8/12/2008 FINAL EXAM **Maths 21a, O. Knill, Summer 2008**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

1) T F The quadratic surface $x^2 + y - z^2 = -5$ is a hyperbolic paraboloid.

Solution:
Write it as $y - 5 = z^2 - x^2$, to see it better.

2) T F There are vectors $\vec{u}$ and $\vec{v}$ such that $|\vec{u} \times \vec{v}| > |\vec{u}||\vec{v}|$.

Solution:
We have a general identity $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(alpha)$ which contradicts the claim.

3) T F $\int_0^{2\pi} \int_0^5 r \, d\theta \, dr$ is the area of a disc of radius 5.

Solution:
There seems nothing wrong with that. But note that the 0 to 5 integration is paired with the $d\theta$.

4) T F If a vector field $\vec{F}(x, y)$ satisfies $\mathrm{curl}(F)(x, y) = Q_x - P_y = 0$ for all points (x, y) in the plane, then $\vec{F}$ is a gradient field.

Solution:
True. We have derived this from Green's theorem.

5) T F The jerk of a parameterized curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is parallel to the acceleration if the curve $\vec{r}(t)$ is a line.

Solution:
The velocity, the acceleration and the jerk are all parallel on a line.

6) T F The curvature of the curve $\vec{r}(t) = \langle 3\sin(t), 0, 3\cos(t)\rangle$ is twice the curvature of the curve $\vec{s}(t) = \langle 6 + 6\sin(t), 6\cos(t), 0\rangle$.

7) T F The curve $\vec{r}(t) = \langle \sin(t), t^2, \cos(t) \rangle$ for $t \in [0, 10\pi]$ is located on a cylinder.

Solution:
Indeed, one can check that $x(t)^2 + z(t)^2 = 1$.

8) T F If a function $f(x,y)$ has the property that $f_x(x,y)$ is zero for all x, y, then f is the constant function.

Solution:
No, for example $f(x,y) = y$ is also a solution too and this solution is not constant.

9) T F If the unit tangent vector $\vec{T}(t)$ of a curve $\vec{r}(t)$ is always parallel to a plane Σ, then the curve is contained in a plane parallel to Σ.

Solution:
Indeed, we never leave the plane which goes through the initial point $r(0)$ because also $r'(t)$ is always parallel to Σ and after integration, the curve $r(t)$ has to be in the plane.

10) T F If (x_0, y_0) is an extremum of $f(x,y)$ under the constraint $x^2 + y^2 = 1$, then the same point is an extremum of $10f(x,y)$ under the same constraint.

Solution:
The point is a solution to the same Lagrange equations.

11) T F At a critical point (x_0, y_0) of a function $f(x,y)$ for which $f_{xx}(x_0, y_0) > 0$, the critical point is always a minimum.

Solution:
No, we also need $D > 0$.

12) T F If a vector field $\vec{F}(x,y)$ is a gradient field, and C is a closed curve which looks like a figure 8, then $\int_C \vec{F} \cdot \vec{dr}$ is zero.

Solution:
This follows from the fundamental theorem of line integrals.

13) T F If C is part of a level curve of a function $f(x,y)$ and $\vec{F} = \langle f_x, f_y \rangle$ is the gradient field of f, then $\int_C \vec{F} \cdot \vec{dr} = 0$.

Solution:
The gradient field is perpendicular to the level curves.

14) T F The divergence of the gradient vector field $\vec{F}(x,y,z) = \nabla f(x,y,z)$ is always the zero function.

Solution:
The divergence of the gradient of f is the Laplacian of f

15) T F The line integral of the vector field $\vec{F}(x,y,z) = \langle x, y, z \rangle$ along a line segment from $(0,0,0)$ to $(1,1,1)$ is $3/2$.

Solution:
By the fundamental theorem of line integrals, we can take the difference of the potential $f(x,y,z) = x^2/2 + y^2/2 + z^2/2$, which is $1/2 + 1/2 + 1/2 = 3/2$.

16) T F The area of a region G can be expressed as a line integral along its boundary.

