Oliver Knill, Harvard Summer School, 2009

Chapter 2. Surfaces and Curves

Section 2.1: Functions, level surfaces, quadrics

A **function of two variables** $f(x, y)$ is usually defined on the entire plane like for $f(x, y) = x^2 + \sin(xy)$. In general, we need to restrict the function to a **domain** D in the plane like for $f(x, y) = 1/y$, where (x, y) is defined everywhere except on the x-axes $y = 0$. The **range** of a function f is the set of values which the function f takes. The function $f(x, y) = 1 + x^2$ for example takes all values ≥ 1. The **graph** of $f(x, y)$ is the set $\{(x, y, f(x, y)) \mid (x, y) \in D\}$. The graph of $f(x, y) = \sqrt{x^2 + y^2}$ on the domain $x^2+y^2 < 1$ is a half sphere. Here are more examples:

function $f(x, y)$	domain D	range $= f(D)$
$f(x, y) = \sin(3x + 3y) - \log(1 - x^2 - y^2)$	open unit disc $x^2 + y^2 < 1$	$[-1, \infty)$
$f(x, y) = f(x, y) = x^2 + y^3 - xy + \cos(xy)$	plane R^2	line
$f(x, y) = \sqrt{4 - x^2 - 2y^2}$	$x^2 + 2y^2 \leq 4$	$[0, 2]$
$f(x, y) = 1/(x^2 + y^2 - 1)$	all except unit circle	all
$f(x, y) = 1/(x^2 + y^2)^2$	all except origin	positive real axis

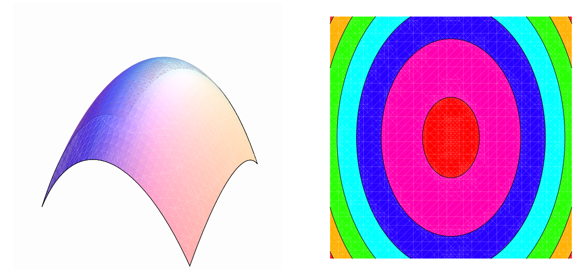
Given a function $f(x, y)$, the set $f(x, y) = c = const$ is called a **contour curve** or **level curve** of f. For example, for $f(x, y) = 4x^2 + 3y^2$ the level curves $f = c$ are ellipses if $c > 0$. Level curves allow to visualize functions of two variables $f(x, y)$.

Example: For $f(x, y) = x^2 - y^2$. the set $x^2 - y^2 = 0$ is the union of the lines $x = y$ and $x = -y$. The set $x^2 - y^2 = 1$ consists of two hyperbola with with their "noses" at the point $(-1, 0)$ and $(1, 0)$. The set $x^2 - y^2 = -1$ consists of two hyperbola with their noses at $(0, 1)$ and $(0, -1)$.

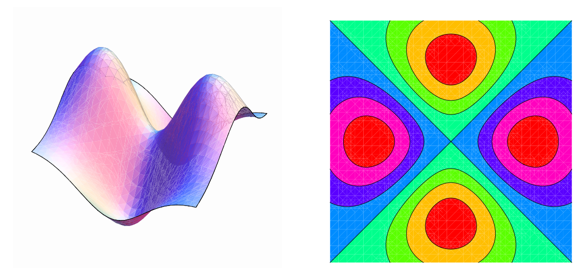
Drawing several contour curves $\{f(x, y) = c\}$ produces a **contour map**. This allows to visualize a function. For example, the contour curves $\sin(xy) = c$ are the same as the contour curves $xy = C$. As in the previous example, they are hyperbola $C \neq 0$ or crossing lines (C=0)

Contour curves are encountered every day: they appear as **isobars**, curves of constant pressure, or **isoclines**, curves of constant field direction, like for example constant wind direction. They can be **isothermes**, curves of constant temperature or **isoheights**, curves of constant height.

