

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

- 1) T F The cosine of the angle between the velocity vector and the acceleration vector is always positive.

Solution:

If you drive with a car on a straight line and break, the acceleration points in the opposite direction than the velocity.

- 2) T F The function $f(t, x) = x^3 - 3xy^2$ satisfies the Laplace partial differential equation $f_{xx} + f_{yy} = 0$.

Solution:

Just differentiate

- 3) T F The velocity vector of the curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at time $t = 0$ is perpendicular to the curve at $\vec{r}(0)$.

Solution:

It is tangent to the curve.

- 4) T F If every point of the plane is a critical point for a function f then f is a constant function.

Solution:

Yes, $f_x = 0$ everywhere means that f is independent of x everywhere and $f_y = 0$ everywhere is equivalent that f is independent of y everywhere.

- 5) T F It is possible that $(1, 1)$ is a local maximum for the function f and $1 = f_{xx} = -f_{yy}$.

Solution:

If $f_{xx} = -f_{yy}$ then

- 6) T F $(0, 0)$ is a local maximum of the function $f(x, y) = 5 - x^8 - y^8$.

Solution:

$(0, 0)$ is a local maximum because the value there is 5 and the function is smaller everywhere else.

- 7) T F If $f(x, y)$ has a local max at the point $(0, 0)$ with discriminant $D > 0$, then $g(x, y) = f(x, y) - x^9 + y^9$ has a local max at $(0, 0)$ too.

Solution:

Adding $x^9 - y^9$ does not change the first and second derivatives.

- 8) T F The value of the function $f(x, y) = \log(e + 3x + 5y)$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - 0.006 + 0.05$.

Solution:

Use the formula for $L(x, y)$.

- 9) T F The curvature of the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = \langle 5, 3 \sin(t), 3 \cos(t) \rangle$ is $1/3$.

Solution:

True.

- 10) T F The chain rule tells that $\frac{d}{dt}f(\vec{r}(t)) = f'(\vec{r}(t))r'(t)$

Solution:

The gradient and dot product are missing.

- 11) T F If the curvature of a curve is zero everywhere, then it is a line.

Solution:

One can see that by the formula but also by physics.

- 12) T F The gradient of f at a point (x_0, y_0, z_0) is orthogonal to the level surface of f which contains (x_0, y_0, z_0) .

Solution:

It is a basic and important fact that ∇f is **perpendicular** to the level surface.

- 13) T F If the Lagrange multiplier λ is negative then the critical point under constraint is a saddle point.

Solution:

The sign of λ has nothing to say about the nature of the critical point.

- 14) T F An aroplane has a velocity vector which happens to coincide with the gradient vector of the pressure at that point. The rate of change of the pressure is positive.

Solution:

Use the chain rule.

- 15) T F The arc length of a curve on $[0, 1]$ can be obtained by integrating up the curvature of the curve along the interval $[0, 1]$.

Solution:

For a line the curvature is zero but the arc length is not zero.

- 16) T F If D is the discriminant at a critical point and $Df_{xx} > 0$ then we either have a saddle point or a local maximum.

Solution:

It is either a saddle point or a local minimum.

- 17) T F The function $f(x, y) = \sin(y)x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xyyxyxy} = 0$.

Solution:

By Clairot's theorem, we can have all three x derivatives at the beginning.

- 18) T F If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum.

Solution:

If $f_{xx}(0,0) > 0$ then on the x-axis the function $g(x) = f(x,0)$ has a local minimum. This means that there are points close to $(0,0)$ where the value of f is larger.

- 19) T F In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{xy} > 0$ to check whether a point is a local minimum.

Solution:

No, f_{xy} can have any sign.

