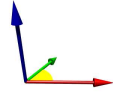


**CROSS PRODUCT**

Maths21a, O. Knill

**CROSS PRODUCT.** The **cross product** of two vectors  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  and  $\vec{w} = \langle w_1, w_2, w_3 \rangle$  is defined as the vector  $\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle$ . To compute it, look at the "determinant computation":



$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix} = i(v_2w_3 - v_3w_2) - j(v_1w_3 - v_3w_1) + k(v_1w_2 - v_2w_1).$$

**DIRECTION OF  $\vec{v} \times \vec{w}$ :**  $\vec{v} \times \vec{w}$  is orthogonal to  $\vec{v}$  and orthogonal to  $\vec{w}$ .

Proof. Check that  $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ .

**LENGTH:**  $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$

Proof. The identity  $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$  can be proven by direct computation. Now,  $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|\cos(\alpha)$ .

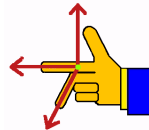
**AREA.** The length  $|\vec{v} \times \vec{w}|$  is the area of the parallelogram spanned by  $\vec{v}$  and  $\vec{w}$ .

Proof. Because  $|\vec{w}|\sin(\alpha)$  is the height of the parallelogram with base length  $|\vec{v}|$ , the area is  $|\vec{v}||\vec{w}|\sin(\alpha)$  which is by the above formula equal to  $|\vec{v} \times \vec{w}|$ .

**EXAMPLE.** If  $\vec{v} = \langle a, 0, 0 \rangle$  and  $\vec{w} = \langle b\cos(\alpha), b\sin(\alpha), 0 \rangle$ , then  $\vec{v} \times \vec{w} = \langle 0, 0, ab\sin(\alpha) \rangle$  which has length  $|ab\sin(\alpha)|$ .

**ZERO CROSS PRODUCT.** We see that  $\vec{v} \times \vec{w}$  is zero if  $\vec{v}$  and  $\vec{w}$  are **parallel**.

**ORIENTATION.** The vectors  $\vec{v}, \vec{w}$  and  $\vec{v} \times \vec{w}$  form a **right handed coordinate system**. The right hand rule is: put the first vector  $\vec{v}$  on the thumb, the second vector  $\vec{w}$  on the pointing finger and the third vector  $\vec{v} \times \vec{w}$  on the third middle finger.



**EXAMPLE.**  $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$  forms a right handed coordinate system.

**DOT PRODUCT** (is a scalar)

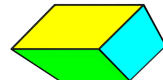
$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$  commutative  
 $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}|\cos(\alpha)$  angle  
 $(a\vec{v}) \cdot \vec{w} = a(\vec{v} \cdot \vec{w})$  linearity  
 $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$  distributivity  
 $\{1, 2, 3\}, \{3, 4, 5\}$  in Mathematica  
 $\frac{d}{dt}(\vec{v} \cdot \vec{w}) = \dot{\vec{v}} \cdot \vec{w} + \vec{v} \cdot \dot{\vec{w}}$  product rule

**CROSS PRODUCT** (is a vector)

$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$  anti-commutative  
 $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin(\alpha)$  angle  
 $(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w})$  linearity  
 $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$  distributivity  
 $\text{Cross}[\{1, 2, 3\}, \{3, 4, 5\}]$  in Mathematica  
 $\frac{d}{dt}(\vec{v} \times \vec{w}) = \dot{\vec{v}} \times \vec{w} + \vec{v} \times \dot{\vec{w}}$  product rule

**TRIPLE SCALAR PRODUCT.** The scalar  $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$  is called the **triple scalar product** of  $\vec{u}, \vec{v}, \vec{w}$ .

**PARALLELEPIPED.**  $[\vec{u}, \vec{v}, \vec{w}]$  is the volume of the parallelepiped spanned by  $\vec{u}, \vec{v}, \vec{w}$  because  $h = \vec{u} \cdot \vec{n}/|\vec{n}|$  is the height of the parallelepiped if  $\vec{n} = (\vec{v} \times \vec{w})$  is a normal vector to the ground parallelogram which has area  $A = |\vec{n}| = |\vec{v} \times \vec{w}|$ . The volume of the parallelepiped is  $hA = \vec{u} \cdot \vec{n}/|\vec{n}| \cdot |\vec{n}| = |\vec{u} \cdot (\vec{v} \times \vec{w})|$ .



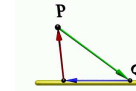
**EXAMPLE.** Find the volume of the parallel epiped which has the one corner  $O = (1, 1, 0)$  and three corners  $P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$  connected to it.

**ANSWER:** The parallelepiped is spanned by  $\vec{u} = (1, 2, 1), \vec{v} = (3, 2, 1),$  and  $\vec{w} = (0, 3, 2)$ . We get  $\vec{v} \times \vec{w} = (1, -6, 9)$  and  $\vec{u} \cdot (\vec{v} \times \vec{w}) = -2$ . The volume is 2.

