Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are needed in this problem.

T **F** For any two nonzero vectors $\vec{v}$ and $\vec{w}$, the vector $((\vec{v}\times\vec{w})\times\vec{v})\times\vec{v}$ is parallel to $\vec{w}$.

Solution:
Take $v=(1,0,0), w=(0,1,0)$ so that $\vec{v}\times\vec{w}=(0,0,1)$ and $(\vec{v}\times\vec{w})\times\vec{v})=(0,1,0)$ and $((\vec{v}\times\vec{w})\times\vec{v})\times\vec{v})=(0,0,1)$.

T **F** $\int_0^2\int_0^{2\pi}(r^2/2)\,d\theta\,dr$ computes the area of a disc of radius 2 in the plane.

Solution:
False. The correct formula is $\int_0^{2\pi}\int_0^2 r\,dr\,d\theta$

T F If a vector field $\vec{F}(x,y)$ satisfies $\text{curl}(F)(x,y)=0$ for all points (x,y) in the plane, then $\vec{F}$ is conservative.

Solution:
True. We have derived this from Greens theorem.

T **F** The velocity and acceleration vectors of a parametrized curve $\vec{r}(t)=(x(t),y(t))$ are always perpendicular at any point $\vec{r}(t)$.

Solution:
False. This is already false for a line $\vec{r}(t)=(t^2,t^2)$, where the velocity and acceleration are parallel.

T F A circle of radius 5 has a smaller curvature than a circle of radius 1.

Solution:
True. The curvature of a circle of radius r is equal to $1/r$.

T F The curve $\vec{r}(t)=(-\sin(t),\cos(t))$ for $t\in[0,\pi]$ is half a circle.

Solution:
True. Indeed, one can check that $\sin^2(t)+\cos^2(t)=1$.

T	F
T (highlighted)	F

The function $\sin(x+t)$ is a solution of the partial differential equation $u_t = u_x + u + u_{xt}$.

Solution:
$u_x = -\cos(x+t)$, $u_t = -\cos(x+t)$ and $u_{xt} + u = 0$.

T | **F** (highlighted)

The length of the curve $\vec{r}(t) = (t^3, t^2)$ parametrized on $1 \leq t \leq 2$ is the value of the integral $\int_1^2 (9t^4 + 4t^2)\, dt$.

Solution:
False. The derivatives have to be squared. The correct answer would be $\int_1^2 \sqrt{9t^4 + 4t^2}\, dt$.

T | **F** (highlighted)

Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then the gradient of g at (x_0, y_0) is perpendicular to the gradient of f at (x_0, y_0).

Solution:
The gradients are **parallel**, not perpendicular.

T | **F** (highlighted)

The directional derivative $D_{\vec{v}} f(x_0, y_0, z_0)$ of $f(x,y,z) = x^2 + y^2 - z^2$ into the direction $\vec{v} = (0,0,1)$ is negative at every point (x_0, y_0, z_0).

Solution:
This directional derivative is $f_z = -2z$. For $z > 0$, this is negative, for $z < 0$, this is positive.

T (highlighted) | F

If a vector field $\vec{F}(x,y)$ is not conservative, we always can find a curve C for which the line integral $\int_C F \cdot dr$ is positive.

Solution:
Indeed. There must exist then a curve for which the line integral is not zero. If this line integral is positive, we have found our curve, if it is negative, we reverse the direction of the curve.

T (highlighted) | F

If C is a closed level curve of a function $f(x,y)$ and $F = (f_x, f_y)$ is the gradient field of f, then $\int_C F \cdot dr = 0$.

Solution:
The gradient field is perpendicular to the level curves.

T | **F** (highlighted)

The divergence of a gradient vector field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is always zero.

Solution:
No, just take a simple example like $f(x,y,z) = x^2$, where div(grad(f) = 2.

T | **F** (highlighted)

The function $f(x,y) = x^2 y^3$ has no critical points.

Solution:
$\nabla f(x,y) = (2xy^3, 3x^2y^2)$, which vanishes at $(0,0)$.