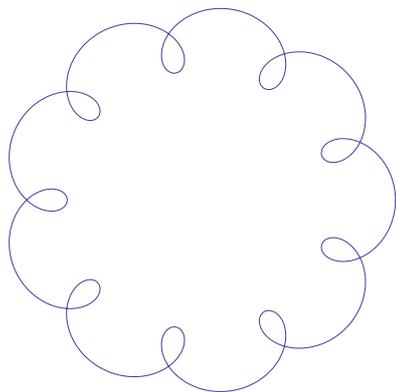
- 20) T F If the curvature of a curve $\vec{r}(t)$ is equal to 1 everywhere and the curve connects the point $(0,0,0)$ with the point $(1,0,0)$, then the arc length is larger than 1.

Solution:

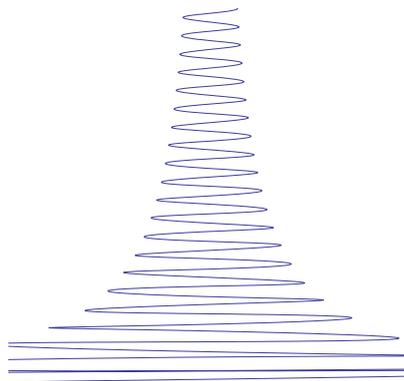
It is even larger than π .

Problem 2) (10 points)

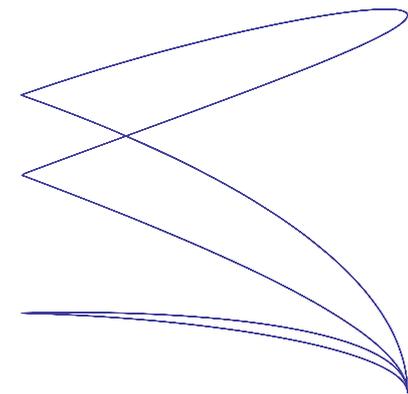
Match the parameterizations with the curves. No justifications are needed.



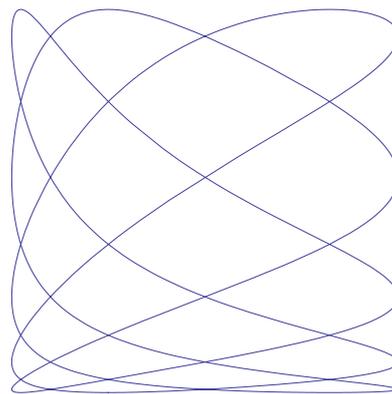
I



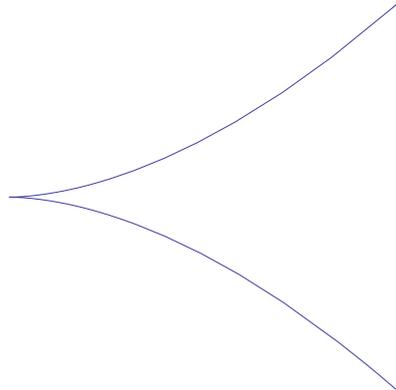
II



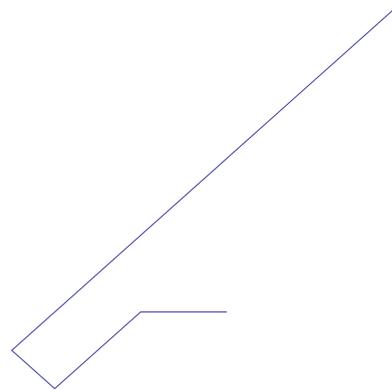
III



IV



V



VI

Enter I,II,III,IV here	Parameterization
	$\vec{r}(t) = \langle 1 - t + -3t , t - 1 - t \rangle$
	$\vec{r}(t) = \langle 1/(2 + \sin(5t)), 1/(2 + \cos(3t)) \rangle$
	$\vec{r}(t) = \langle (\cos(t) + \cos(10t))/5, (\sin(t) + \sin(10t))/5 \rangle$
	$\vec{r}(t) = \langle \cos(7t)/t, \sin(9t)/t + 5t \rangle$
	$\vec{r}(t) = \langle \cos(3t) , \sin(2t) + \sin(t) \rangle$
	$\vec{r}(t) = \langle t^4, t^7 \rangle$

Solution:
VI,IV,I,II,III,V

Problem 3) (10 points)