**DISTANCE POINT-LINE (3D).** If  $P$  is a point in space and  $L = Q + t\vec{u}$  be the line which contains the vector  $\vec{u}$  and the point  $Q$ , then

$$d(P, L) = |\vec{PQ} \times \vec{u}|/|\vec{u}|$$

is the distance between  $P$  and the line  $L$ . The representation of  $L$  as  $r(t) = Q + t\vec{u}$  is called a **parametric equation** of the line.



**PLANE THROUGH 3 POINTS  $P, Q, R$ :**

The vector  $\vec{n} = \vec{PQ} \times \vec{PR}$  is orthogonal to the plane. We will see next week that  $\vec{n} = (a, b, c)$  defines the plane  $ax + by + cz = d$ , with  $d = ax_0 + by_0 + cz_0$  which passes through the points  $P = (x_0, y_0, z_0), Q, R$ .

The cross product appears in many different applications:

**ANGULAR MOMENTUM.** If a mass point of mass  $m$  moves along a curve  $\vec{r}(t)$ , then the vector  $\vec{L}(t) = m\vec{r}(t) \times \vec{r}'(t)$  is called the **angular momentum** of the point. It is coordinate system dependent.

**ANGULAR MOMENTUM CONSERVATION.**

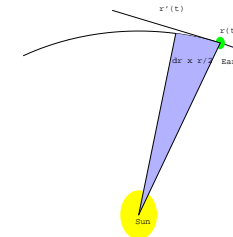
$$\frac{d}{dt}\vec{L}(t) = m\vec{r}''(t) \times \vec{r}'(t) + m\vec{r}'(t) \times \vec{r}''(t) = \vec{r}(t) \times \vec{F}(t)$$

In a central field, where  $\vec{F}(t)$  is parallel to  $\vec{r}(t)$ , we get  $d/dtL(t) = 0$  which means  $L(t)$  is constant.

**TORQUE.** The quantity  $\vec{r}(t) \times \vec{F}(t)$  is also called the **torque**. The time derivative of the **momentum**  $p = m\vec{r}'$  is the **force**  $F$ . the time derivative of the **angular momentum**  $\vec{L} = \vec{r}(t) \times \vec{p}(t) = m\vec{r}(t) \times \vec{r}'(t)$  is the **torque**.

**KEPLER'S AREA LAW.** (Proof by Newton)

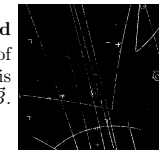
The fact that  $\vec{L}(t)$  is constant means first of all that  $\vec{r}(t)$  stays in a plane spanned by  $\vec{r}(0)$  and  $\vec{r}'(0)$ . The experimental fact that the vector  $\vec{r}(t)$  sweeps over **equal areas in equal times** expresses angular momentum conservation:  $|\vec{r}(t) \times \vec{r}'(t)dt/2| = |\vec{L}dt/m/2|$  is the area of a small triangle. The vector  $\vec{r}(t)$  sweeps over an area  $\int_0^T |\vec{L}|dt/(2m) = |\vec{L}|T/(2m)$  in time  $[0, T]$ .



**MORE PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS:**

The **top**, the motion of a rigid body is describe by the angular momentum  $L$  and the angular velocity vector  $\Omega$  in the body. Then  $\dot{L} = L \times \Omega + M$ , where  $M$  is an external **torque** obtained by external forces.

**Electromagnetism:** (informal) a particle moving along  $\vec{r}(t)$  in a **magnetic field**  $\vec{B}$  for example experiences the force  $\vec{F}(t) = q\vec{r}'(t) \times \vec{B}$ , where  $q$  is the charge of the particle. In a constant magnetic field, the particles move on circles: if  $m$  is the mass of the particle, then  $m\vec{r}''(t) = q\vec{r}'(t) \times \vec{B}$  implies  $m\vec{r}'(t) = q\vec{r}(t) \times \vec{B}$ . Now  $d/dt|\vec{r}|^2 = 2\vec{r} \cdot \vec{r}' = \vec{r} \cdot q\vec{r}'(t) \times \vec{B} = 0$  so that  $|\vec{r}|$  is constant.



**Hurricanes** are powerful storms with wind velocities of 74 miles per hour or more. On the northern hemisphere, hurricanes turn counterclockwise, on the southern hemisphere clockwise. This is a feature of all low pressure systems and can be explained by the Coriolis force. In a rotating coordinate system a particle of mass  $m$  moving along  $\vec{r}(t)$  experience the following forces:  $m\vec{\omega}' \times \vec{r}$  (inertia of rotation),  $2m\vec{\omega} \times \vec{r}'$  (Coriolis force) and  $m\vec{\omega} \times (\vec{\omega} \times \vec{r})$  (Centrifugal force). The Coriolis force is also responsible for the circulation in Jupiter's Red Spot.