Solution:
This is a consequence of Green's theorem and we have seen a few examples.

17) T F The flux of the vector field $\vec{F}(x,y,z) = \langle x, y, -z \rangle$ through the boundary S of a solid ellipsoid E is equal to the volume the ellipsoid.

Solution:
Indeed the divergence of the field is 1 and we can apply the divergence theorem.

18) T F If $\vec{F}$ is a vector field in space and S is a torus surface, then the flux of $\text{curl}(\vec{F})$ through S is 0.

Solution:
This is true by Stokes theorem.

19) | T | **F** |

If the divergence and the curl of a vector field $\vec{F}$ are both zero, then it is a constant field.

Solution:
Take $\vec{F}(x,y,z) = \langle x, -y, 0\rangle$. It has zero curl and zero divergence but is not constant.

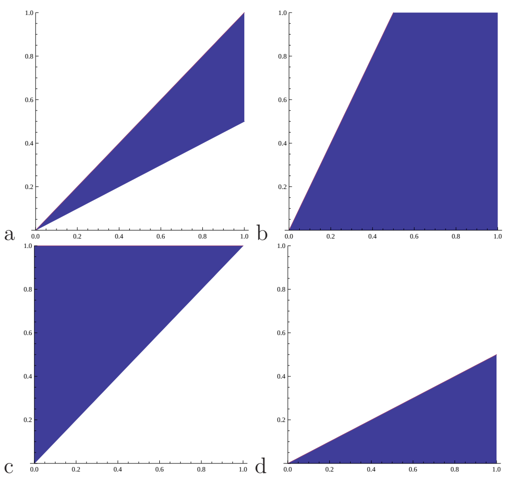
20) | **T** | F |

For any function f, the curl of $\vec{F} = \text{grad}(f)$ is the zero field $\langle 0,0,0\rangle$.

Solution:
$\text{curl}(\text{grad}(f) = \langle 0,0,0\rangle$ is an important identity.

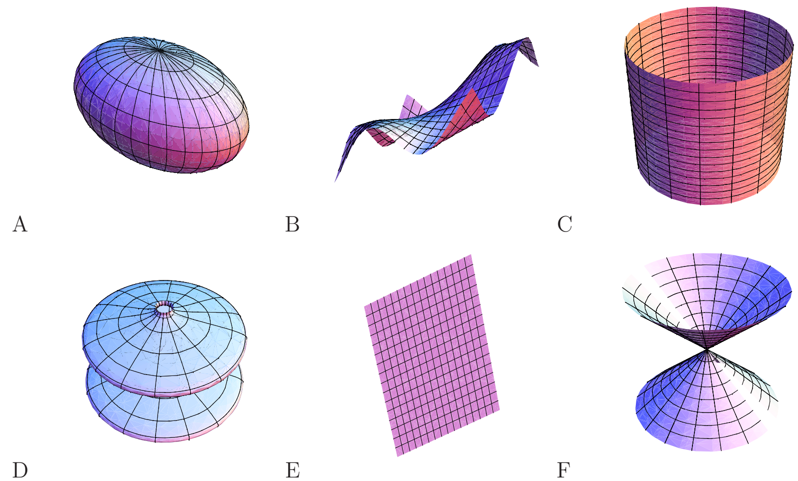
Problem 2) (10 points)

a) (4 points) Match the regions with the corresponding double integrals



Enter a,b,c,d	Function
	$\int_0^1 \int_{x/2}^{x} f(x,y)\, dydx$
	$\int_0^1 \int_0^{y} f(x,y)\, dxdy$
	$\int_0^1 \int_0^{x/2} f(x,y)\, dydx$
	$\int_0^1 \int_{y/2}^{1} f(x,y)\, dxdy$

