7/9/2009 FIRST HOURLY PRACTICE IV Maths 21a, O.Knill, Summer 2009

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points) No justifications are needed.

1) [T] [F] The vector $\vec{v} = \langle 1, -1, 0 \rangle$ is perpendicular to the line $1 - x = 1 - y = 1 - z$.

Solution:
The line contains the vector $\langle 1, 1, 1 \rangle$ which is perpendicular to the vector $\langle 1, -1, 0 \rangle$.

2) [T] [F] The length of the vector $\vec{v} = \langle 1, 0, -1 \rangle$ is 2.

Solution:
It is the square root of 2.

3) [T] [F] $\langle 1, 2, 3 \rangle \times \langle 2, 4, 6 \rangle = \langle 0, 0, 0 \rangle$.

Solution:
The vectors are parallel. So, indeed their cross product is the zero vector.

4) [T] [F] The distance between a point P and a line through the origin O can not be larger than $|OP|$.

Solution:
The distance is the smallest distance between P and any point in the line. So, it is smaller than the distance between P and O.

5) [T] [F] The vectors $\vec{u} = \langle 1, -1, 0 \rangle$ and $\vec{PQ}$ with $P = (1, 1, 1)$ and $Q = (2, 2, 2)$ are parallel.

Solution:
They are perpendicular, not parallel

6) [T] [F] The surface $y^2 - z^2 = 1$ is an elliptical paraboloid.

Solution:
It is a cylindrical hyperboloid.

7) T **F** If two vectors $\vec{v}, \vec{w}$ are orthogonal, then their cross product is the zero vector.

Solution:
No, their **dot** product is zero.

8) **T** F The surface $x^2 + y^2 - z^2 = 2x$ is called a one-sheeted hyperboloid.

Solution:
Complete the square to get $(x-1)^2 + y^2 - z^2 = 1$.

9) **T** F The set of points which have distance 1 from the x-axis is a cylinder.

Solution:
Yes, it is $y^2 + z^2 = 1$.

10) **T** F If in spherical coordinates a point is given by $(\rho, \theta, \phi) = (1, \pi, \pi/2)$, then its Euclidean coordinates are $(x, y, z) = (-1, 0, 0)$.

Solution:
Just make a picture.

11) **T** F The volume of a parallelepiped spanned by $(1,1,1), (1,0,0)$ and $(0,1,2)$ is equal to 1.

Solution:
The triple scalare product is -1 but the volume is 1.

12) **T** F The two planes $x + y - z = 1$ and $-2x - 2z + 2z = 5$ intersect in a line.

Solution:
The second equation simplifies to $x = -5/2$. The planes intersect in a line. The original intention had been to have the planes parallel (replace -2z by 2z) but we left it like that.

13) T F In cylindrical coordinates, the equation $r^2 = z$ defines a paraboloid.

Solution:
Indeed, $x^2 + y^2 = z$ is an elliptical paraboloid.

14) T F The vector projection of the vector $(1, 1, 1)$ onto the vector $\langle 0, 2, 0\rangle$ is $\langle 0, 1, 0\rangle$.

Solution:
Use the formula or take the dot product with j.

15) T F The point given in cylindrical coordinates as $(r, \theta, z) = (1, \pi/2, 1)$ is in Cartesian coordinates the point $(x, y, z) = (0, 1, 1)$.

Solution:
Just make a picture.

16) T F If $z = g(x, y)$ is a graph then $\vec{r}(u, v) = (u, v, g(u, v))$ is a parameterization of the surface.

Solution:
Graphs are one of the 4 basic surface types we have introduced.

17) T F There is a function $f(x, y, z)$ such that $f(x, y, z) = 1$ is a hyperboloid and $f(x, y, z) = -1$ is a paraboloid.

Solution:
Go through the list of functions which define hyperboloids or paraboloids. Hyperboloids contain squared quantities x, y, z, paraboloids are graphs and one of the variables appears only in a linear fashion.