T (highlighted) | F

If $\vec{F}(x,y) = (y, 2x)$ and $C : \vec{r}(t) = (\sqrt{\cos(t)}, \sqrt{\sin(t)})$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C F \cdot\, ds$ is the area of R.

Solution:
This is a direct consequence of Green's theorem and the fact that the two-dimensional curl $Q_x - P_y$ of $F = (P, Q)$ is equal to 1.

T (highlighted) | F

The flux of the vector field $\vec{F}(x,y,z) = (0, y, 0)$ through the boundary S of a solid torus E is equal to the volume of the torus.

Solution:
It is the **volume** of the solid torus.

T (highlighted) | F

The quadratic surface $x^2 + y^2 + 4x - z^2 = -3$ is a one sheeted hyperboloid.

Solution:
Completion of the square gives the equation $(x+2)^2 + y^2 - z^2 = 1$.

T (highlighted) | F

If $\vec{F}$ is a vector field in space and S is the boundary of a solid torus, then the flux of $\text{curl}(\vec{F})$ through S is 0.

Solution:
This is true by Stokes theorem.

T F

If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) and S is a torus surface, then the flux of F through S is zero.

Solution:
This is a consequence of the divergence theorem.

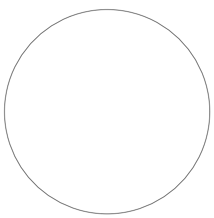
T F

In spherical coordinates, the equation $\rho\cos(\phi) = \rho\sin(\theta)\sin(\phi)$ defines a plane.

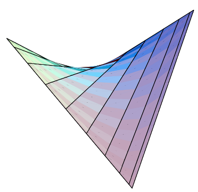
Solution:
True. It is the plane $z = y$.

Problem 2) (10 points) No justifications are needed in this problem.

Match the equations with the objects.



I



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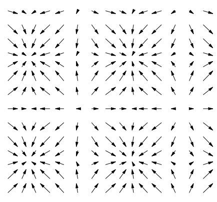

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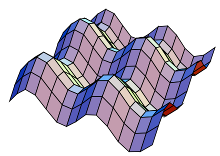

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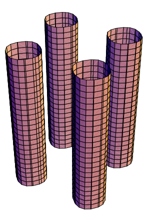

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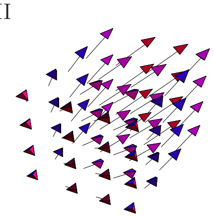

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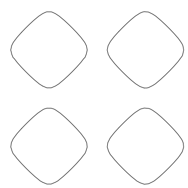
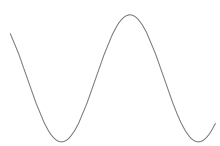


VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$g(x,y,z) = \cos(x) + \sin(y) = 1$
	$y = \cos(x) - \sin(x)$
	$\vec{r}(t) = (\cos(t), \sin(t))$
	$\vec{r}(u,v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
	$\vec{F}(x,y,z) = (\cos(x), \sin(x), 1)$
	$z = f(x,y) = \cos(x) + \sin(y)$
	$g(x,y) = \cos(x) - \sin(y) = 1$
	$\vec{F}(x,y) = (\cos(x), \sin(y))$

Solution:

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
V	$g(x,y,z) = \cos(x) + \sin(y) = 1$
VIII	$y = \cos(x) - \sin(x)$
I	$\vec{r}(t) = (\cos(t), \sin(t))$
II	$\vec{r}(u,v) = (\cos(u), \sin(v), \cos(u)\sin(v))$
VI	$\vec{F}(x,y,z) = (\cos(x), \sin(x), 1)$
IV	$z = f(x,y) = \cos(x) + \sin(y)$
VII	$g(x,y) = \cos(x) - \sin(y) = 1$
III	$\vec{F}(x,y) = (\cos(x), \sin(x))$

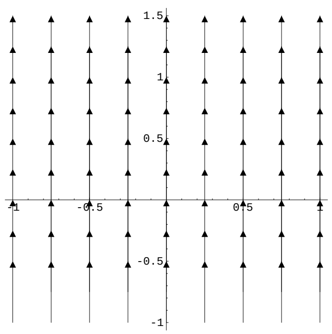
Problem 3) (10 points) No justifications are needed in this problem.

