

## Chapter 6. Integral Theorems

### Section 6.1: Greens theorem

If  $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is a vector field and  $C : \vec{r}(t) = \langle x(t), y(t) \rangle, t \in [a, b]$  is a curve, the **line integral**

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}'(t) dt$$

is a measure of the work done by  $\vec{F}$  along the path.

The **curl** of a 2D vector field  $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is defined as the scalar field

$$\text{curl}(F)(x, y) = Q_x(x, y) - P_y(x, y) .$$

The  $\text{curl}(F)$  measures the **vorticity** of the vector field. One can write  $\nabla \times \vec{F} = \text{curl}(\vec{F})$  because the two dimensional cross product of  $(\partial_x, \partial_y)$  with  $\vec{F} = \langle P, Q \rangle$  is  $Q_x - P_y$ .

**Examples.**

- 1) For  $\vec{F}(x, y) = \langle -y, x \rangle$  we have  $\text{curl}(F)(x, y) = 2$ .
- 2) If  $\vec{F}(x, y) = \nabla f$ , then the curl is zero because if  $P(x, y) = f_x(x, y), Q(x, y) = f_y(x, y)$  and  $\text{curl}(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$  by Clairot's theorem.

**Green's theorem (1827)** If  $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$  is a vector field and  $G$  is a region for which the boundary  $C$  is a curve parametrized so that  $G$  is "to the left". Then

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_G \text{curl}(F) dx dy$$

George Green (1793-1841) was a remarkable physicist of the nineteenth century. He was a self-taught mathematician and miller, whose work greatly contributed to modern physics.

If  $\vec{F}$  is a gradient field, then both sides of Green's theorem are zero:  
 $\int_C \vec{F} \cdot d\vec{r}$  is zero by the fundamental theorem for line integrals. and  $\int \int_G \text{curl}(F) \cdot dA$  is zero because  
 $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$ .

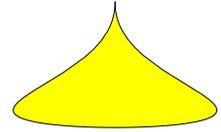
The identity  $\text{curl}(\text{grad}(f)) = 0$  can be checked directly by writing it as  $\nabla \times \nabla f$  and using that the cross product of two identical vectors is 0. Just treat  $\nabla$  as a vector.

**Example:** Find the line integral of  $\vec{F}(x, y) = \langle x^2 - y^2, 2xy \rangle = \langle P, Q \rangle$  along the boundary of the rectangle  $[0, 2] \times [0, 1]$ . Solution:  $\text{curl}(\vec{F}) = Q_x - P_y = 2y - 2y = -4y$  so that  $\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \int_0^1 4y dy dx = 2y^2|_0^1 x|_0^2 = 4$ .

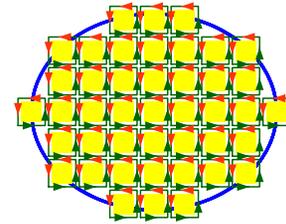
**Example:** Find the area of the region enclosed by

$$\vec{r}(t) = \left\langle \frac{\sin(\pi t)^2}{t}, t^2 - 1 \right\rangle$$

for  $-1 \leq t \leq 1$ . Use Greens theorem with the vector field  $\vec{F} = \langle 0, x \rangle$ .



Why is Greens theorem true? Look first at a small square  $G = [x, x + \epsilon] \times [y, y + \epsilon]$ . The line integral of  $\vec{F} = \langle P, Q \rangle$  along the boundary is  $\int_0^\epsilon P(x + t, y) dt + \int_0^\epsilon Q(x + \epsilon, y + t) dt - \int_0^\epsilon P(x + t, y + \epsilon) dt - \int_0^\epsilon Q(x, y + t) dt$ . This line integral measures the "circulation" at the place  $(x, y)$ . Because  $Q(x + \epsilon, y) - Q(x, y) \sim Q_x(x, y)\epsilon$  and  $P(x, y + \epsilon) - P(x, y) \sim P_y(x, y)\epsilon$ , the line integral is  $(Q_x - P_y)\epsilon^2$  is about the same as  $\int_0^\epsilon \int_0^\epsilon \text{curl}(F) dx dy$ . All identities hold in the limit  $\epsilon \rightarrow 0$ . To prove the statement for a general region  $G$ , chop it into small squares of size  $\epsilon$ . Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the rectangles is a Riemann sum approximation of the double integral.



**Example:**

Green's theorem allows to express the coordinates of the centroid

$$\left( \int \int_G x dA/A, \int \int_G y dA/A \right)$$

as line integrals. With the vector field  $\vec{F} = \langle 0, x^2 \rangle$  we have

$$\int \int_G x dA = \int_C \vec{F} \cdot d\vec{r} .$$

**Example:** An important application of Green is to compute area. With the vector fields  $\vec{F}(x, y) = \langle P, Q \rangle = \langle -y, 0 \rangle$  or  $\vec{F}(x, y) = \langle 0, x \rangle$  have vorticity  $\text{curl}(\vec{F})(x, y) = 1$ . The right hand side in Green's theorem is the **area** of  $G$ :

$$\text{Area}(G) = \int_C x(t)y'(t) dt .$$

**Example:** Let  $G$  be the region under the graph of a function  $f(x)$  on  $[a, b]$ . The line integral around the boundary of  $G$  is 0 from  $(a, 0)$  to  $(b, 0)$  because  $\vec{F}(x, y) = \langle 0, 0 \rangle$  there. The line integral is also zero from  $(b, 0)$  to  $(b, f(b))$  and  $(a, f(a))$  to  $(a, 0)$  because  $N = 0$ . The line integral on  $(t, f(t))$  is  $-\int_a^b \langle -y(t), 0 \rangle \cdot \langle 1, f'(t) \rangle dt = \int_a^b f(t) dt$ . Green's theorem assures that

this is the area of the region below the graph.