18) T F The projection of $\vec{v} = \langle 1, 1, 1\rangle$ onto the vector $\langle 1, 0, 0\rangle$ is $\langle \sqrt{3}, 0, 0\rangle$.

Solution:
It is $(1, 0, 0)$.

19) | T | F | The points satisfying $(x-1)^2 - (y-1)^2 + (z+1)^2 = 1$ form a hyperboloid

Solution:
Indeed, it is a one-sheeted hyperboloid

20) | T | F | The equations $x = y = z$ describe a line which contains the vector $\langle 1, 1, 1 \rangle$.

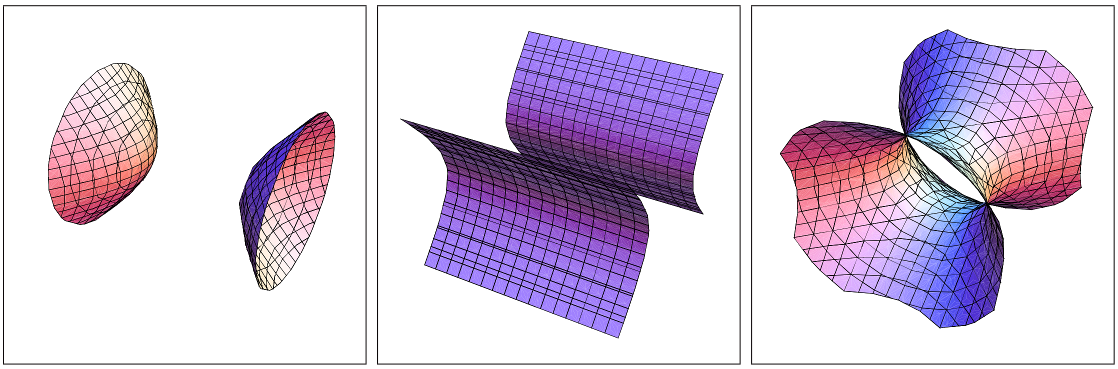
Solution:
Yes, it is a special case of a symmetric equation, where the denominators are $a = 1$,$b = 1$ and $c = 1$.

| | Total |

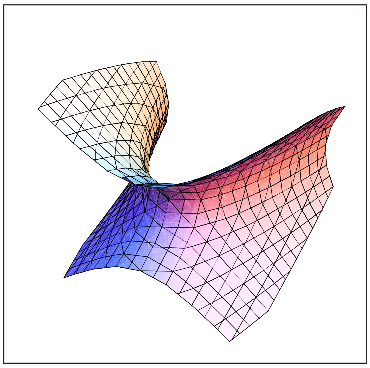
Problem 2) (10 points)

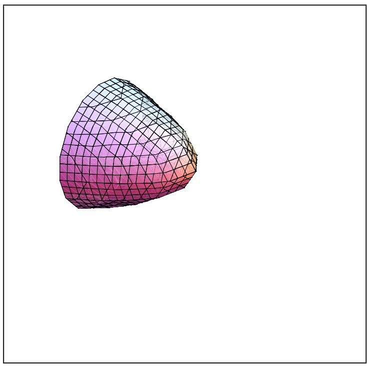
Match the equations $g(x, y, z) = d$ with the surfaces.