The following statements are not complete. Fill in from the pool of words below.

statement	Fill in the letters	statement
The arc length does		on the parametrization.
$\sqrt{48}$ can be estimated by		at $x = 7$. The result is 7-1/14.
The velocity vector is		to the curve.
The discriminant D is		if the point is a saddle point.
The unit normal vector is		to the unit tangent vector.
For a Lagrange minimum, ∇g is		to ∇f .
The curvature does		on the parametrization of the curve.
The trajectory of a ball is a		if we are in free fall.
Arc length is approximated by a		sum if the curve is smooth.
The gradient ∇f is		to the surface $f = c$.

B	negative
U	linear approximation
D	not depend
M	tangent
O	parabola
D	not depend
E	perpendicular
L	parallel
E	orthogonal
R	Rieman

Solution:

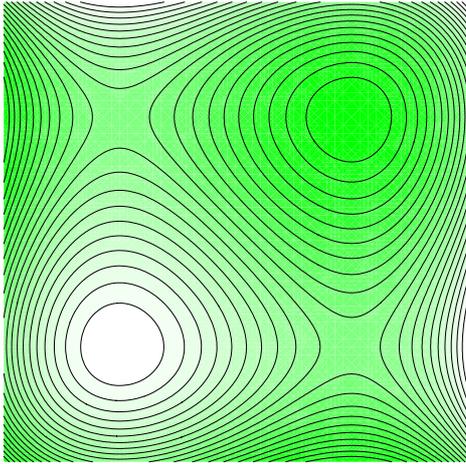
DUMBELDOR (misspelled Dumbledore). The spelling trap had not been intended but some fell in).

Problem 4) (10 points)

The green near one of the holes in the Cambridge Fresh pond golf course has the height

$$f(x, y) = x^3 + y^3 - 3x^2 - 3y^2$$

Find local maxima, local minima or saddle points of this function. Near which point will golf balls most likely end up, if balls like to roll to lower areas.



Solution:

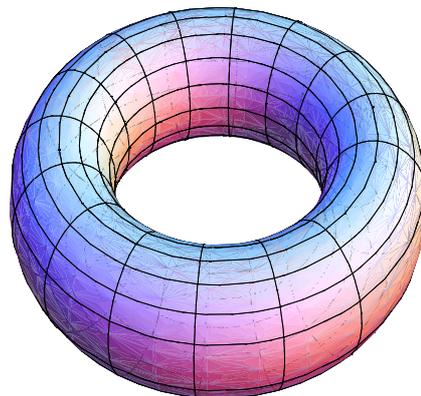
$\nabla f(x, y) = -6x + 3x^2, -6y + 3y^2 = (0, 0)$ so that the critical points are $(0, 0), (0, 2), (2, 0), (2, 2)$. We have $D = 36(x - 1)(y - 1)$ and $f_{xx} = 6x - 6$.

Point	D	f_{xx}	type
$(0, 0)$	$D = 36$	-6	max
$(0, 2)$	$D = -36$	-6	saddle
$(2, 0)$	$D = -36$	6	saddle
$(2, 2)$	$D = 36$	6	min

The ball more likely ends up near the minimum $(2, 2)$.

Problem 5) (10 points)

A torus can be obtained by rotating a circle of radius b around a circle of radius a . The volume of such a torus is $2\pi^2 ab^2$ and the surface area is $4\pi^2 ab$. If we want to find the torus which has minimal surface area while the volume with fixed packing $2\pi^2 a(b^2 + 1)$ is fixed $2\pi^2$, we need to extremize the function $f(a, b) = 4\pi^2 ab$ under the constraint $a + ab^2 = 1$. Find the optimal a, b .



