

VECTOR FIELDS

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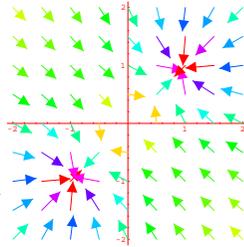
VECTOR FIELDS.

Planar vector field.

A vector field in the plane is a map, which assigns to each point (x, y) in the plane a vector $F(x, y) = (P(x, y), Q(x, y))$.

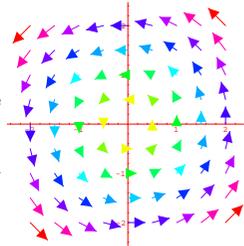
Vector field in space.

A vector field in space is a map, which assigns to each point (x, y, z) in space a vector $F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$.



PLANAR VECTOR FIELD EXAMPLES.

- $F(x, y) = (y, -x)$ is a planar vector field which you see in a picture on the right.
- $F(x, y) = (x-1, y)/((x-1)^2 + y^2)^{3/2} - (x+1, y)/((x+1)^2 + y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. It is shown in the picture above.



GRADIENT FIELD. 2D: If $f(x, y)$ is a function of two variables, then $F(x, y) = \nabla f(x, y)$ is called a gradient field. The same in 3D: gradient fields are of the form $F(x, y, z) = \nabla f(x, y, z)$.

EXAMPLE. $(2x, 2y, -2z)$ is the vector field which is orthogonal to hyperboloids $x^2 + y^2 - z^2 = \text{const}$.

EXAMPLE of a VECTOR FIELD. If $H(x, y)$ is a function of two variables, then $(H_y(x, y), -H_x(x, y))$ is called a **Hamiltonian vector field**. An example is the harmonic Oscillator $H(x, y) = x^2 + y^2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$ is the same as in example 1) above.

WHEN IS A VECTOR FIELD A GRADIENT FIELD (2D)?

$F(x, y) = (P(x, y), Q(x, y)) = \nabla f(x, y)$ implies $Q_x(x, y) = P_y(x, y)$. If this does not hold at some point, F is no gradient field. We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for F to be a gradient field.

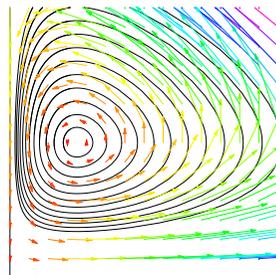
EXAMPLE. VECTOR FIELDS IN BIOLOGY.

Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $-cy(t) + dx(t)y$, where c, d are positive. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction.

A concrete example is the **Volterra-Lotka system**

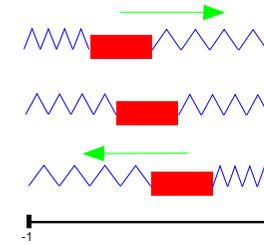
$$\begin{aligned}\dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\end{aligned}$$

Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $(x, y) = (x(t), y(t))$, there is a curve $= \vec{r}(t) = (x(t), y(t))$ through that point for which the tangent $\vec{r}'(t) = (x'(t), y'(t))$ is the vector $(0.4x - 0.4xy, -0.1y + 0.2xy)$.

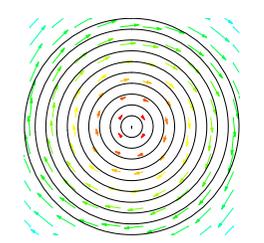


VECTOR FIELDS IN PHYSICS

Newton's law $m\vec{r}'' = F$ relates the acceleration \vec{r}'' of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $m\ddot{x} = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t , then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x, y) , thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.



We don't yet know yet the curve $t \mapsto (x(t), y(t))$, but we know the tangents $(x'(t), y'(t)) = (y(t), -x(t))$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE). More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x, y) , the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r \sin(t), r \cos(t))$ are solutions. They form a family of curves. Can you interpret these solutions physically?



VECTOR FIELDS IN MECHANICS

If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $G(x) = -gm \sin(x)$, where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero.

The Newton equation "mass times acceleration = Force" gives

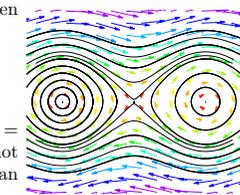
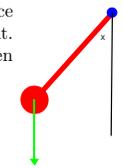
$$\ddot{x}(t) = -g \sin(x(t)) .$$

The equation of motion for the pendulum $\ddot{x}(t) = -g \sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g \sin(x(t))) .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $\vec{r}(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like sin, cos, exp, log etc. However, one still can understand the curves:

Curves on the top of the picture represent situations where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum.



VECTOR FIELDS IN METEOROLOGY. On maps like <http://www.hpc.ncep.noaa.gov/sfc/satsfc.gif> one can see **isotherms**, curves of constant temperature or pressure $p(x, y) = c$. These are level curves. The wind maps are vector fields. $F(x, y)$ is the wind velocity at the point (x, y) . The wind velocity F is not always normal to the **isobars**, the lines of equal pressure p . The scalar pressure field p and the velocity field F depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

$$\frac{d}{dt}F + F \cdot \nabla F = \nu \Delta F - \nabla p + f, \text{div} F = 0$$

(we will see what is Δ, div later.) It is a **partial differential equation** like $u_x - u_y = 0$. Finding solutions is not trivial: 1 Million dollars are given to the person proving that the equations have smooth solutions in space.