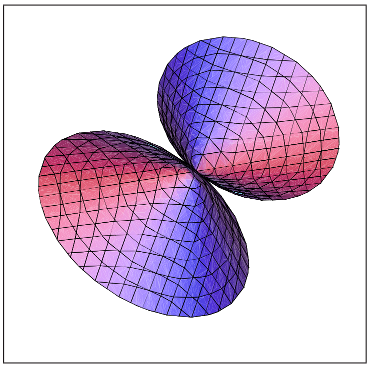
I II III



IV V VI







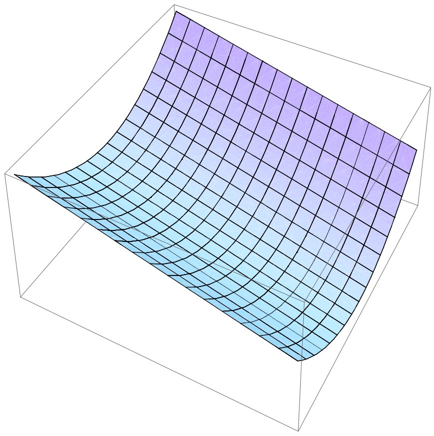
Enter I,II,III,IV,V,VI here	Equation
	$x^2 + y^2 - z^2 = 0$
	$x^2 - y^2 - z^2 = 1$
	$x^2 - y^2 + z^2 = 1$
	$x^2 - y - z^2 = 1$
	$x^2 - z^2 = 1$
	$x + y^2 + z^2 = 0$

Solution:

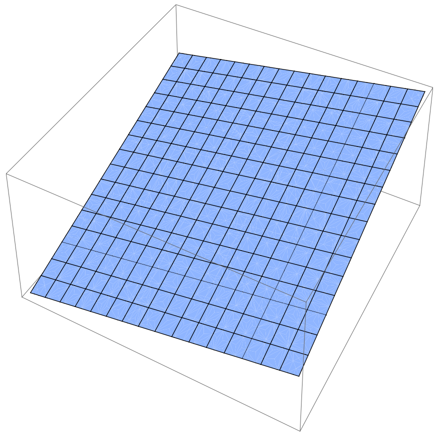
Enter I,II,III,IV,V,VI here	Equation
VI	$x^2 + y^2 - z^2 = 0$
I	$x^2 - y^2 - z^2 = 1$
III	$x^2 - y^2 + z^2 = 1$
IV	$x^2 - y - z^2 = 1$
II	$x^2 - z^2 - 1$
V	$x + y^2 + z^2 = 0$

Problem 3) (10 points)

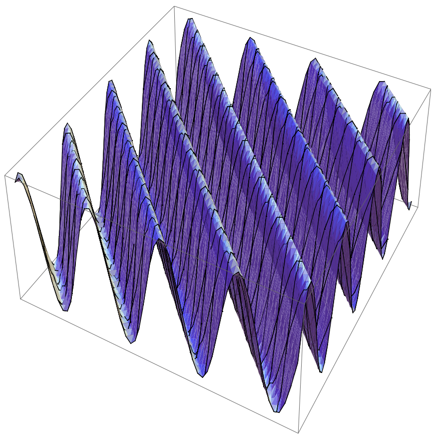
Match the functions with their graphs. No justifications are needed.