b) (6 points) Match the parametrized or implicit surfaces with their definitions

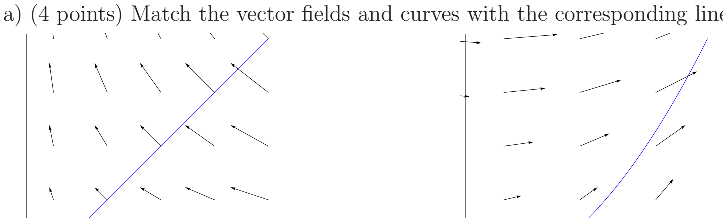


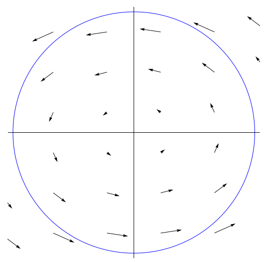
Enter A-F here	Function or parametrization
	$\vec{r}(u,v) = \langle \cos(u), \sin(u), v\rangle$
	$\vec{r}(u,v) = \langle u-v, u+2v, 2u+3v\rangle$
	$x^2 + y^2/3 + z^2/3 = 1$
	$\vec{r}(u,v) = \langle (\sin(v)+1)\cos(u), (\sin(v)+1)\sin(u), v\rangle$
	$z - x + \sin(xy) = 0$
	$x^2 + y^2 - z^2 = 0$

Solution:
a) a c d b
b) C E A D B F

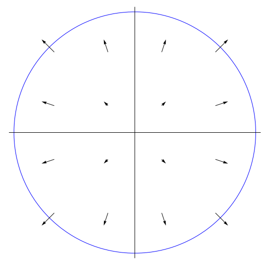
Problem 3) (10 points)

a) (4 points) Match the vector fields and curves with the corresponding line integral





III



IV

Enter I,II,III,IV	Line integral
	$\int_0^{2\pi} \langle \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \; dt$
	$\int_0^{2\pi} \langle -t, t^2 \rangle \cdot \langle 1, 1 \rangle \; dt$
	$\int_0^{2\pi} \langle t^2, t \rangle \cdot \langle 1, 2t \rangle \; dt$
	$\int_0^{2\pi} \langle -3\sin(t), 3\cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \; dt$

b) (6 points) Fill in from following choice: "arc length", "surface area", "chain rule", "volume of parallelepiped", "area of parallelogram", "line integral", "flux integral", "curvature".

Formula	Name of formula or rule or theorem
$\int \int_R \vert\vec{r}_u \times \vec{r}_v\vert \; dudv$	
$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$	
$\int_a^b \vert\vec{r}\,'(t)\vert \; dt$	
$\frac{\vert\vec{r}\,'(t) \times \vec{r}\,''(t)\vert}{\vert\vec{r}\,'(t)\vert^3}$	
$\vert\vec{u} \cdot (\vec{v} \times \vec{w})\vert$	
$\int_0^1 \int_0^1 \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \; dudv$	

Solution:
a) IV, I, II, III
b) surface area, chain rule, arc length, curvature, volume of parallelepiped, flux integral.

Problem 4) (10 points)

of the plane which contains the line and the point.

Solution:
From the symmetric equation, we see that the line contains the vector $\langle 1, 1, 1 \rangle$. The line also contains the point $(1, 2, 3)$ so that the vector $\langle 7, 2, 2 \rangle$ is also in the plane. The cross product $\vec{n} = \langle 1, 1, 1 \rangle \times \langle 7, 2, 2 \rangle$ is $\langle 0, 5, -5 \rangle$ so that the equation of the plane is $5y - 5z = d$. The constant d can be obtained by plugging in a point in the plane, for example $(1, 2, 3)$ so that $d = -5$. The equation of the plane is $5y - 5z = -5$ which can also be written as $\boxed{z - y = 1}$.

Problem 5) (10 points)

Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

Solution:
The critical points are $P = (-2, 0)$ and $Q = (-2, 2)$. The discriminant D at P is $D = -12$ so that P is a saddle point. The Hessian at Q is 12 and $f_{xx} = 2$ which is a local minimum.