Finally, let's mention a theoretical application: if  $\text{curl}(\vec{F}) = 0$  in a simply connected region  $G$ , then  $\vec{F}$  is conservative there: given a closed curve  $C$  in  $G$ , it encloses a region  $R$ . Green's theorem assures that  $\int_C \vec{F} \cdot d\vec{r} = 0$ . So  $\vec{F}$  has the closed loop property in  $G$  and is therefore a gradient field there.

An engineering application of Green's theorem is the **planimeter**, a mechanical device for measuring areas. It had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field  $\vec{F}$  associated to a planimeter which is obtained by placing a unit vector perpendicular to the arm). One can prove that  $\vec{F}$  has vorticity 1. The planimeter calculates the line integral of  $\vec{F}$  along a given curve. Green's theorem assures it is the area.

## Section 6.2: Curl and Div

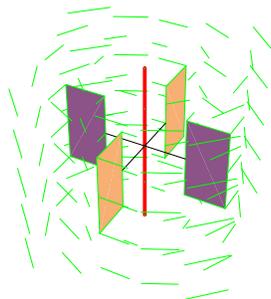
The **curl** of a 3D vector field  $\vec{F} = \langle P, Q, R \rangle$  is defined as the vector field

$$\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

We can write  $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ .

Note that the third component is just the curl of a 2D vector field  $\vec{F} = \langle P, Q \rangle$  is  $Q_x - P_y$ . While the curl in 2 dimensions is a scalar field it is a vector in 3 dimensions. In  $n$  dimensions, it would have dimension  $n(n-1)/2$ .

The curl is usually visualized by a "paddle wheel". If you place such a wheel into the field into the direction  $v$ , its rotation measures  $\vec{F} \cdot \vec{v}$ . Consequently, the direction in which the wheel turns fastest, is the direction of  $\text{curl}(\vec{F})$ . Its angular velocity is the length of the curl. The wheel could actually be used to measure the curl of the vector field at any point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl.



An other important derivative besides curl is the **divergence** of  $\vec{F} = \langle P, Q, R \rangle$  is the scalar field  $\text{div}(\langle P, Q, R \rangle) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$ . The **divergence** of a vector field  $\vec{F} = \langle P, Q \rangle$  is  $\text{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y$ . The curl measure the "vorticity" of a field. The divergence measures the "expansion" of a field.

If a field has zero curl everywhere, the field is called **irrotational**.

If a field has zero divergence everywhere, the field is called **incompressible**.

With the "vector"  $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$ , we can write  $\text{curl}(\vec{F}) = \nabla \times \vec{F}$  and  $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$ . Writing things with the "Nabla vector" is called **Nabla calculus**. This works both in 2 and 3 dimensions.

The following combination of divergence and gradient often appears in physics:

$$\Delta f = \text{divgrad}(f) = f_{xx} + f_{yy} + f_{zz}.$$

It is called the Laplacian of  $f$ . We can write  $\Delta f = \nabla^2 f$  because  $\nabla \cdot (\nabla f) = \text{div}(\text{grad}(f))$ . We can extend the Laplacian also to vectorfields with  $\Delta \vec{F} = \langle \Delta P, \Delta Q, \Delta R \rangle$  and write  $\nabla^2 \vec{F}$ . While direct computations can verify the following identities, they become evident with Nabla calculus from formulas for vectors like  $\vec{v} \times \vec{v} = \vec{0}$ ,  $\vec{v} \cdot \vec{v} \times \vec{w} = 0$  or  $u \times (v \times w) = v(u \cdot w) - (u \cdot v)w$ .

$$\begin{aligned} \text{div}(\text{curl}(\vec{F})) &= 0. & \nabla \cdot \nabla \times \vec{F} &= 0. \\ \text{curlgrad}(\vec{F}) &= \vec{0} & \nabla \times \nabla \vec{F} &= \vec{0}. \\ \text{curl}(\text{curl}(\vec{F})) &= \text{grad}(\text{div}(\vec{F}) - \Delta(\vec{F})). & \nabla \times \nabla \times \vec{F} &= \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}. \end{aligned}$$

These identities can be used to answer questions like:

**Question:** Is there a vector field  $\vec{G}$  such that  $\vec{F} = \langle x + y, z, y^2 \rangle = \text{curl}(\vec{G})$ ?

**Answer:** no because  $\text{div}(\vec{F}) = 1$  is incompatible with  $\text{div}(\text{curl}(\vec{G})) = 0$ .

The derivatives curl, div and grad appear in physics. For example in electrodynamics or fluid dynamics. These equations describe phenomena in life like light, sound and weather.