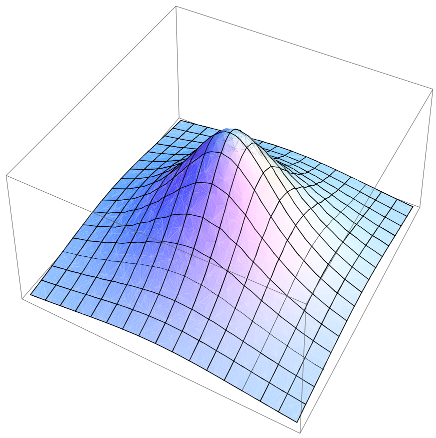
I



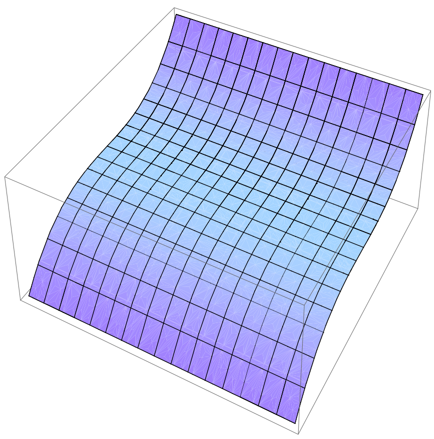
II



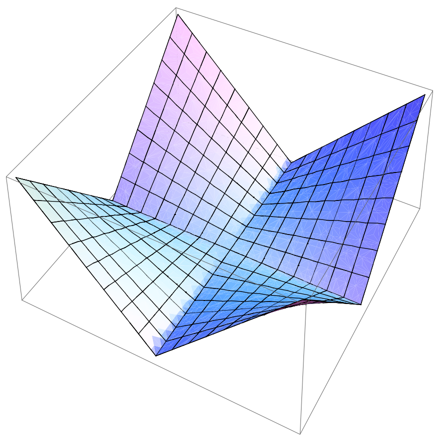
III



IV



V



VI

Enter I,II,III,IV,V or VI here	Equation
	$z = 3x + 5y + 1$
	$z = x/(1 + x^2 + y^2)$
	$z = y^2 - x$
	$z = \|x\| \cdot \|y\|$
	$z = \sin(6(x + y))$
	$z = y^3$

Solution:

Enter I,II,III,IV,V or VI here	Equation
II	$z = 3x + 5y + 1$
IV	$z = x/(1 + x^2 + y^2)$
I	$z = y^2 - x$
VI	$z = \lvert x\rvert \cdot \lvert y\rvert$
III	$z = \sin(6(x + y))$
V	$z = y^3$

Problem 4) (10 points)

Au usual, $\text{proj}_{\vec{v}}(\vec{w})$ is the vector projection of $\vec{w}$ onto the vector $\vec{v}$ and $\text{comp}_{\vec{v}}(\vec{w})$ is the scalar projection. Compute:

a) $(4, 5, 1) \cdot (1, -1, 1)$

b) $(-1, 1, 3) \times (1, 1, 1)$

c) $(2, 1, 3) \cdot ((3, 4, 5) \times (1, 1, 3))$.

d) $\vec{\text{proj}}_{\langle 1,0,0\rangle} \langle 7, 3, 2\rangle$

e) $|\langle 0, 3, 4\rangle| + \text{comp}_{\langle 1,0,0\rangle} \langle 3, 4, 5\rangle$

Solution:

a) $\boxed{4 - 5 + 1 = 0}$.

b) $\boxed{(-2, 4, -2)}$.

c) $((3, 4, 5) \times (2, 1, 3)) = (7, -4, -1)$ and the result is $\boxed{7}$.

d) This is a vector projection: $\boxed{(7, 0, 0)}$.

e) $5 + 3 = \boxed{8}$.

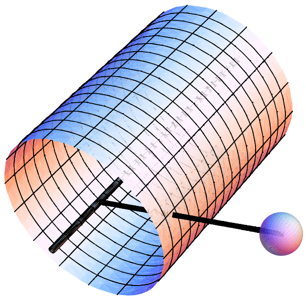
Problem 5) (10 points)

We want to find the distance between the point $P = (5, 0, -5)$ and the cylinder which has an axes going through the points $A = (1, 1, 1)$ and $B = (0, 2, 1)$ and radius 1. To do so:

a) (4 points) Find first a parameterization $\vec{r}(t) = Q + t\vec{v}$ of the line.

b) (4 points) Find the distance between P and the line.

c) (2 points) Now find the distance between P and the cylinder.



Solution:
a) $\vec{v} = \vec{AB} = (\langle 0,2,1\rangle - \langle 1,1,1\rangle) = \langle -1,1,0\rangle$ is in the line. Because we can take $Q = A$, a parameterization is $\vec{r}(t) = \langle 1,0,-1\rangle + t\langle -1,1,0\rangle$. We can write it also as $\boxed{\vec{r}(t) = \langle 1-t, t, (-1)\rangle}$. (note that there are other parametrizations which are ok too).
b) We use the distance formula:

