7/3/2008 IMPLICIT AND PARAMETRIC SURFACES **O.Knill,Maths21a**

This is part 3 (of 3) of the homework which is due July 8 at the beginning of class.

SUMMARY:

- $g(x,y,z) = 0$ **implicit surface**.
- $x = r\cos(\theta)$
  $y = r\sin(\theta)$
  $z = z$
  cylindrical coordinates
- $x = \rho\sin(\phi)\cos(\theta)$
  $y = \rho\sin(\phi)\sin(\theta)$
  $z = \rho\cos(\phi)$
  spherical coordinates
- $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ defines a **parametric surface**.
- Switching from implicit to parametric is not trivial in general. For spheres, planes, surfaces of revolution and graphs we can do it.

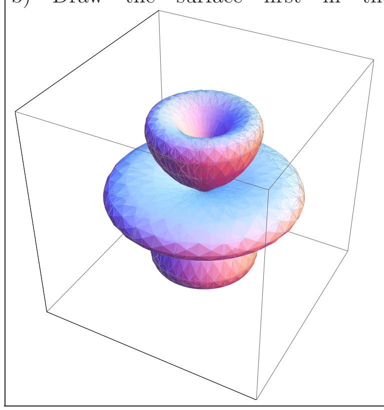
EXAMPLES:

- $x^2 + y^2 + z^2 = \rho^2$ sphere
- $r = 1$, **cylinder**.
- $\rho = 1$, **sphere**.
- $r = z$, **cone**
- $\vec{r}(u,v) = P + u\vec{u} + v\vec{v}$ **plane**
- $\vec{r}(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ **sphere**
- $\vec{r}(u,v) = (\cos(u), \sin(u), v)$ **cylinder**
- $\vec{r}(u,v) = (u, v, f(u,v))$ **graph** of f

Homework Problems

1) (4 points)
a) What is the equation for the surface $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = \sin(3\phi)$ in spherical coordinates (ρ, θ, ϕ).

Solution:
a) $r^2 - 5r\cos(\theta) = z^2$.
b) Draw the surface first in the rz plane. Here you see the picture.



2) (4 points) Plot the surface with the parametrization $\vec{r}(u,v) = (v^2\cos(u), v^2\sin(u), v)$, where $u \in [0, 2\pi]$ and $v \in \mathbf{R}$.

Solution:
It is a surface of revolution, very thin at the origin. The shape is a parabola but it is bent the other way round as in the paraboloid.

3) (4 points) Find a parametrization for the plane which contains the points $P = (3,7,1)$,$Q = (1,2,1)$ and $R = (0,3,4)$.

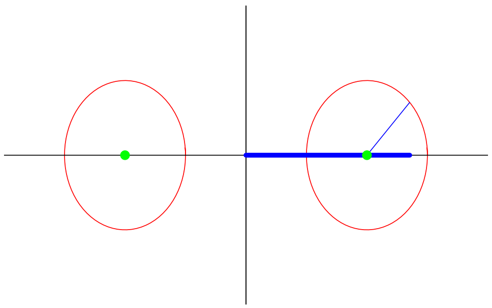
Solution:
Take $r(t) = P + s(Q-P) + t(R-P)$. $\vec{r}(s,t) = (3 - 2s - 3t, 7 - 5s - t, 1 + -3s + 3t)$.

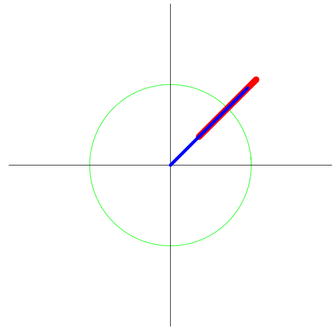
4) (4 points) Find two different parametrisations of the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1$. For one of the parametrizations assume that the surface is a graph. For the other, use angles similarly as for for the sphere.

Solution:
Here are three possible parametrizations:
1) $\vec{r}(u,v) = (u, v, -\sqrt{1 - 2u^2 - 4v^2}$.
2) $\vec{r}(\theta,\phi) = (\sin(\phi)\cos(\theta)/\sqrt{2}, \sin(\phi)\sin(\theta)/2, \cos(\phi))$.

5) (4 points) Find a parametrisation of the **torus** which is obtained as the set of points which have distance 1 from the circle $(2\cos(\theta), 2\sin(\theta), 0)$, where θ is the angle occuring in cylindrical and spherical coordinates.

Hint: Keep $u = t$ as one of the parameters and let r the distance of a point on the torus to the z-axis. This distance is $r = 2 + \cos(\phi)$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = \sin(\phi)$. To finish the parametrization problem, you have to translate back from cylindrical coordinates $(r,\theta,z) = (2+\cos(\phi), \theta, \sin(\phi))$ to Cartesian coordinates (x,y,z). Write down your result in the form $\vec{r}(\theta,\phi) = (x(\theta,\phi), y(\theta,\phi), z(\theta,\phi))$.





Solution:
$\vec{r}(\theta,\phi) = ((2+\cos(\phi))\cos(\theta), (2+\cos(\phi))\sin(\theta), \sin(\phi))$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

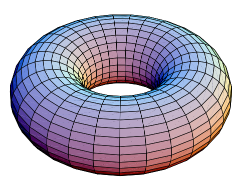
1) Try to graph without a computer the surface $r = f(\theta,\phi) = (2+\sin(3\theta))(2+\cos(2\phi))$ (It is a **graph** in spherical coordinates (r,ϕ,θ).)

Hint. Do it in stages. First graph $r = 2$ (the sphere), then $r = (2+\sin(3\theta))$, then draw a sketch of the final surface.

2) How would you design analogues of spherical or cylindrical coordinates in 4 dimensions?

3) Sketch the surface $r(u,v) = (2+2v\cos(\pi u))\cos(2\pi u), (2+2v\cos(\pi u))\sin(2\pi u), v\sin(\pi u))$.

4) The **torus** is obtained by bending and gluing the ends of a cylinder together.



The **Klein bottle** is obtained in the same way, however, the ends are put together with opposite directions. This can not be achieved without self-intersection. Take one end of the tube, bend it, enter the tube first to match the ends in the opposite direction as for the torus. Can you find a parametrisation $r(u,v)$ for this surface? On the handout for this lecture, you find a parametrization of the same surface but which looks different. The idea is to have a parametrization which produces the "bottle".