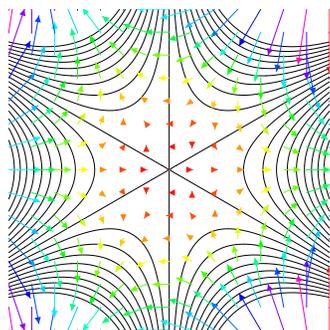
$\text{div}(B) = 0$	No monopoles	there are no magnetic monopoles.
$\text{curl}(E) = -\frac{1}{c}B_t$	Faraday's law	change of magnetic flux induces voltage
$\text{curl}(B) = \frac{1}{c}E_t + \frac{4\pi}{c}j$	Ampère's law	current or change of E produces magnetic field
$\text{div}(E) = 4\pi\rho$	Gauss law	electric charges are sources for electric field

The constant  $c$  is the speed of light. In fluid dynamics, we are interested in the velocity field  $F = v$ , the density  $\rho$  and the potential function  $f$  if it exists.

Incompressibility	$\text{div}(v) = 0$	incompressible fluids, in 2D: $v = \text{grad}(u)$
Irrotational	$\text{curl}(v) = 0$	irrotation fluids
Incompressible Irrotational	$\Delta(f) = 0, v = \nabla f$	incompressible, irrotational fluids
Continuity equation	$\dot{\rho} + \text{div}(\rho v) = 0$	no fluid gets lost

An example of a field, which is both incompressible and irrotational is obtained by taking a function  $f$  which satisfies  $\Delta f = 0$ , like  $f(x, y) = x^3 - 3xy^2$ , then define  $F = \nabla f$ . In that case, this gives

$$\vec{F}(x, y) = \langle 3x^2 - 3y^2, -6xy \rangle.$$



In dimension  $d$ , there are  $d$  fundamental derivatives:

$1 \mapsto 1$	$f'$	derivative
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$1 \mapsto 2$	$\nabla f$	gradient
$2 \mapsto 1$	$\nabla \times F$	curl

$1 \mapsto 3$	$\nabla f$	gradient
$3 \mapsto 3$	$\nabla \times F$	curl
$3 \mapsto 1$	$\nabla \cdot F$	divergence

**Example:** If we rotate the vector field  $\vec{F} = \langle P, Q \rangle$  by 90 degrees  $= \pi/2$ , we get a new vector field  $\vec{G} = \langle -Q, P \rangle$ . The integral  $\int_C \vec{F} \cdot ds$  becomes a **flux**  $\int_\gamma G \cdot dn$  of  $G$  through the boundary of  $R$ , where  $dn$  is a normal vector with length  $|r'|dt$ . With  $\text{div}(\vec{F}) = (P_x + Q_y)$ , we see that

$$\text{curl}(\vec{F}) = \text{div}(\vec{G}).$$

Green's theorem now becomes

$$\int \int_R \text{div}(\vec{G}) \, dx dy = \int_C \vec{G} \cdot \vec{dn},$$

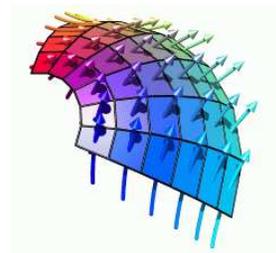
where  $dn(x, y)$  is a normal vector at  $(x, y)$  orthogonal to the velocity vector  $\vec{r}'(x, y)$  at  $(x, y)$ . This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. Note that this is just Green's theorem disguised. There are only 2 basic integral theorems in the plane: Green's theorem and the fundamental theorem of line integrals. The just mentioned variant of Greens theorem can explain however **divergence** means in two dimensions: it is an **expansion rate** similar than the original Green theorem explains that curl is a **rotation rate**: Take a small square in the plane. Greens theorem tells us that the curl of  $F$  integrated over that square is the line integral along the boundary. The divergence version of Greens theorem tells us that the divergence of  $\vec{F}$  integrated over the square is the flux of the field through the boundary.

The curl at a point  $(x, y)$  is the average work of the field done along a small circle of radius  $r$  around the point in the limit when the radius of the circle goes to zero.

The divergence at a point  $(x, y)$  is the average flux of the field through a small circle of radius  $r$  around the point in the limit when the radius of the circle goes to zero.

## Section 6.3: Flux and Stokes Theorem

If  $S$  is a surface in space, and  $\vec{F}$  is a vector field, the integral  $\int \int_S \vec{F} \cdot d\vec{S}$  is called the **flux** of  $\vec{F}$  through the surface  $S$ .

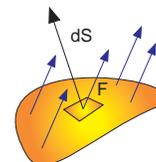


Assume  $S$  is parametrized as  $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$  over a domain  $G$  in the  $uv$ -plane.

The **flux integral** of  $\vec{F}$  through  $S$  is the double integral

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv.$$

The short hand notation  $d\vec{S} = (\vec{r}_u \times \vec{r}_v) \, dudv$  represents an infinitesimal vector in the normal direction to the surface.