In this problem, vector fields F are written as $F = (P,Q)$. We use abbreviations $\text{curl}(F) = Q_x - P_y$. When stating $\text{curl}(F) = 0$, we mean that $\text{curl}(F)(x,y) = 0$ vanishes for **all** (x,y). Similarly, we say $\text{div}(F)$ if $\text{div}(F)(x,y) = P_x(x,y) + Q_y(x,y) = 0$ for all x, y.

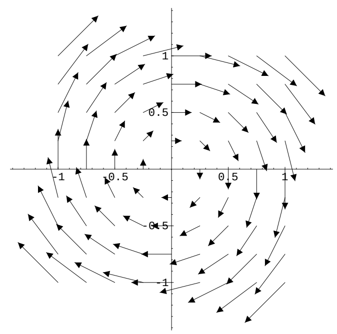
Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV and mark also the places, indicating the vanishing of $\text{curl}(F)$.

Vectorfield	I	II	III	IV	curl$(F)=0$	div$(F)=0$
$\mathbf{F}(x,y)=(2,x)$						
$\mathbf{F}(x,y)=(y,-x)$						
$\mathbf{F}(x,y)=(0,5)$						
$\mathbf{F}(x,y)=(-x,-y)$						

I



II



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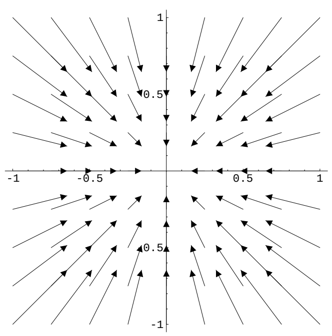
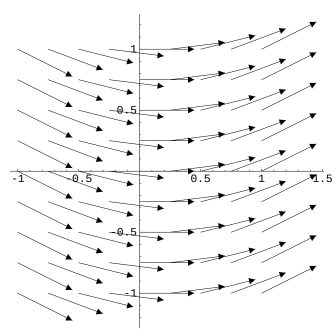


IV



Solution:

Vectorfield	I	II	III	IV	curl$(F)=0$	div$(F)=0$
$\mathbf{F}(x,y)=(2,x)$				X		X
$\mathbf{F}(x,y)=(y,-x)$		X				X
$\mathbf{F}(x,y)=(0,5)$	X				X	X
$\mathbf{F}(x,y)=(-x,-y)$			X		X	

Problem 4) (10 points)

Find the distance between the two parametrized lines

$$\vec{r}_1(t) = (0,0,0) + t(1,1,1)$$

and

$$\vec{r}_2(s) = (1,-1,1) + s(0,0,1)\ .$$

Hint. You can do this problem using a formula derived in the first week of the course. Alternatively, you are also welcome to solve this problem by minimizing the function $f(t,s) = |\vec{r}_1(t) - \vec{r}_2(s)|^2 = (t-1)^2 + (t+1)^2 + (t-1-s)^2$ of the two variables t,s.

Solution:
1. solution. Use the formula. The point $P=(0,0,0)$ is on the first line, the point $Q=(1,-1,1)$ is on the second line. With $u=(1,1,1)$ and $v=(0,0,1)$, we have

$$d = \frac{(u\times v)\cdot(P-Q)|}{|u\times v|} = \frac{|(1,-1,0)\cdot(1,-1,1)|}{|(1,-1,0)|} = \sqrt{2}\ .$$

2. Solution. $f(t,s) = 3t^2+3+s^2-2ts-2t+2s$ has the gradient $\nabla f(t,s) = (6t-2s-2, 2s-2t+2) = (0,0)$ gives the critical point $(t,s)=(0,-1)$. The Hessian matrix at this point is $\begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$ which has determinant $D=2$. Because $f_{tt}(0,-1) = 6 > 0$, the critical point is a local minimum.