$P = (-2, 0)$	D=-12	irrelevant	saddle point
$Q = (-2, 2)$	D=12	$f_{xx} = 2$	local minimum

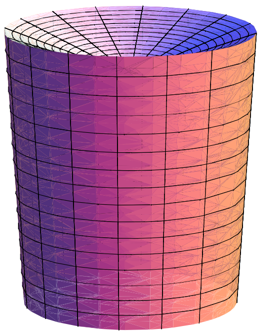
Problem 6) (10 points)

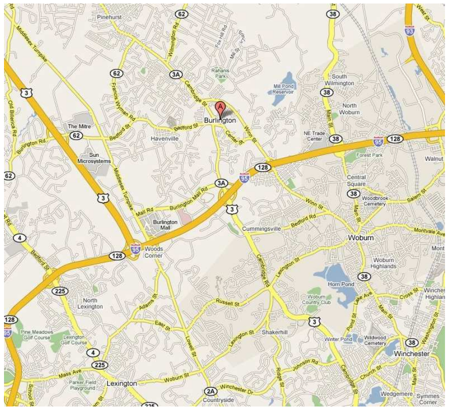
The hyperbolic paraboloid $x^2 - y^2 - 3z = 0$ contains the point $P = (1, 1, 0)$ and the point $Q = (3, 0, 3)$. Find the tangent planes to the surface at P and Q and find a parametrization $\vec{r}(t)$ of the line of intersection of these two planes.

Solution:
The gradient of $g(x,y,z) = x^2 - y^2 - z = 0$ is $(2x, -2y, -3)$ which is $(2,-2,-3)$ at the first point and $(6,0,-3)$ at the second point. The first tangent plane equation is $\boxed{4x - 2y - 3z = 3}$. The second tangent plane equation is $\boxed{6x - 3z = 9}$. To get the line of intersection, we find a point in the intersection, like $(3,0,3)$ and the cross product $\langle 4,-2,-3\rangle \times \langle 6,0,-3\rangle = \langle 6,-6,12\rangle$ of the normal vectors to get a direction in the line. $\boxed{\vec{r}(t) = (3,0,3) + t\langle 1,-1,2\rangle}$. Note that there are many solutions since the initial position on the line and the length of the vector in the line can change.

Problem 7) (10 points)

A water reservoir in Burlington, MA (the map to the right is centered there) is bounded by a solid cylinder $x^2 + y^2 \leq 1$. It has as the roof the cone $x^2 + y^2 = (z-6)^2$ and is bounded from below by the xy-plane $z = 0$. What is the volume of the reservoir?





Solution:
If R denotes the unit disc, we have the triple integral

$$\int\int_R \int_0^{6+\sqrt{x^2+y^2}} 1 \; dzdydx$$

which of course is integrated best in cylindrical coordinates:

$$\int^1 \int^{2\pi} \int^{6-r} r \; dz \; d\theta \; dr$$

Problem 8) (10 points)

Find the maxima and minima of the function $f(x,y) = x^2 - y^2$ on the parabola $x + y^2 = 1$ using the Lagrange multiplier method.

Solution:
The Lagrange equations are

$$\begin{aligned} 2x &= \lambda \\ -2y &= 2\lambda y \\ x + y^2 &= 1 \end{aligned}$$

The second equation can be solved by $y = 0$, a case which was often forgotten. In that case the third equation gives $x = 1$.
If y is not zero, we can divide the second equation by y. Dividing now the second by the first equation gives $-1/x = 2$ so that $x = -1/2$. Plugging into the third gives $y = \pm\sqrt{3/2}$. The solutions are $(1,0), (-1/2, -\sqrt{3/2}), (-1/2, \sqrt{3/2})$. By plugging in f values, we see that the first is a maximum, the other two points are minima. As usual with Lagrange methods we can not use the second derivative test to check for maxima and minima. In summary: $\boxed{(1,0) \text{ is a maximum, and } (-1/2, \sqrt{3/2}) \text{ as well as } (-1/2, -\sqrt{3/2}) \text{ are minima.}}$

Problem 9) (10 points)

Compute the surface area of the surface $\vec{r}(u,v) = \langle u^3, v^3, u^3 - v^3\rangle$ parametrized so that (u,v) is in the unit disc.