The interpretation is that if  $\vec{F}$  = fluid velocity field, then  $\int \int_S \vec{F} \cdot d\vec{S}$  is the amount of fluid passing through  $S$  in unit time.

Example: Let  $\vec{F}(x, y, z) = \langle 0, 1, z^2 \rangle$  and let  $S$  be the upper half sphere with

$$\vec{r}(u, v) = \langle \cos(u) \sin(v), \sin(u) \sin(v), \cos(v) \rangle$$

,  $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}$  so that  $\vec{F}(\vec{r}(u, v)) = \langle 0, 1, \cos^2(v) \rangle$  and

$$\int_0^{2\pi} \int_0^\pi \langle 0, 1, \cos^2(v) \rangle \cdot \langle \cos(u) \sin^2(v), \sin(u) \sin^2(v), \cos(v) \sin(v) \rangle \, dudv.$$

The integral is  $\int_0^{2\pi} \int_{\pi/2}^\pi \sin^2(v) \sin(u) + \cos^3(v) \sin(v) \, dudv$ . The first integral is zero. The second is  $\int_{\pi/2}^\pi \cos^3 v \sin(v) \, dv = -\cos^4(v)/4|_0^{\pi/2} = 1/4$ .

Because  $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$  is a unit vector normal to the surface and on the surface,  $\vec{F} \cdot \vec{n}$  is the normal component of the vector field with respect to the surface. One could write therefore also  $\int \int_S \vec{F} \cdot d\vec{S} = \int \int_S \vec{F} \cdot \vec{n} \, dS$  where  $dS$  is the surface element we know from when we

computed surface area. The function  $\vec{F} \cdot \vec{n}$  is the scalar projection of  $\vec{F}$  in the normal direction. Whereas the formula  $\iint 1 \, dS$  gave the area of the surface with  $dS = |\vec{r}_u \times \vec{r}_v| \, dudv$ , the flux integral weights each area element  $dS$  with the normal component of the vector field with  $\vec{F}(\vec{r}(u, v)) \cdot \vec{n}(\vec{r}(u, v))$ . For actual computations, we of course refrain from this, since there is more work to compute  $\vec{n}$ . We just compute the vectors  $\vec{F}(\vec{r}(u, v))$  and  $\vec{r}_u \times \vec{r}_v$  and integrate its dot product over the domain.

**Example.** Calculate the flux of the vector field  $\vec{F}(x, y, z) = \langle 1, 2, 4z \rangle$  through the paraboloid  $z = x^2 + y^2$  lying above the region  $x^2 + y^2 \leq 1$ .

We can parametrize with polar coordinates  $\vec{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), r^2 \rangle$  where  $\vec{r}_r \times \vec{r}_\theta = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle$  and  $\vec{F}(\vec{r}(u, v)) = \langle 1, 2, 4r^2 \rangle$ . We get  $\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2 \cos(v) - 4r^2 \sin(v) + 4r^3) \, dr d\theta = 2\pi$ .

Here is a concept problem: Compute the flux of  $\vec{F}(x, y, z) = \langle 2, 3, 1 \rangle$  through the torus parameterized as  $\vec{r}(u, v) = \langle (2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v) \rangle$ , where both  $u$  and  $v$  range from  $0$  to  $2\pi$ .

No computation is needed. Think about what the flux means.

**Stokes theorem:** Let  $S$  be a surface with boundary curve  $C$  and let  $F$  be a vector field. Then

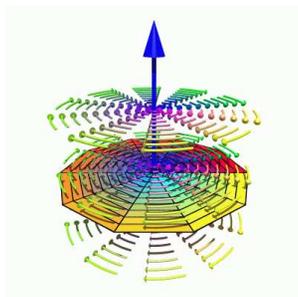
$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}.$$

Note that the orientation of  $C$  is such that if you walk along the surface (your head points into the direction of the normal  $\vec{r}_u \times \vec{r}_v$ ), then the surface to your left.

Example: Let  $\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$  and let  $S$  be the upper semi hemisphere, then  $\text{curl}(\vec{F})(x, y, z) = \langle 0, 0, 2 \rangle$ . The surface is parameterized by  $\vec{r}(u, v) = \langle \cos(u) \sin(v), \sin(u) \sin(v), \cos(v) \rangle$  on  $G = [0, 2\pi] \times [0, \pi/2]$  and  $\vec{r}_u \times \vec{r}_v = \sin(v) \vec{r}(u, v)$  so that  $\text{curl}(\vec{F})(x, y, z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v) \sin(v) 2$ . The integral  $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v) \, dv du = 2\pi$ .

The boundary  $C$  of  $S$  is parameterized by  $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$  so that  $d\vec{r} = \vec{r}'(t) dt = \langle -\sin(t), \cos(t), 0 \rangle dt$  and  $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \sin(t)^2 + \cos(t)^2 = 1$ . The line integral  $\int_C \vec{F} \cdot d\vec{r}$  along the boundary is  $2\pi$ .

If  $S$  is a surface in the  $xy$ -plane and  $\vec{F} = \langle P, Q, 0 \rangle$  has zero  $z$  component, then  $\text{curl}(\vec{F}) = \langle 0, 0, Q_x - P_y \rangle$  and  $\text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y \, dx dy$ .



