7/24/2008 SECOND HOURLY PRACTICE II Maths 21a, O. Knill, Summer 2008

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) | T | F | The function $f(t, x, y) = (y + t)\cos(x - t)$ satisfies the partial differential equation $f_{tt} = f_{xx} + f_{yy}$ which is called the two dimensional wave equation.

Solution:
Indeed this is a solution. The partial differential equation is called two-dimensional wave equation.

2) | T | F | The velocity of the curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at time $t = 0$ is 1.

Solution:
The velocity is a vector.

3) | T | F | There exists a function $f(x, y)$ of two variables which every point (n, m) with integer n and integer m is a critical point.

Solution:
For example, $f(x, y) = \cos(\pi x)\cos(\pi y)$.

4) | T | F | If $f_x(x, y) = f_y(x, y) = f_{xx}(x, y) = f_{yy}(x, y) = f_{xy}(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

Solution:
False, f could be constant.

5) | T | F | $(0, 0)$ is a local minimum of the function $f(x, y) = x^6 + y^6$.

Solution:
$(0, 0)$ is a local minimum because the value there is 0 and the function is positive.

6) | T | F | If $f(x, y)$ has a local max at the point $(0, 0)$ with discriminant $D > 0$, then $g(x, y) = f(x, y) - x^9 + y^9$ has a local max at $(0, 0)$ too.

Solution:
Adding $x^9 - y^9$ does not change the first and second derivatives.

7)	T	F	The value of the function $f(x,y) = \sqrt{1+3x+5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:
Use formula for $L(x,y)$.

8)	T	F	The curve $\vec{r}(t) = (x(t), y(t), z(t)) = (\sin(t)t^5, \sin(t)t^5, \sin(t)t^5)$ is a line in space.

Solution:
True.

9)	T	F	The chain rule can be written in the form $\frac{d}{dt} f(\vec{r}(t)) = D_{\vec{\mathrm{r}}\,'(\mathrm{t})} f(\vec{r}(t))$

Solution:
By definition $D_v f = \nabla f \cdot f$. The chain rule is $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$.

10)	T	F	The curvature of the curve $\vec{r}(t) = (\cos(5t), \sin(5t)$ for $t = 0$ is 5 times the curvature of the curve $r(t) = (\cos(t), \sin(t))$ for $t = 0$.

Solution:
Both curves lead to the same circle.

11)	T	F	The gradient of f at a point (x_0, y_0, z_0) is tangent to the level surface of f which contains (x_0, y_0, z_0).

Solution:
It is a basic and important fact that ∇f is **perpendicular** to the level surface.

12)	T	F	If the Lagrange multiplier λ is positive, then the critical point under constraint is a local minimum.

Solution:
The sign of λ has nothing to say about the nature of the critical point.

13)	T	F	If the directional derivative $D_{\vec{v}}f(1,1) = 0$ for all vectors $\vec{v}$, then $(1,1)$ is a critical point of $f(x,y)$.

Solution:
Especially, $D_{\nabla f}(f) = |\nabla f|^2 = 0$ so that $\nabla f = (0,0,0)$.

14)	T	F	The arc length of the curve $r(t) = \langle t, \cos(t) \rangle$ from $t = 0$ to $t = 2\pi$ is the integral $\int_0^{2\pi} \sqrt{t^2 + \cos^2(t)}\, dt$.

Solution:
The correct integral is $\int_0^{2\pi} \sqrt{1 + \sin^2(t)}\, dt$

15)	T	F	For any curve $\vec{r}(t)$, the vectors $\vec{r}\,''(t)$ and $\vec{r}\,'(t)$ are always perpendicular to each other.

Solution:
This is most of the time wrong.

16)	T	F	Every critical point (x,y) of a function $f(x,y)$ for which the discriminant D is not zero is either a local maximum or a a local minimum.

Solution:
The second derivative test give for negative D that we have a saddle point.

17)	T	F	The function $f(x,y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:
By Clairot's theorem, we can have all three x derivatives at the beginning.

18)	T	F	If $(0,0)$ is a critical point of $f(x,y)$ and the discriminant D is zero but $f_{xx}(0,0) < 0$ then $(0,0)$ can not be a local minimum.

