

This is part 1 (of 2) of the weekly homework. It is due Tuesday August 6. This is an exception. In the last week, homework is due daily.

SUMMARY.

- $F(x, y) = (P(x, y), Q(x, y))$ **vector field in the plane.**
- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$ **line integral** of F along curve $C : t \mapsto r(t)$.
- Example: $C : r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ (circle), $F(x, y) = (-y, x)$. $\int_C F \cdot dr = \int_0^{2\pi} (r(t)) \cdot r'(t) dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt = \int_0^{2\pi} 1 dt = 2\pi$.
- $\text{curl}(P, Q) = Q_x - P_y$. **Curl** for 2D vector fields.
- C **positively oriented boundary** of the region D : (the region is "to the left" when you follow the boundary).
- $\int_C F \cdot dr = \int_D \text{curl}(F) dA$ **Greens theorem.**
Written out: $\int_C F(r(t)) \cdot r'(t) dt = \int_D (Q_x - P_y) dx dy$.
- $\int_C x dy$ **area** of D , C is the positively oriented boundary of D :
Example: C unit circle: $x(t) = \cos(t), dy = \cos(t) dt$, Area = $\int_0^{2\pi} \cos^2(t) dt = \pi$.
- $\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ the **curl** of $\vec{F} = \langle P, Q, R \rangle$.
- The curl of a gradient field vanishes: $\text{curl}(\nabla f) = (0, 0, 0)$.
- $(u, v) \mapsto r(u, v) = (x(u, v), y(u, v), z(u, v))$ **surface.**
- $\int \int_S F \cdot dS = \int \int_R F(r(u, v)) \cdot (r_u \times r_v) dA$ **flux integral.**
- $\int \int_S \text{curl}(F) \cdot dS = \int_C F \cdot dr$ **Stokes theorem**, C : boundary of S , oriented so that "surface is to your left" if your head points in the normal direction.

Homework Problems

- 1) (4 points) Calculate the line integral $\int_C 2(y + x \sin(y), x^2 \cos(y) - 3y^2) dr$ along a triangle C with edges $(0, 0)$, $(1, 0)$ and $(1, 1)$ using Green's theorem.

Solution:

$\text{curl}(F)(x, y) = 4x \cos(y) - 2 - 2x \cos(y) = 2x \cos(y) - 2$. By Green's theorem, we have to integrate this function over the region R enclosed by the triangle:

$$\int_0^1 \int_0^x 2x \cos(y) - 2 dy dx = \int_0^1 (2x \sin(x) - 2x) dx = -1 - 2 \cos(1) + 2 \sin(1).$$

- 2) (4 points) Evaluate the line integral of the vector field $F(x, y) = (xy^2, x^2)$ along the rectangle with vertices $(0, 0)$, $(2, 0)$, $(2, 3)$, $(0, 3)$ in two ways. Do this by calculating the line integral as

well as using Greens theorem.

Solution:

Integrating $\text{curl}(F) = 2x - 2xy$ over the rectangle gives

$$\int_0^2 \int_0^3 2x - 2xy dy dx = \int_0^2 6x - 9x dx = -12/2 = -6$$

The line integral is a sum of four line integrals: $\int_0^2 (t0^2, t^2) \cdot (1, 0) dt = 0$ and $\int_0^3 (2t^2, 4) \cdot (0, 1) dt = 12$ and $-\int_0^2 (9t, t^2) \cdot (1, 0) dt = -18$ as well as $\int_0^3 (0t^2, 0) \cdot (0, 1) dt = 0$. The sum is also -6 .

- 3) (4 points) Find the area of the region bounded by the hypocycloid $\vec{r}(t) = (\cos^3(t), \sin^3(t))$ using Green's theorem. The curve is parameterized by $t \in [0, 2\pi]$.

Solution:

Take a vector field $F(x, y) = (0, x)$ which has the curl 1. Then by Green the area is the line integral $\int_0^{2\pi} (0, \cos^3(t)) \cdot (-3 \cos^2(t) \sin(t), 3 \sin^2(t) \cos(t)) dt = 3 \int_0^{2\pi} \cos^4(t) \sin^2(t) dt = 3 \int_0^{2\pi} \sin^2(2t)/4(\cos(2t) + 1)/2 = 3/8\pi$.