We see that for a surface which is flat, Stokes theorem is a consequence of Green's theorem. If we put the coordinate axis so that the surface is in the  $xy$ -plane, then the vector field  $F$  induces a vector field on the surface such that its 2D curl is the normal component of  $\text{curl}(F)$ . The reason is that the third component  $Q_x - P_y$  of  $\text{curl}(\vec{F}) \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$  is the two dimensional curl:  $\vec{F}(\vec{r}(u, v)) \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$ . If  $C$  is the boundary of the surface, then  $\iint_S \vec{F}(\vec{r}(u, v)) \cdot \langle 0, 0, 1 \rangle \, dudv = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$ .

A general surface can be approximated by a polyhedron consisting of a mesh of small triangles. When summing up line integrals along all these parallelograms, the line integrals inside the surface cancel and only the integral along the boundary remain. On the other hand, the sum of the fluxes of the curl through boundary adds up to the flux through the surface.

**Stokes theorem was found by Ampère in 1825.** George Gabriel Stokes (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840.



George Gabriel Stokes

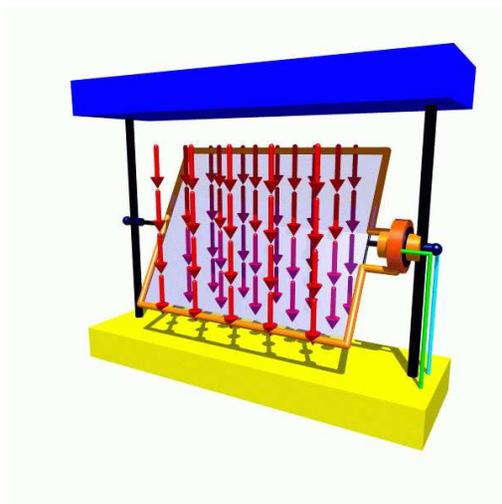


André Marie Ampere

**Problem:** Calculate the flux of the curl of  $\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$  through the surface parameterized by  $\vec{r}(u, v) = \langle \cos(u) \cos(v), \sin(u) \cos(v), \cos^2(v) + \cos(v) \sin^2(u + \pi/2) \rangle$ . Because the surface has the same boundary as the upper half sphere, the integral is again  $2\pi$  as in the above example.

For every surface bounded by  $C$  the flux of  $\text{curl}(\vec{F})$  through through the surface is the same. The flux of the curl of a vector field through a surface  $S$  depends only on the boundary of  $S$ . Compare this with the earlier statement that for every curve between two points  $A, B$  the line integral of  $\text{grad}(f)$  along  $C$  is the same. The line integral of the gradient of a function of a curve  $C$  depends only on the end points of  $C$ .

Stokes theorem is important in physics. Here is an example: The electric field  $E$  and the magnetic field  $B$  are linked by a Maxwell equation  $\text{curl}(\vec{E}) = -\frac{1}{c} \dot{B}$ . Take a closed wire  $C$  which bounds a surface  $S$  and consider  $\int_S B \cdot dS$ , the flux of the magnetic field through  $S$ . Its change can be related with a voltage using Stokes theorem:  $d/dt \int_S B \cdot dS = \int_S \dot{B} \cdot dS = \int_S -c \text{curl}(\vec{E}) \cdot d\vec{S} = -c \int_C \vec{E} \cdot d\vec{r} = U$ , where  $U$  is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire, we get an electric voltage. This happens in a power-generator like an alternator in a car. In practical implementations, the wire is turned inside a fixed magnet.



## Section 6.4: Divergence theorem

**Divergence theorem.** Let  $G$  be a solid region in space. Assume the boundary of the solid is a surface  $S$ . Let  $\vec{F}$  be a vector field. Then

$$\iiint_G \operatorname{div}(\vec{F}) \, dV = \iint_S \vec{F} \cdot d\vec{S}.$$

The orientation of  $S$  is such that the normal vector  $\vec{r}_u \times \vec{r}_v$  points **outside** the solid  $G$ .

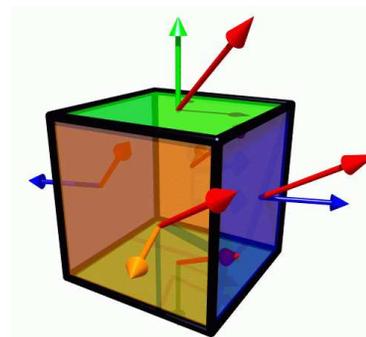
Example: Let  $\vec{F}(x, y, z) = \langle x, y, z \rangle$  and let  $S$  be sphere. The divergence of  $\vec{F}$  is the constant function  $\operatorname{div}(\vec{F}) = 3$  and  $\iiint_G \operatorname{div}(\vec{F}) \, dV = 3 \cdot 4\pi/3 = 4\pi$ . The flux through the boundary is  $\iint_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v \, dudv = \iint_S |\vec{r}(u, v)|^2 \sin(v) \, dudv = \int_0^\pi \int_0^{2\pi} \sin(v) \, dudv = 4\pi$  also.