Problem 5) (10 points)

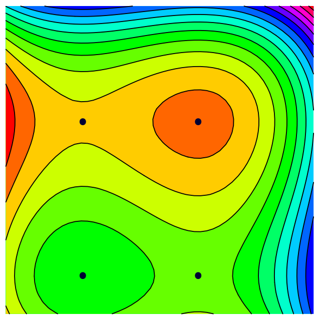
Find all the critical points of the function

$$f(x,y) = \frac{x^3}{3} + \frac{x^2}{2} - 2x + \frac{y^3}{3} + y^2 - 3y$$

and classify them by telling from each of them, whether it is a local maximum, a local minimum or a saddle point.

Solution:
The gradient is $\nabla f(x,y) = (x^2+x-2, y^2+2y+3)$. This gradient vanishes if $x = 1$ or -2 and $y = 1$ or $y = -3$. So, there are four critical points $(1,-1),(1,3),(-2,-1),(-2,3)$. The Hessian matrix is $H(x,y) = \begin{bmatrix} 2x-1 & 0 \\ 0 & 2y \end{bmatrix}$.

point	discriminant	f_{xx}	nature
$(-2,-3)$	D= 12	-3	max
$(-2,1)$	D= -12	-3	saddle
$(1,-3)$	D= -12	3	saddle
$(1,1)$	D= 12	3	min

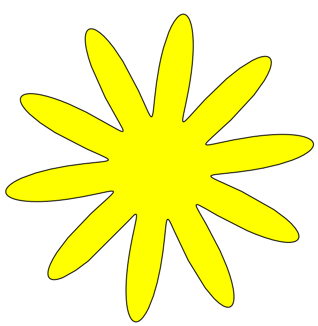


Problem 6) (10 points)

Find the area $\int\int_R 1 \, dxdy$ of the 10 legged "sea star" R, enclosed by the polar curve

$$r(\theta) = 2 + \sin(10\,\theta) ,$$

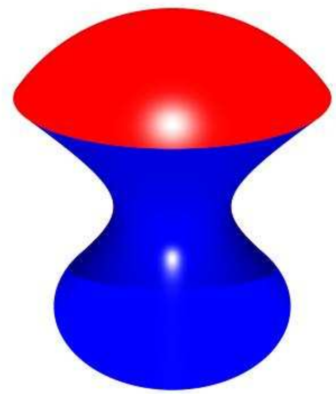
where $\theta \in [0, 2\pi]$. The photo to the right shows a real sea star.

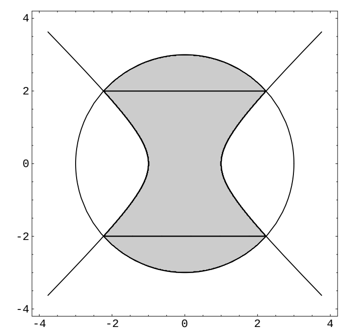


Solution:
$\int_0^{2\pi}\int_0^{2+\sin(10\theta)} r \, dr d\theta = \int_0^{2\pi}(\sin^2(10\theta)+4+4\sin(10\theta))/2 \, d\theta$ which is equal to $(\pi+8\pi)/2 =$ $\boxed{= 9\pi/2}$.

Problem 7) (10 points)

Find the volume of the intersection of the interior of the one sided hyperboloid $x^2+y^2-z^2 \leq 1$ with the solid ball enclosed by the sphere $x^2+y^2+z^2 \leq 9$.