- 4) (4 points) Evaluate the flux integral $\int \int_S \text{curl}(F) \cdot dS$ for $F(x, y, z) = (xy, yz, zx)$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[1, 0] \times [0, 1]$ and has an upward orientation. Evaluate the same flux integral as in the previous question also using Stokes theorem.

Hint. The boundary C can be built up using 4 curves:

$$C_1 : \vec{r}(t) = (t, 0, 4 - t^2), \vec{r}'(t) = (1, 0, -2t).$$

$$C_2 : \vec{r}(t) = (1, t, 3 - t^2), \vec{r}'(t) = (0, 1, -2t).$$

$$C_3 : \vec{r}(t) = (t, 1, 4 - t^2 - 1), \vec{r}'(t) = (1, 0, -2t).$$

$$C_4 : \vec{r}(t) = (0, t, 4 - t^2), \vec{r}'(t) = (0, 1, -2t).$$

Watch the orientation of each of these curves, when doing each line integral.

Solution:

$\text{curl}(F) = (-y, -z, -x)$. The parametrization $\vec{r}(u, v) = (u, v, 4 - u^2 - v^2)$ gives $r_u \times r_v = (2u, 2v, 1)$ and $\text{curl}(F)(r(u, v)) = (-v, u^2 + v^2 - 4, -u)$. The flux integral is $\int_0^1 \int_0^1 (-2uv + 2v(u^2 + v^2 - 4) - u) dv du = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$.

The boundary C consists of the 4 curves given in the hint. The line integrals are:

$$I : \int_0^1 (0, 0, 4t - t^3) \cdot (1, 0, -2t) dt = -8/3 + 2/5 = -34/15.$$

$$II : \int_0^1 (t, 3t - t^3, 3 - t^2) \cdot (0, 1, -2t) dt = 1/4 - 3/2 = -5/4.$$

$$III : \int_0^1 (t, 3 - t^2, 3t - t^3) \cdot (1, 0, -2t) dt = -11/10.$$

$$IV : \int_0^1 (0, t(4 - t^2), 0) \cdot (0, 1, -2t) dt = 7/4$$

$I + II - III - IV = -34/15 - 5/4 + 11/10 - 7/4 = -25/6$. (The line integrals III and IV along C_3, C_4 were taken negative because the curves are traced backwards.)

- 5) (4 points) If S is the surface $x^6 + y^6 + z^6 = 1$ and assume F is a smooth vector field. Show that $\int \int_S \text{curl}(F) \cdot dS = 0$.

Solution:

The flux of $\text{curl}(F)$ through a closed surface is zero by Stokes theorem and the fact that the surface does not have a boundary.

One can see this also by cutting the surface in two pieces and apply Stokes to both pieces.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

- 1) The planimeter calculates the area: the **planimeter vector field** $F(x, y) = (P(x, y), Q(x, y))$ is defined by attaching a unit vector orthogonal to the vector $(x-a, y-b)$ at (x, y) , where (a, b) is the "knee" of the planimeter. The wheel rotation is the line integral of F along the boundary of R . By **Green's theorem**, this integral is the double integral of $\text{curl}(F)$ over R . The planimeter vector field is explicitly given by $F(x, y) = (P(x, y), Q(x, y)) = (-(y - b(x, y)), (x - a(x, y)))$. Furthermore, $\text{curl}(F) = Q_x - P_y$ is equal to $2 + (-a_x - b_y)$ which is 2 plus the curl of the vector field $G(x, y) = (b(x, y), -a(x, y))$. Show that $\text{curl}(G) = -1$. For more information see <http://www.math.duke.edu/education/ccp/materials/mvcalc/green/>
- 2) Let D be a region bounded by a simple closed path C in the plane. Use Green's theorem to prove that the coordinates of the center of mass are $(\int_C x^2 dy / (2A), -\int_C y^2 dx / (2A))$, where A is the area of D .