Solution:

The Lagrange equations $\nabla f(a, b) = \lambda \nabla g(a, b), g(a, b) = 1$ are

$$\begin{aligned} 4\pi^2 b &= \lambda(b^2 + 1) \\ 4\pi^2 a &= 2\lambda ab \\ a + ab^2 &= 1 \end{aligned}$$

Dividing the first equation by the second gets rid of λ and leaves the two equations for the unknown a, b :

$$\begin{aligned} b/a &= (b^2 + 1)/(2ab) \\ a + ab^2 &= 1 \end{aligned}$$

The first equation is $ab^2 = a$ and since the second equation excludes $a = 0$ we have $b^2 = 1$ or $b = 1$ ($b \geq 0$ because it is a radius). The last equation gives $a = 1/2$.

Problem 6) (10 points)

- a) Find the arc length of the curve $\vec{r}(t) = \langle t^2, 2t^3/3, 1 \rangle$ from $t = -1$ to $t = 1$.
- b) What is the curvature of the curve at time $t = 1$? The formula for the curvature is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Solution:

a) $r'(t) = \langle 2t, 2t^2, 0 \rangle$ and $|r'(t)| = \sqrt{4t^2 + 4t^4} = 2|t|\sqrt{1+t^2}$. Note the absolute value because the velocity is always nonnegative.

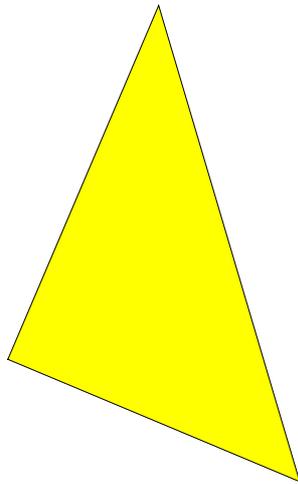
$$\int_{-1}^1 2|t|\sqrt{1+t^2} dt = 2 \int_0^1 2t\sqrt{1+t^2} dt = \frac{4}{3}(1+t^2)^{3/2} \Big|_0^1 = 4(\sqrt{2}^3 - 1)/3.$$

b) $r'(1) = \langle 2, 2, 0 \rangle$ and $r''(1) = \langle 2, 4, 0 \rangle$ so that

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{|\langle 0, 0, 4 \rangle|}{\sqrt{8}^3} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}.$$

Problem 7) (10 points)

A right angle triangle has the side lengths $x = 0.999$ and $y = 1.00001$. Estimate the value of the hypotenuse $f(x, y) = \sqrt{x^2 + y^2}$ using linear approximation.



Solution:

We estimate near $(x_0, y_0) = (1, 1)$. The linearization function $L(x, y)$ is defined as

$$L(x, y) = f(1, 1) + \nabla f(1, 1) \cdot \langle x - 1, y - 1 \rangle$$

Now $\nabla f(x, y) = \langle x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2} \rangle$ and $\nabla f(1, 1) = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$ we have

$$L(x, y) = \sqrt{2} + (x - 1)/\sqrt{2} + (y - 1)/\sqrt{2}.$$

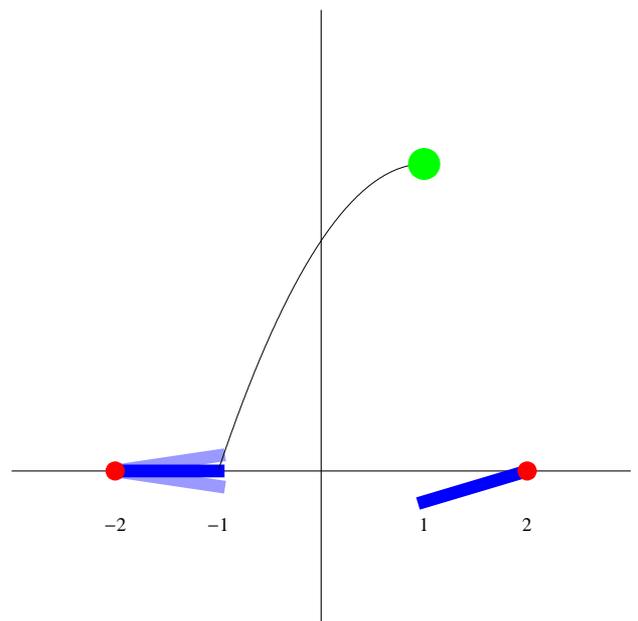
For the values given, we have $x - 1 = -0.001$, $y - 1 = 0.00001$ and so the estimate

$$\sqrt{2} - 0.001/\sqrt{2} + 0.00001/\sqrt{2}.$$

Problem 8) (10 points)

A pinball machine is tilted in such a way that a ball in the xy plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}'(0) = \langle 1/2, 5 \rangle$.