Solution:
If $f_{xx}(0,0) < 0$ then on the x-axis the function $g(x) = f(x,0)$ has a local maximum. This means that there are points close to $(0,0)$ where the value of f is larger.

19) T F In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

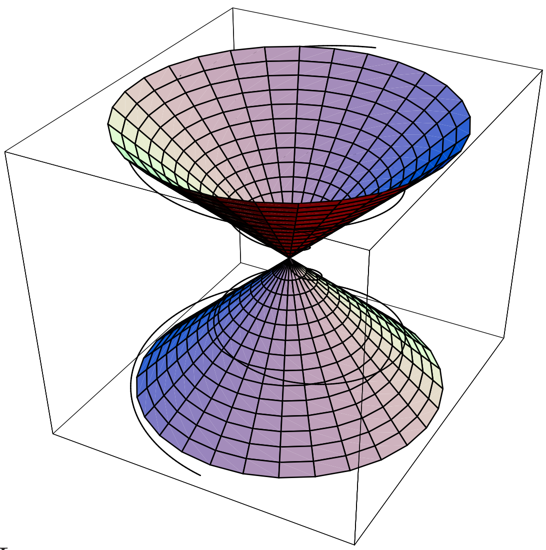
Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

20) T F The arc-length of $\vec{r}(t) = (0,0,1) + t(0,3,4)$ with $t \in [0,2]$ is 10.

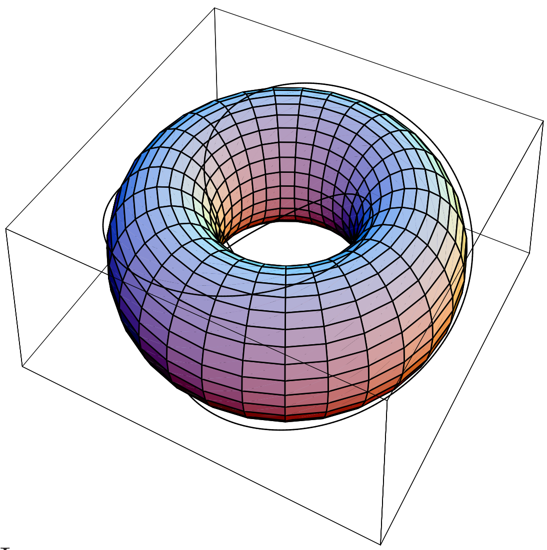
Solution:
The speed is 5. The path has length $\int_0^2 5\,dt = 10$.

Problem 2) (10 points)

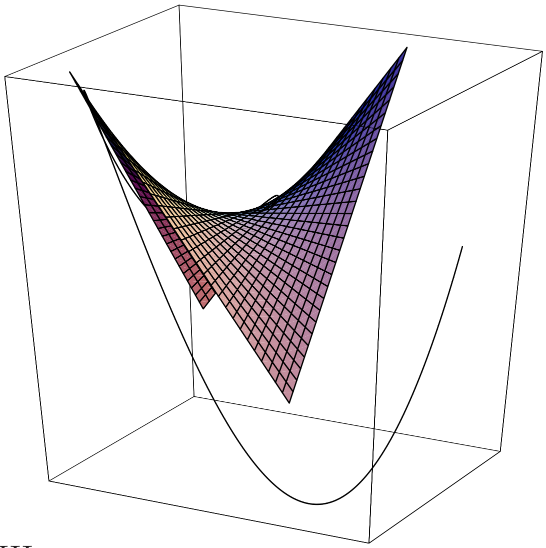
Match the parameterizations with the curves. In this problem, the curves are located on parameterized surfaces with two parmeters, but of course, the curves themselves have only one parameter. No justifications are needed.