Assume  $\rho$  is the density of a fluid with velocity field  $\vec{F}$ . The flux  $\iint_S \vec{F} \cdot d\vec{S}$  of the fluid through the closed surface  $S$  bounding a region  $G$  is the minus the rate of change of the mass  $-(d/dt) \iiint_G \rho \, dV$  inside  $G$ . This is because the flux through the boundary is the amount of mass leaving  $G$  in unit time. But this flux is by Gauss theorem equal to  $\iiint_G \operatorname{div}(\vec{F}) \, dV$ . Therefore,  $\iiint_G (\frac{d}{dt}\rho + \operatorname{div}(\vec{F})) \, dV = 0$ . Because this is true for all regions  $G$ , the partial differential equation  $\dot{\rho} + \operatorname{div}(\vec{F})$  holds. It is called the **continuity equation**. It gives a meaning to  $\operatorname{div}(\vec{F})$  as the rate of change of "density" at this point. Take a small cube and let it flow with the field. If it expands in volume, then the divergence is positive, the density becomes smaller.

Example. What is the flux of the vector field  $\vec{F}(x, y, z) = \langle 2x, 3z^2 + y, \sin(x) \rangle$  through the solid  $G = [0, 3] \times [0, 3] \times [0, 3] \setminus ([0, 3] \times [1, 2] \times [1, 2] \cup [1, 2] \times [0, 3] \times [1, 2] \cup [0, 3] \times [0, 3] \times [1, 2])$  which is a cube where three perpendicular cubic holes have been removed.

Answer: Use the divergence theorem:  $\operatorname{div}(\vec{F}) = 2$  and so  $\iiint_G \operatorname{div}(\vec{F}) \, dV = 2 \iiint_G \, dV = 2 \operatorname{Vol}(G) = 2(27 - 7) = 40$ . Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

To the proof of the divergence theorem: Consider a small box  $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$ . The flux of  $\vec{F} = \langle P, Q, R \rangle$  through the the faces perpendicular to the  $x$ -axes is  $[\vec{F}(x+dx, y, z) \cdot \langle 1, 0, 0 \rangle + \vec{F}(x, y, z) \cdot \langle -1, 0, 0 \rangle] dydz = P(x+dx, y, z) - P(x, y, z) = P_x dx dy dz$ . Similarly, the flux through the  $y$ -boundaries is  $P_y dy dx dz$  and the flux through the two  $z$ -boundaries is  $P_z dz dx dy$ . The total flux through the faces of the cube is  $(P_x + P_y + P_z) dx dy dz = \operatorname{div}(\vec{F}) dx dy dz$ . A general solid can be approximated by a union of small cubes. The sum of the fluxes over all the cubes is the sum of the fluxes through the faces which do not touch an other box. Fluxes through adjacent sides cancel. The sum of all the infinitesimal fluxes of the cubes is the flux through the boundary of the union. The sum of all the  $\operatorname{div}(\vec{F}) dx dy dz$  is a Riemann sum approximation for the integral  $\iiint_G \operatorname{div}(\vec{F}) \, dV$ . In the limit, where  $dx, dy, dz$  goes to zero, it leads to the divergence theorem.

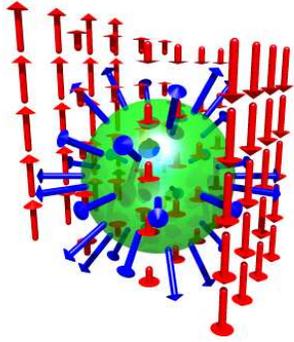


Problem. Find the flux of  $\operatorname{curl}(F)$  through a torus if  $\vec{F} = \langle yz^2, z + \sin(x) + y, \cos(x) \rangle$  and the torus has the parametrization

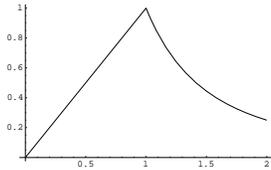
$$\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi) \rangle.$$

The answer is 0 because the divergence of  $\operatorname{curl}(F)$  is zero. By the divergence theorem, the flux is zero.

Similarly as Green's theorem allowed to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field  $\vec{F}(x, y, z) = \langle x, 0, 0 \rangle$  which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid:  $\iint_{\partial G} \langle x, 0, 0 \rangle \cdot d\vec{S} = \operatorname{Vol}(G)$ .



Example: How heavy are we, at distance  $r$  from the center of the earth? The law of gravity can be formulated as  $\text{div}(\vec{F}) = 4\pi\rho$ , where  $\rho$  is the mass density. We assume that the earth is a ball of radius  $R$ . By rotational symmetry, the gravitational force is normal to the surface:  $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/\|\vec{x}\|$ . The flux of  $\vec{F}$  through a ball of radius  $r$  is  $\int_{S_r} \vec{F}(x) \cdot d\vec{S} = 4\pi r^2 \vec{F}(r)$ . By the **divergence theorem**, this is  $4\pi M_r = 4\pi \int \int_{B_r} \rho(x) dV$ , where  $M_r$  is the mass of the material inside  $S_r$ . We have  $(4\pi)^2 \rho r^3 / 3 = 4\pi r^2 \vec{F}(r)$  for  $r < R$  and  $(4\pi)^2 \rho R^3 / 3 = 4\pi r^2 \vec{F}(r)$  for  $r \geq R$ . Inside the earth, the gravitational force  $\vec{F}(r) = 4\pi\rho r/3$ . Outside the earth, it satisfies  $\vec{F}(r) = M/r^2$  with  $M = 4\pi R^3 \rho / 3$ .