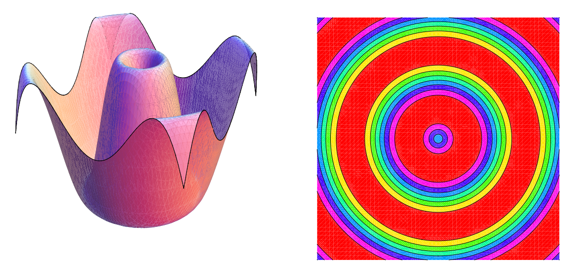
Example: The function $f(x, y) = 1 - 2x^2 - y^2$ as contour curves $f(x, y) = 1 - 2x^2 + y^2 = c$ which are ellipses $2x^2 + y^2 = 1 - c$ for $c < 1$.



Example: Lets look at the function $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$. While we can not find explicit expressions for the contour curves $(x^2 - y^2)e^{-x^2-y^2} = c$, we can draw the curves with the help of a computer:



Example: The surface $z = f(x, y) = \sin(\sqrt{x^2 + y^2})$ has concentric circles as contour curves.



In applications, we sometimes have to deal with functions which are not continuous. When plotting the rate of change of temperature of water in relation to pressure and volume for example, one experiences **phase transitions**. Mathematicians have tamed discontinuous events with a mathematical field called "catastrophe theory".

A function $f(x,y)$ is called **continuous** at (a,b) if $f(a,b)$ is finite and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that along any curve $\vec{r}(t)$ with $\vec{r}(0) = (a,b)$, we have $\lim_{t\to 0} f(\vec{r}(t)) = f(a,b)$. Continuity for functions of more than two variables is defined in the same way. Continuity is not always easy to check. Lets look at some examples:

Example: For $f(x,y) = (xy)/(x^2+y^2)$, we have $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous.

Example: For $f(x,y) = (x^2y)/(x^2+y^2)$, it is better to describe the function using polar coordinates: $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous.

A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. The function $g(x,y,z) = 2+\sin(xyz)$ is an example. It could define the temperature distribution in space. We can no more draw a graph of g because that would be an object in 4 dimensions. But we can visualize it differently by drawing **contour surfaces** $g(x,y,z) = c$, where c is constant. For example, for $f(x,y,z) = 4x^2+3y^2+z^2$, the contour surfaces are ellipsoids.

Many surfaces can be described as level surfaces. If this is the case, we call this an **implicit description** of a surface. Here are some examples we know already:

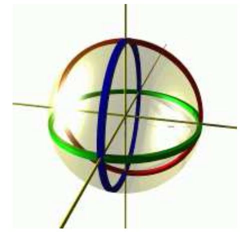
Spheres: The level surfaces of $g(x,y,z) = x^2+y^2+z^2$ are spheres.

Graphs: For $g(x,y,z) = z - f(x,y)$ we have the level surface $g = 0$ which is the graph $z = f(x,y)$ of a function of two variables. For example, for $g(x,y,z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2+y^2$ of the function $f(x,y) = x^2+y^2$ which is a paraboloid. Note however that most surfaces of the form $g(x,y,z) = c$ can not be written as graphs. The sphere is an example, where we need two graphs to cover it.

Planes: $ax+by+cz = d$ is a plane. With $\vec{n} = \langle a,b,c\rangle$ and $\vec{x} = \langle x,y,z\rangle$, we can rewrite the equation $\vec{n}\cdot\vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n}\cdot\vec{x}_0 = d$. so that $\vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$. This means that every vector $\vec{x}-\vec{x}_0$ in the plane is orthogonal to $\vec{n}$.

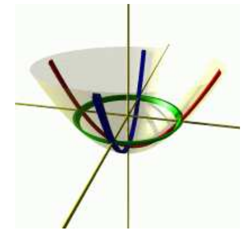
Quadrics: If the function depends only quadratically on variables, that is if $f(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+m$ then the surface $f(x,y,z) = 0$ is called a **quadric**. Lets look at a few of them:

Sphere



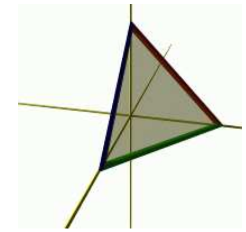
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

