DIVERGENCE THEOREM — Maths21a, O. Knill

DIV. The **divergence** of a vector field F is $\mathrm{div}(P,Q,R) = \nabla \cdot F = P_x + Q_y + R_z$. It is a scalar field.
The flux integral of a vector field F through a surface $S = r(R)$ was defined as $\int\int_S F \cdot dS = \int\int_R F(r(u,v)) \cdot r_u \times r_v \; dudv$.
Recall also that the integral of a scalar function f on a region R is $\int\int\int_R f \; dV = \int\int\int_G f(x,y,z) \; dxdydz$.

GAUSS THEOREM or DIVERGENCE THEOREM. Let G be a solid in space. Assume the boundary of the solid is a surface S. Let F be a vector field. Then

$$\int\int\int_G \mathrm{div}(F) \; dV = \int\int_S F \; \cdot dS \; .$$

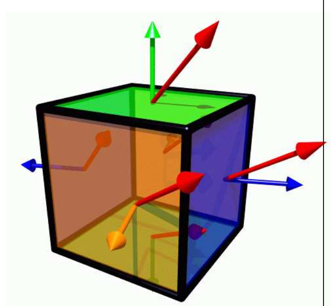
The orientation of S is such that the normal vector $r_u \times r_v$ points **outside** of G.

EXAMPLE. Let $F(x,y,z) = (x,y,z)$ and let S be sphere. The divergence of F is constant 3 and $\int\int\int_G \mathrm{div}(F) \;\; dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S r \cdot r_u \times r_v \;\; dudv = \int\int_S |r(u,v)|^2 \sin(v) \; dudv = \int_0^\pi \int_0^{2\pi} \sin(v) \; dudv = 4\pi$ also.

CONTINUITY EQUATION. If ρ is the density of a fluid with velocity field $\vec{F}$, the flux $\int\int_S \vec{F} \cdot dS$ of the fluid through a closed surface S bounding a region G is the minus the rate of change of the mass $-(d/dt)\int\int\int_G \rho dV$ inside G. This is because the flux through the boundary is the amount of mass leaving G in unit time. But this flux is by Gauss theorem equal to $\int\int\int_G \mathrm{div}(\vec{F}) \; dV$. Therefore, $\int\int\int_G (\frac{d}{dt}\rho + \mathrm{div}(\vec{F}) \; dV = 0$. Because this is true for all regions G, the partial differential equation $\dot{\rho} + \mathrm{div}(\vec{F})$ holds. It is called the **continuity equation**. It gives a meaning to $\mathrm{div}(\vec{F})$ as the rate of change of "density" at this point. Take a a small cube and let it flow with the field. If it expands in volume, then the divergence is positive, the density becomes smaller.

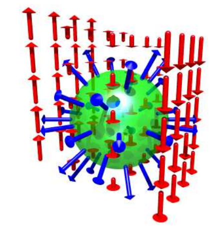
EXAMPLE. What is the flux of the vector field $F(x,y,z) = (2x, 3z^2 + y, \sin(x))$ through the box $G = [0,3] \times [0,2] \times [-1,1]$?
Answer: Use the divergence theorem: $\mathrm{div}(F) = 2$ and so $\int\int\int_G \mathrm{div}(F) \; dV = 2\int\int\int_G dV = 2\mathrm{Vol}(G) = 24$.
Note: Often, it is easier to evaluate a three dimensional integral than a flux integral because the later needs a parameterization of the boundary, which requires the calculation of $r_u \times r_v$ etc.

PROOF OF THE DIVERGENCE THEOREM. Consider a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. Call the sides orthogonal to the x axis **x**-boundaries etc. The flux of $F = (P,Q,R)$ through the x-boundaries is $[F(x+dx,y,z) \cdot (1,0,0) + F(x,y,z) \cdot (-1,0,0)]dydz = P(x+dx,y,z) - P(x,y,z) = P_x dxdydz$. Similarly, the flux through the y-boundaries is $P_y dydxdz$ and the flux through the z-boundary is $P_z dzdxdy$. The total flux through the boundary of the box is $(P_x + P_y + P_z)dxdydz = \mathrm{div}(F)dxdydz$.

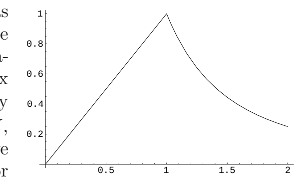


For a general body, approximate it with a union of small little cubes. The sum of the fluxes over all the little cubes is sum of the fluxes through the sides which do not touch an other box (fluxes through touching sides cancel). The sum of all the infinitesimal fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\mathrm{div}(F)dxdydz$ is a Riemann sum approximation for the integral $\int\int\int_G \mathrm{div}(F)dxdydz$. In the limit, where dx, dy, dz goes to zero, we obtain Gauss theorem.