We are now at the end of the course. Lets have an overview over the integral theorems:

The fundamental theorem for line integrals, Green's theorem, Stokes theorem and divergence theorem are all incarnation of a single theorem, which has the form  $\int_A dF = \int_{\delta A} F$ , where  $dF$  is a **exterior derivative** of  $F$  and  $\delta A$  is the **boundary** of  $A$ . They all generalize the fundamental theorem of calculus.

### Fundamental theorem of line integrals

If  $C : \vec{r}(t), t \in [a, b]$  is a curve and  $f$  is a function. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Consequences.

- 1) For closed curves, the line integral  $\int_C \nabla f \cdot d\vec{r}$  is zero.
- 2) Gradient fields are **path independent**: if  $\vec{F} = \nabla f$ , then the line integral between two points  $P$  and  $Q$  does not depend on the path connecting the two points.
- 3) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus**  $\int_a^b f'(x) dx = f(b) - f(a)$
- 4) The theorem justifies the name **conservative** for gradient vector fields.
- 5) The term "potential" was coined by George Green who lived from 1783-1841.

**Problem.** Let  $f(x, y, z) = x^2 + y^4 + z$ . Find the line integral of the vector field  $\vec{F}(x, y, z) = \nabla f(x, y, z)$  along the path  $\vec{r}(t) = \langle \cos(5t), \sin(2t), t^2 \rangle$  from  $t = 0$  to  $t = 2\pi$ .

**Solution.**  $\vec{r}(0) = \langle 1, 0, 0 \rangle$  and  $\vec{r}(2\pi) = \langle 1, 0, 4\pi^2 \rangle$  and  $f(\vec{r}(0)) = 1$  and  $f(\vec{r}(2\pi)) = 1 + 4\pi^2$ . The fundamental theorem of line integral gives  $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = 4\pi^2$ .

**Green's theorem.** If  $R$  is a region in the plane with boundary  $C$  and  $\vec{F} = \langle P, Q \rangle$  is a vector field, then

$$\int \int_R \text{curl}(\vec{F}) dx dy = \int_C \vec{F} \cdot d\vec{r}.$$

### Remarks.

- 1) It can be used to switch from double integrals to one dimensional integrals.
- 2) The curve is oriented in such a way that the region is to the left.
- 3) The boundary of the curve can consist of piecewise smooth pieces.
- 4) If  $C : t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$ , the line integral is  $\int_a^b \langle P(x(t), y(t)), Q(x(t), y(t)) \rangle \cdot \langle x'(t), y'(t) \rangle dt$ .
- 5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
- 6) If  $\text{curl}(\vec{F}) = 0$  in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
- 7) Taking  $\vec{F}(x, y) = \langle -y, 0 \rangle$  or  $\vec{F}(x, y) = \langle 0, x \rangle$  gives **area formulas**.

**Problem.** Find the line integral of the vector field  $\vec{F}(x, y) = \langle x^4 + \sin(x) + y, x + y^3 \rangle$  along the path  $\vec{r}(t) = \langle \cos(t), 5 \sin(t) + \log(1 + \sin(t)) \rangle$ , where  $t$  runs from  $t = 0$  to  $t = \pi$ .

**Solution.**  $\text{curl}(\vec{F}) = 0$  implies that the line integral depends only on the end points  $(0, 1), (0, -1)$  of the path. Take the simpler path  $\vec{r}(t) = \langle -t, 0 \rangle, -1 \leq t \leq 1$ , which has velocity  $\vec{r}'(t) = \langle -1, 0 \rangle$ . The line integral is  $\int_{-1}^1 \langle t^4 - \sin(t), -t \rangle \cdot \langle -1, 0 \rangle dt = -t^5/5|_{-1}^1 = -2/5$ .

**Remark** We could also find a potential  $f(x, y) = x^5/5 - \cos(x) + xy + y^5/4$ . It has the property that  $\text{grad}(f) = F$ . Again, we get  $f(0, -1) - f(0, 1) = -1/5 - 1/5 = -2/5$ .

**Stokes theorem.** If  $S$  is a surface in space with boundary  $C$  and  $\vec{F}$  is a vector field, then

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

**Remarks.**

- 1) Stokes theorem allows to derive Greens theorem: if  $\vec{F}$  is  $z$ -independent and the surface  $S$  is contained in the  $xy$ -plane, one obtains the result of Green.
- 2) The orientation of  $C$  is such that if you walk along  $C$  and have your head in the direction of the normal vector  $\vec{r}_u \times \vec{r}_v$ , then the surface to your left.
- 3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).
- 4) The flux of the curl of a vector field does not depend on the surface  $S$ , only on the boundary of  $S$ . This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve.
- 5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

**Problem.** Compute the line integral of  $\vec{F}(x, y, z) = \langle x^3 + xy, y, z \rangle$  along the polygonal path  $C$  connecting the points  $(0, 0, 0)$ ,  $(2, 0, 0)$ ,  $(2, 1, 0)$ ,  $(0, 1, 0)$ .