Paraboloid



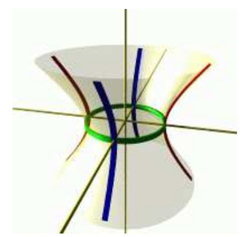
$(x-a)^2+(y-b)^2-c = z$

Plane



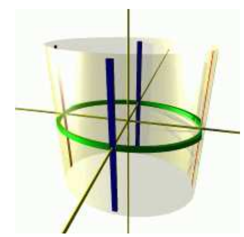
$ax+by+cz = d$

One sheeted Hyperboloid



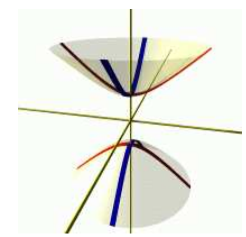
$(x-a)^2+(y-b)^2-(z-c)^2 = r^2$

Cylinder



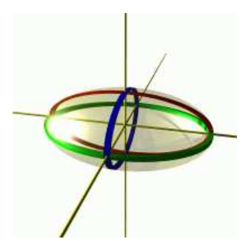
$(x-a)^2+(y-b)^2 = r^2$

Two sheeted Hyperboloid



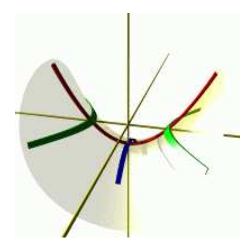
$(x-a)^2+(y-b)^2-(z-c)^2 = -r^2$

Ellipsoid



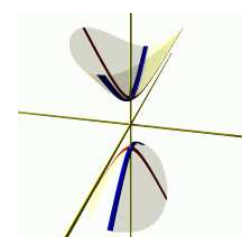
$x^2/a^2+y^2/b^2+z^2/c^2 = 1$

Hyperbolic paraboloid



$x^2-y^2+z = 1$

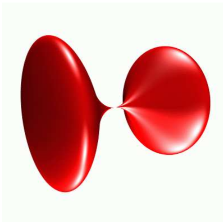
Elliptic hyperboloid

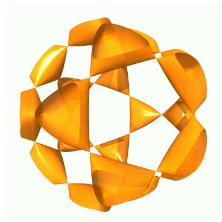


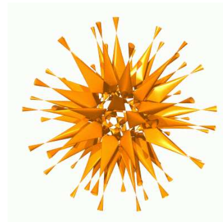
$x^2/a^2+y^2/b^2-z^2/c^2 = 1$

To understand a contour surface, it always helps to look at **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.

Higher order polynomial surfaces can be intriguingly beautiful. If the function involves only multiplications of variables x, y, z and $x \to f(x, x, x)$ has degree d, then it is called a **degree** d **polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **quartics**, degree 4 surfaces **quintics**, degree 10 surfaces **decics** and so on.





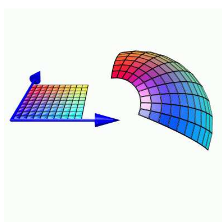


Section 2.2: Parametrized surfaces

There is a second, fundamentally different way to describe surfaces. It is called **parametrization** of a surface. This is achieved with a vector-valued function

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle \ .$$

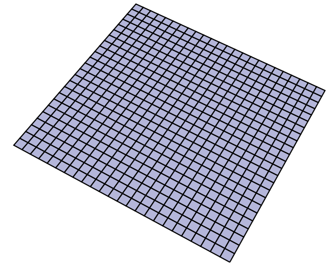
It three functions $x(u, v), y(u, v), z(u, v)$ of two variables. Because two variables u and v are involved, the map $\vec{r}$ is often called uv-map.



If we keep the first parameter u constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ traces a curve the surface. These curves are called **grid curves**. This can be useful to draw the surfaces. We will discuss parametrized curves next.

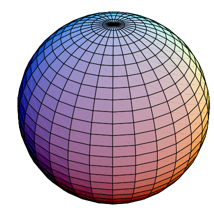
The world of parametric surfaces is fantastic. You will explore this land with the help of the computer algebra system Mathematica. There are 4 important examples:
1. **Planes**.
Parametric: $\vec{r}(s, t) = \vec{OP} + s\vec{v} + t\vec{w}$
Implicit: $ax + by + cz = d$.

We can change from parametric to implicit using the cross product $\vec{n} = \vec{v} \times \vec{w}$. We can change from implicit to parametric by finding three points P, Q, R on the surface and forming $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.