Solution:
The simplest solution is to use cylindrical coordinates and compute the complement of the object with respect to the sphere. Then

$$V = 4\pi 3^3/3 - \int_{-2}^{2}\int_0^{2\pi}\int_{-\sqrt{1+z^2}}^{\sqrt{9-z^2}} r \, dr d\theta dz$$

This is $36\pi - 2\pi\int_{-2}^{2}[(9-z^2)-(1+z^2)]/2 \, dz = 36\pi - \pi\int_{-2}^{2}(10-2z^2) \, dz = \boxed{44\pi/3}$. An other possibility is to the middle part $-2 \leq z \leq 2$ of the solid seperately and add to this the volumes of the upper and lower caps which have equal volume. It is possible to use spherical coordinates. But cylindrical coordinates are again simpler:

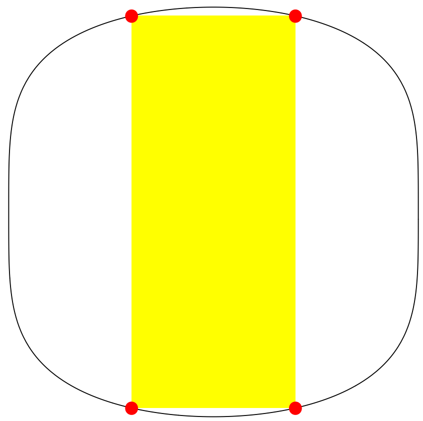
$$V = (2\pi)\int_{-2}^{2}(1+z^2)/2 \, drdz + 2[(2\pi)\int_2^3 (9-z^2)/2 \, drdz] = (2\pi)14/3 + 2[(2\pi)4/3] = \frac{44\pi}{3} .$$

Again, the answer is $\boxed{44\pi/3}$.

Problem 8) (10 points)

What is the largest rectangle with corners at $A = (-x, -y), B = (x, -y), C = (x, y), D = (-x, y)$ which can be placed into the region

$$g(x, y) = x^2 + 2y^4 \leq 1 \;?$$



Hint. You have to extremize the function $f(x, y) = 4xy$ which gives the area of the rectangle under some constraint.

Solution:
The Lagrange equations are

$$\begin{aligned} 4y &= \lambda 2x \\ 4x &= \lambda 8y^3 \\ x^2 + 2y^4 &= 1 \end{aligned}$$

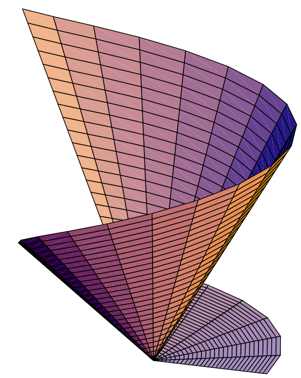
Eliminating $\lambda = 2y/x$ in the first equation and putting this into the second equation $x = \lambda 2y^3$ gives $4y^4 = x^2$. Plugging this into the last equation gives $6y^4 = 1$ and so $\boxed{x = \sqrt{\frac{2}{3}}, y = 1/6^{1/4}}$.

Problem 9) (10 points)

A surface S is parametrized as $\vec{r}(t, s) = (s\cos(t), s\sin(t), st)$, where $t \in [0, 3\pi]$ and $s \in [0, 1]$.

a) (6 points) Find the surface area of the surface S.

b) (4 points) The boundary of the surface is the helix $\vec{r}(t) = (\cos(t), \sin(t), t)$ for $0 \leq t \leq 3\pi$. Find the length of this curve.