- a) Compute the trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ of the ball.
- b) The right flipper can reach the interval $[1, 2]$ on the x -axes. Can the player hit the ball when it comes back again to the base line $y = 0$?



Solution:

a) We know the acceleration $\vec{r}''(t) = \langle 0, -2 \rangle$. Integrating once gives the velocity

$$\vec{r}'(t) = \langle 0, -2t \rangle + \langle 1/2, 5 \rangle .$$

Integrating again gives

$$\vec{r}(t) \langle x(t), y(t) \rangle = \langle -1 + t/2, 5t - t^2 \rangle .$$

b) We have $y(t) = 0$ for $t = 0$ and $t = 5$. At this time, the ball is at $(3/2, 0)$. This is right in the middle of the right flipper. The player will hit the ball.

Problem 9) (10 points)

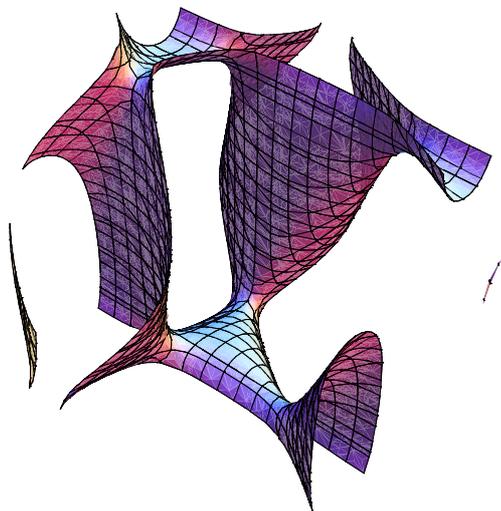
Find the tangent plane to the surface

$$\sin(x + y) - \cos(z - x) + \sin(y) = -1$$

at the point $(0, \pi, 0)$.

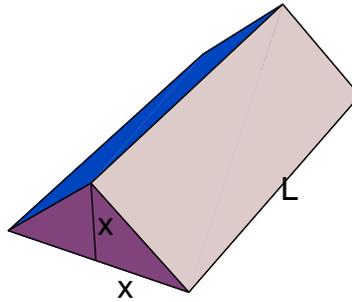
Solution:

We compute the gradient $\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$ with $f_x = \cos(x + y) + \sin(x - z)$, $f_y = \cos(y) + \cos(x + y)$ and $f_z = -\sin(x - z)$. At the point $(0, \pi, 0)$ we have $\nabla f(0, \pi, 0) = \langle -1, -2, 0 \rangle = \langle a, b, c \rangle$. The equation for the tangent plane is $ax + by + cz = d$ where d is obtained by plugging in $(x, y, z) = (0, \pi, 0)$. The equation is $-x - 2y = -2\pi$ which can also be written as $x + 2y = 2\pi$.



Problem 10) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x, y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x, y) = 2x^2 + 4xy$ under the constraint $g(x, y) = x^2y = 1$. Solve the problem with the Lagrange method.



Solution:

The Lagrange equations

$$\nabla f = \lambda \nabla g, g = 1$$

are

$$\begin{aligned} 4x + 4y &= \lambda 2xy \\ 4y &= \lambda x^2 \\ xy^2 &= 1 \end{aligned}$$

Eliminating λ gives $(4x + 4y)/4x = \lambda 2xy/\lambda x^2$ or $1 + y/x = 2y/x$ so that $1 = y/x$. The only critical point with positive x, y is $(1, 1)$. The minimum of f is $f(1, 1) = 6$. The minimal surface area is $6\sqrt{2}$.