**Solution.** The path  $C$  bounds a surface  $S : \vec{r}(u, v) = \langle u, v, 0 \rangle$  parameterized by  $R = [0, 2] \times [0, 1]$ . By Stokes theorem, the line integral is equal to the flux of  $\text{curl}(\vec{F})(x, y, z) = \langle 0, 0, -x \rangle$  through  $S$ . The normal vector of  $S$  is  $\vec{r}_u \times \vec{r}_v = \langle 1, 0, 0 \rangle \times \langle 0, 1, 0 \rangle = \langle 0, 0, 1 \rangle$  so that  $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 \langle 0, 0, -u \rangle \cdot \langle 0, 0, 1 \rangle dudv = \int_0^2 \int_0^1 -u dudv = -2$ .

**Divergence theorem** If  $S$  is the boundary of a region  $B$  in space with boundary  $S$  and  $\vec{F}$  is a vector field, then

$$\int \int \int_B \text{div}(\vec{F}) dV = \int \int_S \vec{F} \cdot d\vec{S}.$$

**Remarks.**

- 1) The divergence theorem is also called **Gauss theorem**.
- 2) It can be helpful to determine the flux of vector fields through surfaces.
- 3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
- 4) For divergence free vector fields  $\vec{F}$ , the flux through a closed surface is zero. Such fields  $\vec{F}$  are also called **incompressible** or **source free**.

**Problem.** Compute the flux of the vector field  $\vec{F}(x, y, z) = \langle -x, y, z^2 \rangle$  through the boundary  $S$  of the rectangular box  $[0, 3] \times [-1, 2] \times [1, 2]$ .

**Solution.** By Gauss theorem, the flux is equal to the triple integral of  $\text{div}(F) = 2z$  over the box:  $\int_0^3 \int_{-1}^2 \int_1^2 2z dx dy dz = (3-0)(2-(-1))(2-1) = 27$ .

Finally, lets mention how all these theorems fit together. It turns out that in  $d$ -dimensions there

are  $d$  theorems. We have here seen the situation in dimension 2 and 3, but it would continue. The fundamental theorem of line integrals generalizes directly to higher dimensions: just define line integrals in  $d$  dimensions. Also the divergence theorem (aka Gauss theorem) generalizes directly since an  $n$ -dimensional integral in  $n$  dimensions is obvious to do as is the concept of divergence. The generalization of curl and flux is more subtle, since in 4 dimensions already, the curl of a vector field is a 6 dimensional object. It is a  $d(d-1)/2$  dimensional object in general.

dimension	1D objects	2D objects	3D objects	4D objects
1	FTC	-	-	-
2	FTL	Green	-	-
3	FTL	Stokes	Gauss	-
4	FTL	Stokes 1	Stokes 2	Gauss

To formulate calculus in dimensions larger than 3, one needs also to generalize the derivatives grad, curl and div. Look at the Pascal triangle: in one dimensions, there is a derivative  $f(x) \rightarrow f'(x)$  from scalar to scalar functions. It corresponds to the entry  $1-1$  in the Pascal triangle. The next entry  $1-2-1$  corresponds to differentiation in two dimensions, where we have the gradient  $f \rightarrow \nabla f$  mapping a scalar function to a vector field with 2 components. And we have a curl,  $F \rightarrow \text{curl}(F)$  which corresponds to the transition  $2-1$ . The situation in three dimensions is captured by the entry  $1-3-3-1$  in the Pascal triangle. The first derivative  $1-3$  is the gradient. The second derivative  $3-3$  is the curl and the third derivative  $3-1$  is the divergence. In four dimensions, we would have to look at  $1-4-6-4-1$ . The first derivative  $1-4$  is still the gradient. Then we have a first curl, which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps an object with 6 components back to a vector field we would have to look at  $1-4-6-4-1$ . The first derivative  $1-4$  is still the gradient. Then we have a first curl  $4-6$ , which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps such so called 2 forms back to vector fields  $6-4$  and then there is the divergence  $4-1$  which maps a vector field back to a scalar field. When setting up calculus in higher dimensions, one talks about **differential forms** instead of scalar fields vector fields. If  $\vec{F}$  is a one form, then  $\text{curl}(\vec{F})$  is a 2-form, which happens in three dimensions to be a vector field again. The divergence of a 2-form is a 3 form, which is a scalar field again. The general formalism defines a derivative  $d$  called **exterior derivative** on differential forms as well as integration of such  $k$  forms on  $k$  dimensional objects. There is a boundary operation  $\delta$  which maps a  $k$ -dimensional object into a  $k-1$  dimensional object. This boundary operation is dual to differentiation and they both satisfy the same relation  $dd(F) = 0$  and  $\delta\delta G = 0$ . Differentiation and integration are linked by the general Stokes theorem

$$\int_{\delta G} F = \int_G dF$$

which becomes a single theorem generalizing the fundamental theorem of calculus. It is the **fundamental theorem of multivariable calculus**.