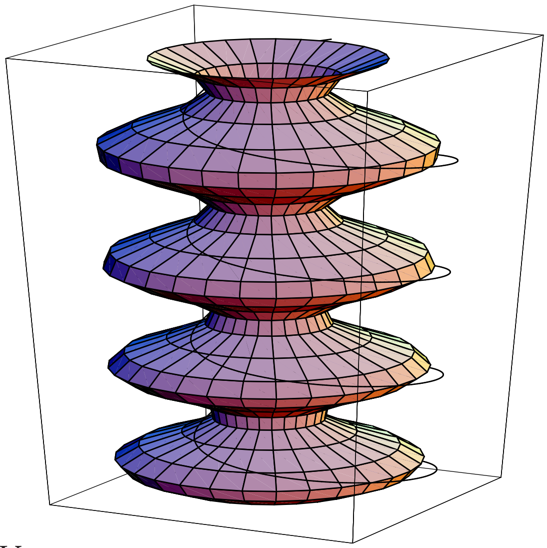
I



II



III



IV

Enter I,II,III,IV here	Parameterization
	$\vec{r}(t) = (t\sin(t), t\cos(t), t)$
	$\vec{r}(t) = (2 + \cos(2t)\cos(3t), (2 + \cos(2t))\sin(3t), \sin(2t))$
	$\vec{r}(t) = (t\cos(t), t\sin(t), t^2\cos(t)\sin(t))$
	$\vec{r}(t) = ((2 + \sin(t))\sin(t), (2 + \sin(t))\cos(t), t)$

Solution:
I,II,III,IV. The equations were already in the correct order. It is one of the $4! = 24$ possibilities.

Problem 3) (10 points)

Tell from each of the 10 following objects, whether they are a vector or a scalar. No justifications are needed. Each correct answer is 1 point. All objects are defined for objects in space $\mathbf{R}^3$.

object	vector	scalar
curvature		
arc length		
velocity		
speed		
acceleration		
unit tangent vector		
jerk		
discriminant		
directional derivative		
gradient		

Solution:
Only velocity, acceleration, unit tangent vector, jerk and the gradient are vectors. The other are scalars.

Problem 4) (10 points)

Find all the critical points of

$$f(x,y) = x^3 + y^3 - 3x - 12y$$

and indicate whether they are local maxima, local minima or saddle points.

Solution:
$\nabla f(x,y) = (-3 + 3x^2, -12 + 3y^2) = (0,0)$ so that the critical points are $(0,1),(0,-1),(1,1),(1,-1)$. We have $D = 36xy$ and $f_{xx} = 6x$.

Point	D	f_{xx}	type
$(-1,-2)$	$D = 72$	-6	max
$(-1,2)$	$D = -72$	6	saddle
$(1,-2)$	$D = -72$	6	saddle
$(1,2)$	$D = 72$	6	min

Problem 5) (10 points)

When Ramanujan, an amazing mathematician who was born in India was sick in the hospital and the English mathematician Hardy visited him, Ramanujan asked "whats up?" Hardy answered. "Nothing special. Even the number of the taxi cab was boring: 1729". Ramanujan answered: "No, that is a remarkable number. It is the smallest number, which can be written in two different ways as a sum of two perfect cubes. Indeed $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

a) (5 points) Find the linearization $L(x, y, z)$ of the function $f(x, y, z) = x^3 + y^3 - z^3$ at the point $(9, 10, 12)$.

b) (5 points) Use the technique of linear approximation to estimate $9.001^3+10.02^3-12.001^3$.

Solution:
We make a linear approximation of $f(x, y, z) = x^3 + y^3 - z^3$ at the point $(9, 10, 12)$. We have $\nabla f(x, y, z) = (3x^2, 3y^2, -3z^2)$ which is $(273, 300, 432)$. a) $L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$. b) $L(9.001, 10.02, 12.001) = 1 + 273 \cdot 0.001 + 300 \cdot 0.02 - 432 \cdot 0.001$.
To the grading: since the numerics was a bit challenging without computer, we would not have taken off any points for numerical computation errors here. We are not all Ramanujans.

Problem 6) (10 points)

a) (5 points) Find the equation $ax + by + cz = d$ for the tangent plane to the level surface

$$f(x, y, z) = x^3 + y^3 - z^3 = 1$$

at the point $(1, 1, 1)$. Note that this is the same Ramanujan function as in the previous problem.

b) (5 points) If we intersect the level surface $f(x, y, z) = 1$ with the plane $z = 2$, we obtain the equation for an implicit curve $x^3 + y^3 = 9$. It is a level curve for the function $g(x, y) = x^3 + y^3$. Find the tangent line to this curve at the point $(1, 2)$.

Solution:
a) We have $\nabla f(1,1,1) = (3,3,-3)$ so that the plane is $3x + 3y - 3z = 3$.
b) We have $\nabla g(x,y) = (3x^2, 3y^2)$ and $\nabla g(1,2) = (3,12)$. The line has the form $3x+12y = 27$.

