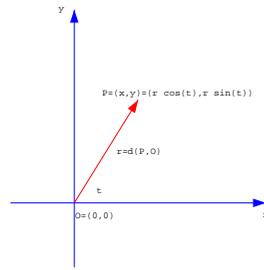


POLAR INTEGRATION

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POLAR COORDINATES. A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \arctg(y/x)$. We have $x = r \cos(\theta), y = r \sin(\theta)$.



Footnote: Note that $\theta = \arctg(y/x)$ defines the angle θ only up to an addition of π . The points (x, y) and $(-x, -y)$ would have the same θ . In order to get the correct θ , one could take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$ as Mathematica does, where $\pi/2$ is the value when $y/x = \infty$, and add π if $x < 0$ or $x = 0, y < 0$. In Mathematica, you can get the polar coordinates with $(r, \theta) = (\text{Abs}[x + Iy], \text{Arg}[x + Iy])$.

POLAR CURVES. A general polar curve is written as $(r(t), \theta(t))$. It can be translated into x, y coordinates: $x(t) = r(t) \cos(\theta(t)), y(t) = r(t) \sin(\theta(t))$.

POLAR GRAPHS. Curves which are graphs when written in polar coordinates are called **polar graphs**.

EXAMPLE. $r(\theta) = \cos(3\theta)$ is the which belongs to the class of **roses** $r(t) = |\cos(nt)|$. These curves are also called **rhododenea**.

EXAMPLE. If $y = 2x + 3$ is a line, then the equation gives $r \sin(\theta) = 2r \cos(\theta) + 3$. Solving for $r(t)$ gives $r(\theta) = 3/(\sin(\theta) - 2 \cos(\theta))$. The line is also a polar graph.

EXAMPLE. The polar curve $r(\theta) = \frac{a(1-\epsilon^2)}{1+\epsilon \cos(\theta)}$ of the ellipse (see Kepler). The ellipse is a polar graph.

EXAMPLE. The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of a **limaçon** $r(\theta) = 1 + b \sin(\theta)$.

POLAR COORDINATES. For many regions, it is better to use polar coordinates for integration:

$$\int \int_R f(x, y) \, dx dy = \int \int_R g(r, \theta) r \, dr d\theta$$

For example if $f(x, y) = x^2 + x^2 + xy$, then $g(r, \theta) = r^2 + r^2 \cos(\theta) \sin(\theta)$.

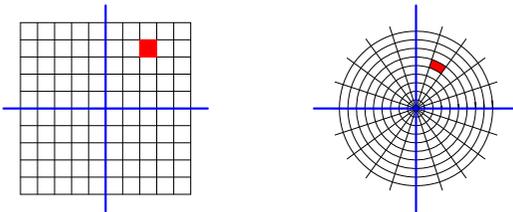
EXAMPLE. We had computed area of the disc $\{x^2 + y^2 \leq 1\}$ using substitution. Was quite a mess. It is easier to do that integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r \, dr d\theta = 2\pi r^2/2|_0^1 = \pi.$$

WHERE DOES THE FACTOR "r" COME FROM?

A small rectangle R with dimensions $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta)$

$(r \cos(\theta), r \sin(\theta))$ to a sector segment S in the (x, y) plane. It has approximately the area $r d\theta dr$. It is small for small r .

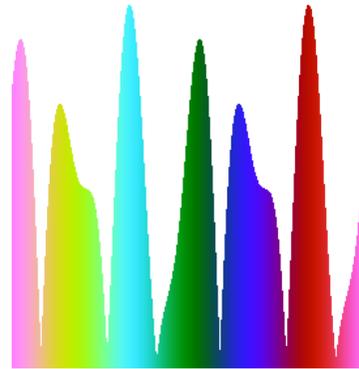


AN OTHER EXPLANATION (for people who have seen some linear algebra). The map translating from polar coordinates to Cartesian coordinates $(x, y) = T(r, \theta) = (r \cos(\theta), r \sin(\theta))$ has as a linear approximation the matrix

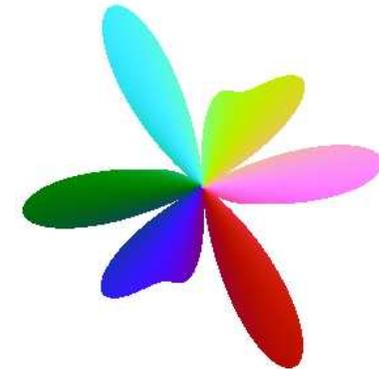
$$DT(x, y) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r \sin(\theta) & r \cos(\theta) \end{bmatrix}$$

which has determinant r . A small rectangle in the (r, θ) -plane of area dA will have in the (x, y) plane the area $r dA$.

ROSES. We can now integrate over type I or type II regions in the (θ, r) plane. Examples are **roses**: $\{(\theta, r) | 0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ .



The region R in the $\theta - r$ coordinates is a type I region



The region $S = T(R)$ in the $x - y$ coordinates is neither a type I nor a type II region.

EXAMPLE. Find the area of the region $\{(\theta, r(\theta)) | r(\theta) \leq |\cos(3\theta)|\}$.

$$\int \int_R y \, dx dy = \int_0^{2\pi} \int_0^{|\cos(3\theta)|} r \, dr d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2} d\theta = \pi/2$$

EXAMPLE. Integrate $f(x, y) = y\sqrt{x^2 + y^2}$ over the semi annulus $R = \{(x, y) | 1 < x^2 + y^2 < 4, y > 0\}$.

Solution.

$$\int_1^2 \int_0^\pi r \sin(\theta) r \, r \, d\theta dr = \int_1^2 r^3 \int_0^\pi \sin(\theta) \, d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^\pi \sin(\theta) \, d\theta = 15/2$$

For integration problems, where the region is part of an annulus, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r \cos(\theta), y = r \sin(\theta)$.

THE SUPER-CURVE. The Belgian Biologist Johan Gielis came up in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n2}}{b} \right)^{-1/n3}$$

```
S[m_, n1_, n2_, n3_, a_, b_] := Module[{t},
  f1[t_] := Abs[Cos[m t/4]/a]^n1;
  f2[t_] := Abs[Sin[m t/4]/b]^n2;
  f[t_] := (f1[t]^2 + f2[t]^2)^(1/n3);
  r[t_] := If[f[t] == 0, {0, 0}, {Cos[t], Sin[t]}/f[t]];
  ParametricPlot[r[t], {t, -2Pi, 2Pi}, PlotRange -> All,
  PlotPoints -> 1000, Axes -> False, AspectRatio -> 1, Frame -> False];
];
R1 := Random[Integer, {3, 20}]; RR := 20 Random[];
T := S[R1, RR, RR, RR, RR, RR];
```

It is called the **super-curve** because it can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes" (see later).

The super-curve generalizes the **superellipse** which had been discussed in 1818 by Lamé and helps to tackle one of the more intractable problems in biology: describing form. A twist: Gielis has patented his discovery described in "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003). To the right you see the Mathematica code.