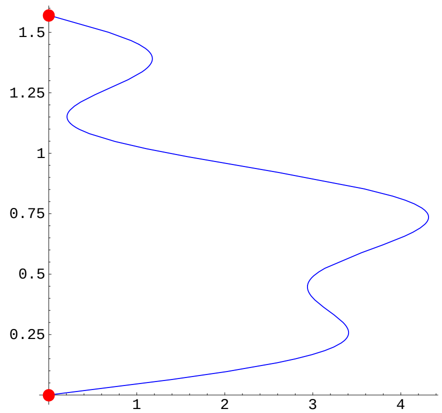


Solution:
a) $r_t = (-s\sin(t), s\cos(t), s)$ and $r_s = (\cos(t), \sin(t), t)$. We have $|r_t \times r_s| = s\sqrt{(2 + t^2)}$. The surface area is $\int_0^1 \int_0^{2\pi} s\sqrt{(2 + t^2)}\, dtds$. After integrating over s, we obtain the integral $\boxed{\frac{1}{2}\int_0^{2\pi} \sqrt{2 + t^2}\, dt}$. This answer gives already full credit. But the integral can be evaluated explicitly sing integration by parts with $du = 1, v = \sqrt{2 + t^2})$ so that $\int \sqrt{2 + t^2}\, dt = t\sqrt{2 + t^2}/2 + \text{arcsinh}(t/\sqrt{2})$. The definite integral is $\boxed{3\pi\sqrt{2 + 9\pi^2}/4 + \text{arcsinh}(3\pi/\sqrt{2})}$.

b) The length is $\int_0^{3\pi} |r'(t)|\, dt = \int_0^{3\pi} \sqrt{2}\, dt = \boxed{3\pi\sqrt{2}}$.

Problem 10) (10 points)

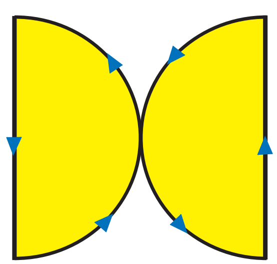
Find the line integral of the vector field $\vec{F}(x, y) = (x^{30} + y, y^{50} + x)$ along the path $\vec{r}(t) = (4\sin(\pi\sin(t)) + \sin(10t), t)$ with $0 \leq t \leq \pi/2$.



Solution:
There are two possibilities to solve this problem. The first solution is to find the potential $f(x, y)$ which has F as a gradient field. It is $f(x, y) = \frac{x^{31}}{31} + \frac{y^{51}}{51} + xy$.
By the fundamental theorem of line integrals, the result is $f(0, \pi/2) - f(0, 0) = \boxed{\frac{\pi^{51}}{2^{51} \cdot 51}}$.
The second solution is to note that since the vector field has vanishing curl, it is conservative and the line integral along the path C is the same as the line integral along the path $\vec{r}(t) = (0, t)$. We obtain the same answer by computing this line integral.

Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x, y) = (-y + e^{xy}y, e^{xy}x)$ along the boundary C of the butterfly shaped region R inside the unit square. The curve C consists of circular arcs which connect the points $(0, 0), (1/2, 1/2), (1, 0), (1, 1), (1/2, /1, 2), (0, 1), (0, 0)$ in that order.



Solution:
The curl is 1 so that by Green's theorem, the line integral is the area of R which is $\boxed{\pi/4}$, the area of a disk of radius $1/2$.

Problem 12) (10 points)

Let S be the part of the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$$

satisfying $z > 0$, parametrized so that the normal vectors on S point upwards. Find the flux of the curl of $\vec{F}(x, y, z) = (-y + \sin(x^2) + \sin(\sin(z)), x + \cos(y^5), 3 + \sin(z^3))$ through the surface S.

Hint. It can be very helpful to see first that the curl of $\vec{F}$ is the same as the curl of $\vec{G}(x, y, z) = (-y + \sin(\sin(z)), x, 3)$.

Solution:
We have to compute the line integral of G along the boundary which is the ellipse $x^2/9 + y^2/4$ in the xy-plane. This ellipse is parametrized by $\vec{r}(t) = (3\cos(t), 2\sin(t), 0)$ so that $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = (-2\sin(t), 3\cos(t), 3) \cdot (-3\sin(t), 2\cos(t), 0) = 6\sin^2(t) + 6\cos^2(t) = 6$ and the line integral is $\boxed{12\pi}$.

Problem 13) (10 points)

What is the flux of the vector field

$$\vec{F}(x, y, z) = (3x + 5y^2 z, x, y^3 + \sin(xy^3))$$

through the boundary S of the rectangular cube $E = \{1 \leq x \leq 3, 2 \leq y \leq 5, -1 \leq z \leq 1\}$?

Solution:
We use the divergence theorem $\int\int_S F \cdot dS = \int\int\int \text{div}(F)\, dV$. The divergence is 3 so that the flux integral is 3 times the volume of the box which is $\boxed{36}$.