Problem 7) (10 points)

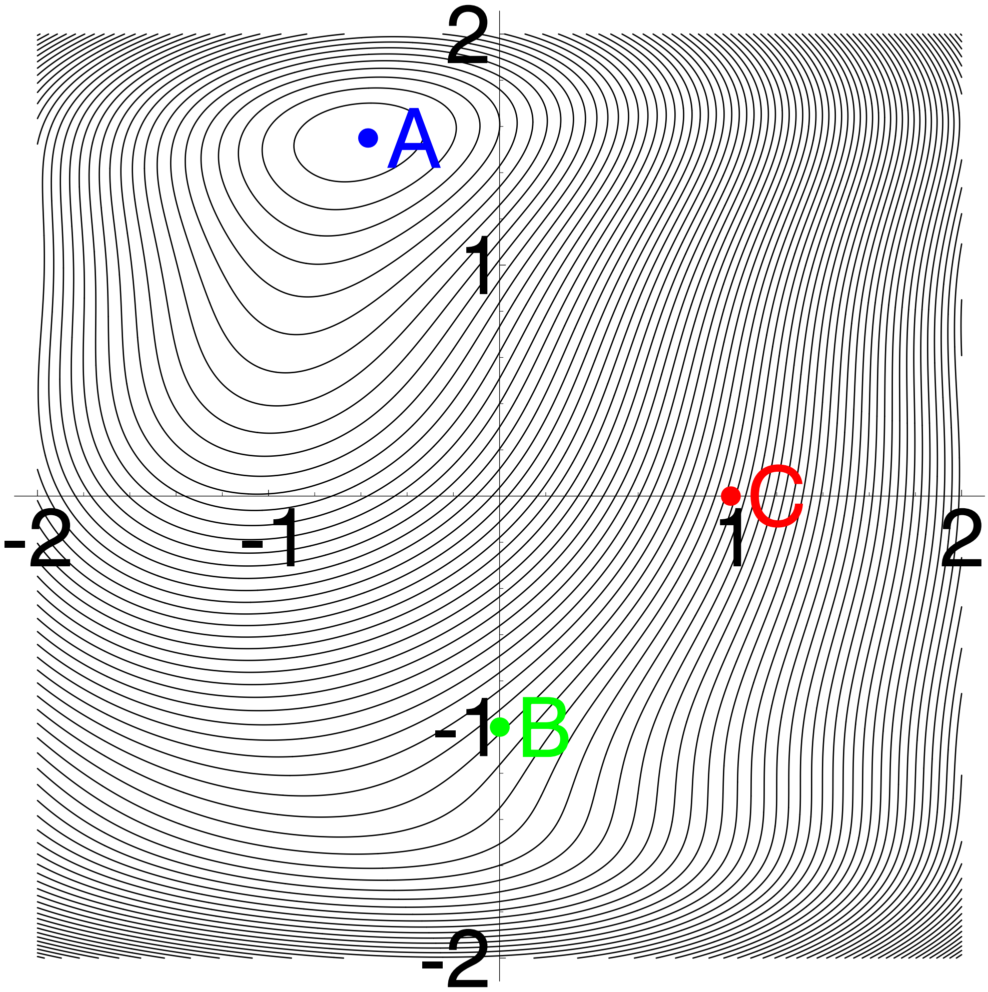
A function $f(x,y)$ of two variables describes the height of a mountain. You don't know the function but you see its level curves. The mountain has its peak at the point A on the picture.

a) (3 points) At which point does f take its maximum under the constraint $x = 0$.

b) (3 points) At which point does f take its maximum under the constraint $y = 0$.

c) (4 points) At which of the points B or C is the length of the gradient vector larger?

Note: As always, you have to give explanations to get full credit. The points in a) and b) are not necessarily marked. Give it up to an accuracy of 1/2. For example, an answer in a) or b) could look like $(0.5, 1.5)$.



Solution:
a) The point is about at $(0, 1.5)$. this is a point, where the level curve is tangent to the constrained curve $x = 0$. The fact that the point is closest to the mountain is only an accident here.
b) The point is near $(-1, 0)$. Again it is a point, where the level curve is tangent to the constrained curve $y = 0$.
c) If level curves are close to each other this means that the gradient is large and that the surface is steep at this point in the direction of the gradient. At the point C, the steepness is larger.

