int_0^\epsilon P(x+t,y)dt + \int_0^\epsilon Q(x+\epsilon, y+t)\ dt - \int_0^\epsilon P(x+t, y+\epsilon)\ dt - \int_0^\epsilon Q(x, y+t)\ dt$. (Note also that this line integral measures the "circulation" at the place (x,y).)

Because $Q(x+\epsilon, y) - Q(x,y) \sim Q_x(x,y)\epsilon$ and $P(x, y+\epsilon) - P(x,y) \sim P_y(x,y)\epsilon$, the line integral is $(Q_x - P_y)\epsilon^2$ is about the same as $\int_0^\epsilon \int_0^\epsilon \text{curl}(F)\ dxdy$. All identities hold in the limit $\epsilon \to 0$.
To prove the statement for a general region R, we chop it into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the rectangles is a Riemann sum approximation of the double integral.



APPLICATION: CALCULATING DOUBLE INTEGRALS.

Sometimes, the reverse is true and it is harder to calculate the double integral. An example is to determine the area of a polygon with sides $(x_1, y_1), ..., (x_n, y_n)$ (see problem 21, section 13.4 in the book). In that case, there is a closed formula for the area: $A = \frac{1}{2}\sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i)$, something a computer can do evaluate very fast and which does not involve any integration. This formula is used for example in computer graphics.

APPLICATION: FINDING THE CENTROID OF A REGION.

See homework. Green's theorem allows to express the coordinates of the centroid

$$(\int\int_R x\ dA/A, \int\int_R y\ dA/A)$$

as line integrals. One just has to find the right vector fields for each coordinate. For example, to verify

$$\int\int_R x\ dA = \int_C F\ dr$$

take the vector field $F\ dr = x^2 dy$.

APPLICATION: AREA FORMULAS.

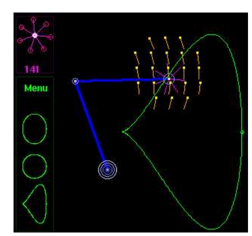
The vector fields $F(x,y) = (P,Q) = (-y, 0)$ or $F(x,y) = (0,x)$ have vorticity curl$(F(x,y)) = 1$. The right hand side in Green's theorem is the **area** of R:

$$\text{Area}(R) = \int_C -y\ dx = \int_C x\ dy$$

EXAMPLE. Let R be the region under the graph of a function $f(x)$ on $[a,b]$. The lineintegral around the boundary of R is 0 from $(a,0)$ to $(b,0)$ because $F(x,y) = 0$ there. The lineintegral is also zero from $(b,0)$ to $(b, f(b))$ and $(a, f(a))$ to $(a,0)$ because $N = 0$. The line integral on $(t, f(t))$ is $-\int_a^b ((-y(t), 0) \cdot (1, f'(t))dt = \int_a^b f(t)\ dt$. Green's theorem assures that this is the area of the region below the graph.

APPLICATION. THE PLANIMETER.

The planimeter is a mechanical device for measuring areas: in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engeneering to measure the size of profiles. There is a vector field F associated to a planimeter (put a vector of length 1 orthogonally to the arm). One can prove that F has vorticity 1. The planimeter calculates the line integral of F along a given curve. Green's theorem assures it is the area.



The picture to the right shows a Java applet which allows to explore the planimeter (from a CCP module by O. Knill and D. Winter, 2001).
To explore the planimeter, visit the URL http://www.math.duke.edu/education/ccp/materials/mvcalc/green/