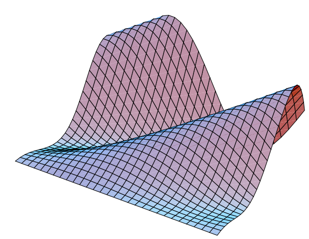
2. **The sphere**:
Parametric: $\vec{r}(u, v) = (a, b, c) + \langle\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)\rangle$.
Implicit: $(x - a)^2 + (y - b)^2 + (z - c)^2 = \rho^2$.

We can go from parametric to implicit by reading off the radius.
Implicit to Parametric: know it



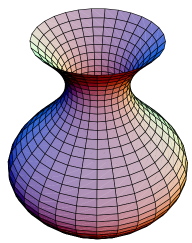
3. **Graphs**:
Parametric: $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$
Implicit: $z - f(x, y) = 0$.

Parametric to Implicit: think about $z = f(x, y)$
Implicit to Parametric: use x and y as the parameterizations.



4. **Surfaces of revolution**:
Parametric: $r(u, v) = (g(v)\cos(u), g(v)\sin(u), v)$
Implicit: $\sqrt{x^2 + y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.

Parametric to Implicit: read off the function $g(z)$ the distance to the z-axis.
Implicit to Parametric: use the function g.



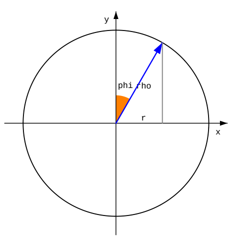
A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(y/x)$. We have $(x, y) = (r\cos(\theta), r\sin(\theta))$. Note that $\theta = \text{arctg}(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ would have the same θ. In order to get the correct θ, one could take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the value when $y/x = \infty$, and add π if $x < 0$ or $x = 0, y < 0$. Representing points

$$(x, y, z) = (r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$$

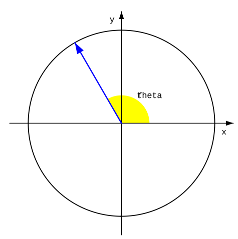
in space is called using **cylindrical coordinates**. Here are some level surfaces in cylindrical coordinates: $r = 1$ is a cylinder, $r = |z|$ is a double cone, $\theta = 0$ is a half plane, $r = \theta$ rolled sheet of paper and and $r = 2 + \sin(z)$ is an example of a surface of revolution.

Spherical coordinates use the distance ρ to the origin as well as two angles θ the polar angle and ϕ, the angle between the vector and the z axis. A point has the coordinates $(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))$

The distance $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ from the right picture.



$$x = \rho\cos(\theta)\sin(\phi),$$
$$y = \rho\sin(\theta)\sin(\phi),$$
$$z = \rho\cos(\phi)$$



Here are some level surfaces described in spherical coordinates: $\rho = 1$ is a sphere, the surface $\phi = \pi/2$ is a single cone, $\rho = \phi$ is an apple shaped surface and $\rho = 2 + \cos(3\theta)\sin(\phi)$ is a bumpy sphere.

Section 2.3: Parametrized curves

If $x(t), y(t)$ are two functions of a variable t, which is defined in a **parameter interval** $I = [a, b]$, then $\vec{r}(t) = \langle f(t), g(t)\rangle$ is called a **parametric curve** in the plane. The functions $x(t), y(t)$ are called **coordinate functions**. For example, $\vec{r}(t) = \langle\cos(3t), \sin(5t)\rangle$ describes such a curve.

In three dimensions, we describe curves with three functions. The parametrization is $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ and traces a **space curve**. We always think of the **parameter** t as **time**. For a fixed time t, we have a point $(x(t), y(t), z(t))$ in space. As t varies, this point moves along the curve.

Example: If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = \langle t, f(t)\rangle$ traces the graph of $f(x)$. For example, for $f(x) = x^2 + 1$, the curve is a parabola.

Example: With $x(t) = \cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows a **circle**. We can see this from $x(t)^2 + y(t)^2 = 1$.

Example: With $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$ we get a **space curve** $\vec{r}(t) = \langle\cos(t), \sin(t), t\rangle$. It traces a **helix**.