Problem 8) (10 points)

Olivers great-grand-dad Emil Frech Hoch (1874-1947) founded the car company "Frech-Hoch" in Switzerland. The company produces cars and trucks. The company still exists today and produces specialized vehicles.

Assume the revenue of the company is $f(x,y) = x^2 + 2y^2$, where x is the number of cars and y is the number of trucks produced per year. The production is constrained by the amount of steel available. Trucks need twice as much steel leading to $g(x,y) = x + 2y = 1$. Use the Lagrange multiplier method to find the optimal production rate.

Solution:
The Lagrange equations are

$$\begin{aligned} 2x &= \lambda \\ 4y &= 2\lambda \\ x + 2y &= 1 \end{aligned}$$

THe first two equations gives $x = y$ and lead to $x = y = 1/3$. It turns out that the only solution to this Lagrange problem is $(1/3, 1/3)$. While it is an optimum it turns out be a minimum. A more realistic assumption would have been that $x^2 + 2y^2$ is a commodity which one wants to minimize, like for example production trash. Some people suggested to consider the point $(1,0)$ which is the maximum if one also has the constraints $x \geq 0$ and $y \geq 0$.

Problem 9) (10 points)

During a bus ride over the country side, a kid throws an unfinished apple out of the window at time $t = 0$. The bus drives at that time with a speed of 40 (meters per second) in the x direction. The apple is thrown from the hand with an initial velocity of $(0,1,1)$ meters per second from a height of 4 meters. We chose a coordinate system such that the initial position of the apple is $\vec{r}(0) = (0,0,4)$. You can assume the acceleration of the earth is $(0,0,-10)$ meters per seconds^2.

a) (4 points) Find the parameterization of the path $\vec{r}(t) = (x(t), y(t), z(t))$ of the apple after the drop.

b) (3 points) At which time does it hit the earth, if one assumes the apple falls without being slowed down by the air resistance?

c) (3 points) Where does it hit the earth?

Solution:
The initial position is $r(0) = (0, 0, 1)$. The initial velocity is $(40, 1, 1)$. The acceleration is $r''(t) = (0, 0, -10)$. By integrating the acceleration and comparing $r'(0)$, we obtain

$$r'(t) = (0, 0, -10t) + (40, 1, 1)$$

Integrating again gives

$$r(t) = (0, 0, 4) + (40, 1, 1)t + (0, 0, -10t^2/2) = (40t, t, t + 4 - 5t^2)$$

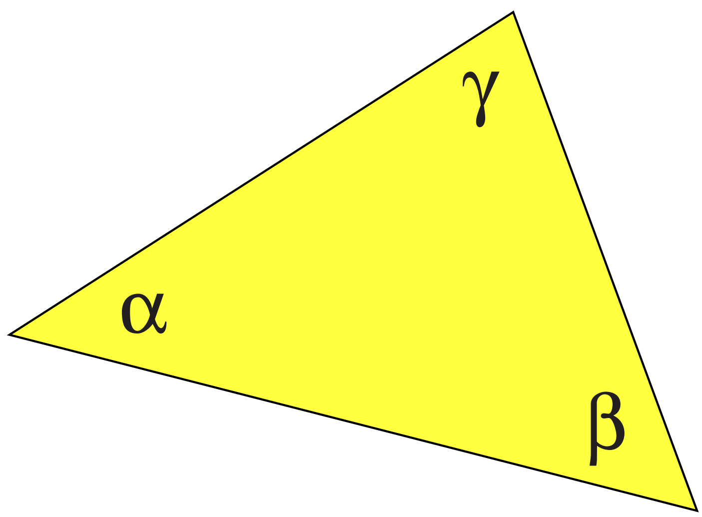
b) We see $t = 1$.
c) The apple of "little Newton" hits the floor at the point $(40, 1, 0)$.

Problem 10) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log\left(\sin(\alpha)\sin(\beta)\sin(\gamma)\right)$$

is maximal?



Solution:
The Lagrange equations are $\cot(\alpha) = \lambda, \cot(\beta) = \lambda, \cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\boxed{\alpha = \beta = \gamma = \pi/3}$ and $\sin(\alpha)\sin(\beta)\sin(\gamma) = (\sqrt{3}/2)^3$.