Solution:
$\vec{r}_u \times \vec{r}_v = \langle -9u^2v^2, 9u^2v^2, 9u^2v^2 \rangle$ with length $\boxed{|\vec{r}_u \times \vec{r}_v| = \sqrt{3}9u^2v^2}$. Integrating this over the unit disc R leads to a double integral problem

$$9\sqrt{3} \int \int_R u^2v^2 \; dudv \ .$$

which is best solved in polar coordinates $u = r\cos(\theta), v = r\sin(\theta)$. Using $2^2\sin(\theta)^2\cos(\theta)^2 = \sin(2\theta)^2$ we get

$$9\sqrt{3} \int_0^{2\pi} \int_0^1 r^5 (1/4) \sin^2(2\theta) \; d\theta dr$$

Using $\int_0^{2\pi} \sin^2(2\theta) \; d\theta = \pi$ we end up with $\boxed{\sqrt{3}3\pi/8}$.

Problem 10) (10 points)

Evaluate the following double integral

$$\int_0^2 \int_{x/2}^1 \cos(y^2) \; dy \; dx \ .$$

Solution:
The integral can not be evaluated as given. Change the order of integration

$$\int_0^1 \int_0^{2y} \cos(y^2) \; dxdy$$

This can now be solved. After evaluating the trivial inner integral we get

$$\int_0^1 2y\cos(y^2) \; dy = \sin(y^2)|_0^1 = \sin(1) \ .$$

The final answer is $\boxed{\sin(1)}$.

Problem 11) (10 points)

Find the value of the line integral

where $\vec{F}(x,y) = \langle y + \sin(\cos(x)), -2x \rangle$ and C is the boundary of the unit circle traversed in the counter clockwise direction.

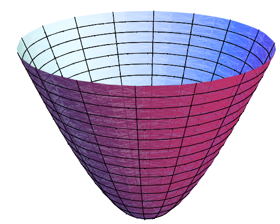
Solution:
By Greens theorem the line integral is the double integral of curl($\vec{F} = Q_x - P_y = -3$ over the unit disc, which is -3 times the area π of the disc. The final result is $\boxed{-6\pi}$.

Problem 12) (10 points)

Find the value of the flux integral

$$\int \int_S \text{curl}(\vec{F})(\vec{r}(u,v)) \; \cdot \vec{r}_u \times \vec{r}_v \; dudv$$

where $\vec{F}(x,y,z) = \langle -y, x, z \rangle$ and S is the part of the two-sheeted hyperboloid $x^2+y^2-z^2 = -1$ which satisfies $1 < z < 2$ and which is oriented so that the normal vector points downwards on S.

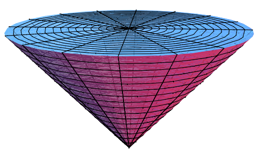


Solution:
By Stokes theorem, we can compute the line integral of $\vec{F}$ along the boundary of the surface S instead. This boundary C is parametrized by $\vec{r}(t) = \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t), 2 \rangle$. The line integral $\int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$ which is $\boxed{6\pi}$.

Let E be the solid which is bounded on the side by the cone $S_1 : x^2 + y^2 = z^2, 0 < z < 1$ and on top by the disc $S_2 = x^2 + y^2 \leq 1, z = 1$. Let $\vec{F}(x, y, z) = \langle 1 + 4x, 2 - 5y, 3 + 2z \rangle$. Find the value of the flux integral

$$\int_S \vec{F}(\vec{r}(u, v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv \; ,$$

where S is the union of the two surfaces S_1 and S_2. The normal vector of S is oriented outwards on $S_1 \cup S_2$.



Solution:
By the divergence theorem, the flux is $\int \int \int_E 1 \; dxdydz$, where E is the cone. The divergence of $\vec{F}$ is constant 1 so that we have to integrate 1 over E which is the volume of the cone. This is best computed in cylindrical coordinates: $\int_0^{2\pi} \int_0^1 \int_0^z r \; drdzd\theta =$ which is $\boxed{\pi/3}$.