VOLUME CALCULATION. Similarly as the planimeter allowed to calculate the area of a region by passing along the boundary, the volume of a region can be determined as a flux integral. Take for example the vector field $F(x,y,z) = (x,0,0)$ which has divergence 1. The flux of this vector field through the boundary of a region is the volume of the region. $\int\int_{\delta G}(x,0,0) \cdot dS = \mathrm{Vol}(G)$.



GRAVITY INSIDE THE EARTH. How much do we weight deep in earth at radius r from the center of the earth? (Relevant in the movie "The core") The law of gravity can be formulated as $\mathrm{div}(F) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $F(x) = F(r)x/||x||$. The flux of F through a ball of radius r is $\int\int_{S_r} F(x) \cdot dS = 4\pi r^2 F(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi \int\int\int_{B_r} \rho(x) \; dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2 \rho r^3/3 = 4\pi r^2 F(r)$ for $r < R$ and $(4\pi)^2 \rho R^3/3 = 4\pi r^2 F(r)$ for $r \geq R$. Inside the earth, the gravitational force $F(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $F(r) = M/r^2$ with $M = 4\pi R^3 \rho/3$.



WHAT IS THE BOUNDARY OF A BOUNDARY? The fundamental theorem for line integral, Green's theorem, Stokes theorem and Gauss theorem are all of the form $\int_A dF = \int_{\delta A} F$, where dF is a derivative of F and δA is a boundary of A. They all generalize the fundamental theorem of calculus. There is some similarity in how d and δ behave:

f scalar field	ddf = curl grad$(f) = 0$	S surface in space	δS is union of closed curves	$\delta\delta S = \emptyset$
F vector field	ddF = div curl$(F) = 0$	G region in space	δG is a closed surface	$\delta\delta G = \emptyset$

The question when $\mathrm{div}(F) = 0$ implies $F = \mathrm{curl}(G)$ or whether $\mathrm{curl}(F) = 0$ implies $G = \mathrm{grad}(G)$ is interesting. We look at it Friday.

STOKES AND GAUSS. **Stokes theorem was found by Ampere in 1825.** George Gabriel Stokes: (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840. **Gauss theorem was discovered 1764 by Joseph Louis Lagrange.** Carl Friedrich Gauss, who formulates also Greens theorem, rediscovers the divergence theorem in 1813. Green also rediscovers the divergence theorem in 1825 not knowing of the work of Gauss and Lagrange.

Carl Friedrich Gauss

George Gabriel Stokes

Joseph Louis Lagrange

André Marie Ampere

GREEN IDENTITIES. If G is a region in space bounded by a surface S and f, g are scalar functions, then with $\Delta f = \nabla^2 f = f_{xx} + f_{yy} + f_{zz}$, one has as a direct consequence of Gauss theorem the **first and second Green identities** (see homework)

$$\int\int\int_G (f\Delta g + \nabla f \cdot \nabla g)\ dV = \int\int_S f\nabla g \cdot dS$$

$$\int\int\int_G (f\Delta g - g\Delta f)\ dV = \int\int_S (f\nabla g - g\nabla f) \cdot dS$$

These identities are useful in electrostatics. Example: if $g = f$ and $\Delta f = 0$ and either $f = 0$ on the boundary S or ∇f is orthogonal to S, then Green's first identity gives $\int\int\int_G |\nabla f|^2\ dV = 0$ which means $f = 0$. This can be used to prove uniqueness for the **Poisson equation** $\Delta h = 4\pi\rho$ when applying the identity to the difference $f = h_1 - h_2$ of two solutions with either **Dirichlet boundary conditions** ($h = 0$ on S) or von Neumann boundary conditions (∇h orthogonal to S).

GAUSS THEOREM IN HIGHER DIMENSIONS. If G is a n-dimensional "hyperspace" bounded by a $(n-1)$ dimensional "hypersurface" S, then $\int_G \text{div}(F) dV = \int_S F \cdot dS$.

DIV. In dimension d, the divergence is defined $\text{div}(F) = \nabla \cdot F = \sum_i \partial F_i / \partial x_i$. The proof of the n-dimensional divergence theorem is done as in three dimensions.

By the way: Gauss theorem in two dimensions is just a version of Green's theorem. Replacing $F = (P, Q)$ with $G = (-Q, P)$ gives $\text{curl}(F) = \text{div}(G)$ and the flux of G through a curve is the line integral of F along the curve. Green's theorem for F is identical to the 2D-divergence theorem for G.