Example: If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in the previous example but the curve is traversed **faster**. The **parameterization** of the curve has changed.

Example: If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve again but we traverse it in the **opposite direction**.

Example: If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a + t(u - a), b + t(v - b), c + t(w - c)\rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q. For example, $\vec{r}(t) = \langle 1 + t, 1 - t, 2 + 3t\rangle$ connects the points $P = (1, 1, 2)$ with $Q = (2, 0, 1)$.

Sometimes it is possible to eliminate the parameter t and write the curve using equations. We need one equation in the plane or two equations in space. We have seen this already with lines.

For example, for $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$, then $x = t\cos(z), y = t\sin(z)$ and $x^2 + y^2 = z^2$. The curve is located on a cone.

Curves describe the paths of particles, celestial bodies, or quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time.. Here are some more places, where curves appear:

Strings or knots are closed curves in space.
Large Molecules like RNA or proteins can be modeled as curves.
Computer graphics: surfaces are represented by mesh of curves.

Typography: fonts represented by Bezier curves.
Space time A curve in space-time describes the motion of particles.
Topology Examples: space filling curves, boundaries of surfaces or knots.

If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle = \langle \dot{x}, \dot{y}, \dot{z}\rangle$ is called the **velocity** at time t. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**. We have:

The velocity vector $\vec{r}\,'(t)$ is tangent to the curve at $\vec{r}(t)$.

Here is an example where velocities, acceleration and jerk are computed:
if $\vec{r}(t) = \langle \cos(3t), \sin(2t), 2\sin(t)\rangle$, then $\vec{r}'(t) = \langle -3\sin(3t), 2\cos(2t), 2\cos(t)\rangle$, $\vec{r}''(t) = \langle -9\cos(3t), -4\sin$
and $\vec{r}'''(t) = \langle 27\sin(3t), 8\cos(2t), -2\cos(t)\rangle$.

Lets look at some examples of velocities and accelerations:

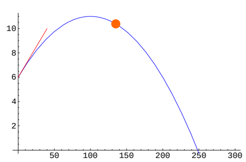
Signals in nerves:	40 m/s	Train:	0.1-0.3 m/s^2
Plane:	70-900 m/s	Car:	3-8 m/s^2
Sound in air:	Mach1=340 m/s	Space shuttle:	$\leq 3G = 30 m/s^2$
Speed of bullet:	1200-1500 m/s	Combat plane F16:	9G=90 m/s^2
Earth around the sun:	30'000 m/s	Ejection from F16:	14G=140 m/s^2.
Sun around galaxy center:	200'000 m/s	Free fall:	1G = 9.81 m/s^2
Light in vacuum:	300'000'000 m/s	Electron in vacuum tube:	10^{15} m/s^2

The **differentiation rules** in one dimension $(f+g)' = f' + g'$ (addition rule) $(cf)' = cf'$, $(fg)' = f'g + fg'$ (Leibniz), $(f(g))' = f'(g)g'$ (chain rule) generalize for vector-valued functions: $(\vec{v} + \vec{w})' = \vec{v}' + \vec{w}'$, $(c\vec{v})' = c\vec{v}'$, $(\vec{v}\cdot\vec{w})' = \vec{v}'\cdot\vec{w} + \vec{v}\cdot\vec{w}'$ $(\vec{v}\times\vec{w})' = \vec{v}'\times\vec{w} + \vec{v}\times\vec{w}'$ (Leibniz), $(\vec{v}(f(t)))' = \vec{v}'(f(t))f'(t)$ (chain rule).

The process of differentiation of a curve can be reversed. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

Lets look the example of **free fall**. If $\vec{r}''(t) = \langle 0, 0, -10\rangle$, $\vec{r}'(0) = \langle 0, 1000, 2\rangle$, $\vec{r}(0) = \langle 0, 0, h\rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2\rangle$.



Section 2.4: Arc length and curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then

$$\int_a^b |\vec{r}\,'(t)|\, dt$$

is called the **arc length of the curve**. It can be written out. For space curves for example we have

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

Example: The arc length of the circle $\vec{r}(t) = \langle \cos(t), \sin(t)\rangle$ parameterized by $0 \leq t \leq 2\pi$ is $2pi$ because the speed $|\vec{r}'(t)|$ is constant and equal to 1.

Example: The helix $\vec{r}(t) = (\cos(t), \sin(t), t)$ has velocity $\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$ and constant speed $|\vec{r}\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.

Example: Compute the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2\rangle .$$

for $1 \leq t \leq 2$. Because $\vec{r}'(t) = \langle 1, 1/t, t\rangle$, we have $\vec{r}'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$. We have $L = \int_1^2 \frac{1}{t} + t\, dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, wec an call it "opportunity curve".

Example: What is the arc length of the curve $\vec{r}(t) = \langle \cos^3(t), \sin^3(t)\rangle$? We have $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi} (3/2)\sin(2t)\, dt = 6$.

Example: Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3\rangle$ for $-1 \leq t \leq 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2}\, dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^1 |x|\sqrt{1+x^2}\, dx = 2\int_0^1 x\sqrt{1+x^2}\, dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2} - 1)/3$.

Example: The arc length of an **epicycle** $\vec{r}(t) = \langle t + \sin(t), \cos(t)\rangle$ parameterized by $0 \leq t \leq 2\pi$. We have $|\vec{r'}(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)}\, dt$. A **substitution** $t = 2u$ gives $L = \int_0^{\pi} \sqrt{2 + 2\cos(2u)}\, 2du = \int_0^{\pi} \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\, 2du = \int_0^{\pi} \sqrt{4\cos^2(u)}\, 2du = 4\int_0^{\pi} |\cos(u)|\, du = 8$.

Example: the arc length of the **catenary** $\vec{r}(t) = \langle t, \cosh(t)\rangle$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1 ,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**.

Because a parameter change $t = t(s)$ corresponds to a substitution in the integration which does not change the integral, we have

The arc length is independent of the parameterization of the curve.

Example: the circle parameterized by $\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\ dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ leads to the arc length integral $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\ dt$ which can only be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**. The **curvature** if a curve at the point $\vec{r}(t)$ is defined as

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} \, .$$

The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

Example: The curve $\vec{r}(t) = (t, f(t))$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

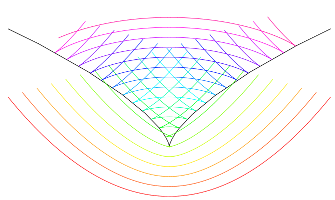
For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **tangent vector**
$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **unit normal vector** and
$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **binormal vector**

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

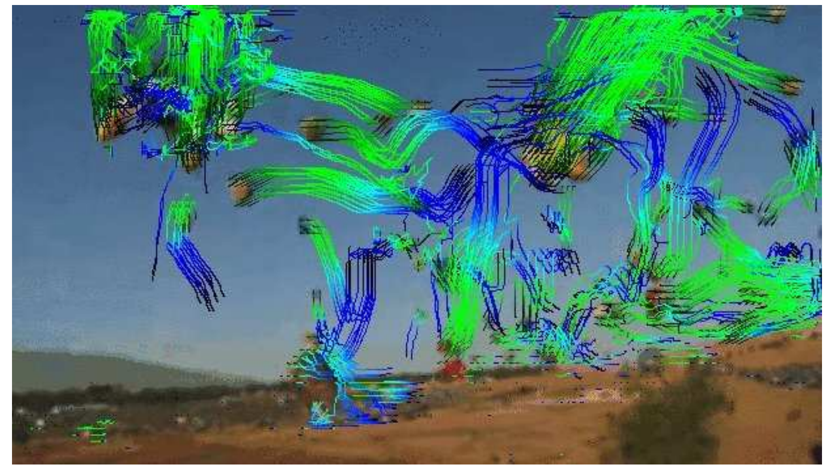
Here is an application of curvature: If a curve $\vec{r}(t)$ represents a wavefront and $\vec{n}(t)$ is a unit vector normal to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.



A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

which we prove in class to be equivalent to the definition. Finally, lets mention that curvature is important also in computer vision. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to corners of objects in the picture. This is useful when trying to track or identify objects.



Tracking balloons in a movie taken at a baloon festival in Albuquerque. The program computes curvature in order to identify intersting points, then tracks them over time.