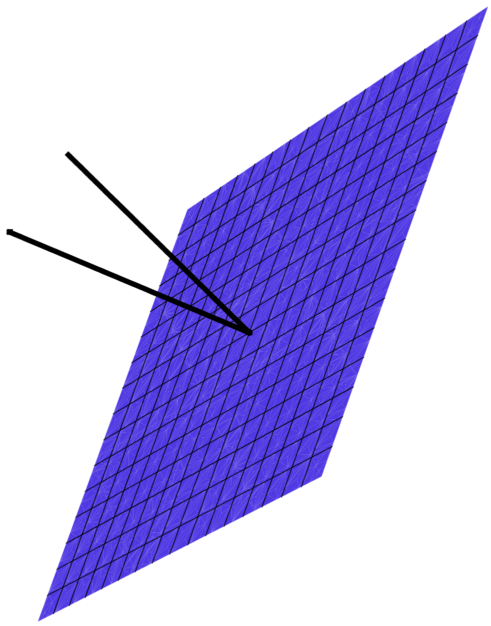
$$\frac{|\vec{AP} \times \vec{v}|}{|\vec{v}|} = \frac{|\langle 4,-1,-6\rangle \times \langle -1,1,0\rangle|}{|\langle -1,1,0\rangle|} = \frac{|\langle 6,6,3\rangle|}{\sqrt{2}} .$$

The answer is $\boxed{9/\sqrt{2}}$.
c) Because the distance to the cylinder is 1 less, the final answer is $\boxed{9/\sqrt{2} - 1}$.

Problem 6) (10 points)

The angle between a line and a plane is defined as $\pi/2 - \alpha$, if α is the angle between the normal vector to the plane and a vector in the line.

Find the angle between the plane $x + y - z/2 = 1$ and the line $\vec{r}(t) = \langle 1,1,1\rangle + t\langle 1,1,3\rangle$.



Solution:
The normal vector to the plane is $\vec{n} = \langle 1, 1, -1/2 \rangle$. We have

$$\cos(\alpha) = \frac{|\langle 1,1,3\rangle \cdot \langle 1,1,-1/2\rangle|}{||\langle 1,1,3\rangle| \cdot |\langle 1,1,-1/2\rangle|} = 1/(3\sqrt{11} \; .$$

So that $\alpha = \arccos(1/3\sqrt{11})$ and the answer is $\boxed{\pi/2 - \arccos(\frac{1}{3\sqrt{11}})}$.

Problem 7) (10 points)

Find the implicit equation

$$ax + by + cz = d$$

of the plane which contains the line

$$\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle -1, 1, 1\rangle + t\langle 3, 4, 5\rangle$$

and the point $Q = (7, 7, 9)$.

Solution:
The vector $\vec{v} = \langle 3, 4, 5 \rangle$ as well as the vector $\vec{PQ} = \langle 8, 6, 8 \rangle$ are in the plane. Their cross product $\vec{v} \times \vec{PQ} = \langle 2, 16, -14 \rangle$ is perpendicular to the plane. The plane has therefore the equation $2x + 16y - 14z = d$. The constant $d = 0$ can be obtained by plugging in a point. Fro example, the point $\langle -1, 1, 1 \rangle$. The final answer is $\boxed{2x + 16y - 14z = 0}$.

Problem 8) (10 points)

If you cut the plane $x + 3y + 2z = 6$ by the coordinate planes, we get a triangle.

a) (4 points) What are the vertices A, B, C of this triangle?

b) (6 points) Find the area of this triangle.

Solution:
a) The points are the intercepts. They can be obtained by putting $x = y = 0$ or $x = z = 0$ or $y = z = 0$. The points are $A = (6, 0, 0), B = (0, 2, 0), C = (0, 0, 3)$.
b) The area is half of the length of the cross product of the side vectors of the triangle because a parallelogram is made up of two equal triangles. Take the cross product of AB with AC which is

$$\langle -6, 2, 0 \rangle \times \langle -6, 0, 3 \rangle = \langle 6, 18, 12 \rangle = 6\langle 1, 3, 2 \rangle \ .$$

The length of this vector, $6\sqrt{14} = \sqrt{504}$ is twice the area of the triangle. The result is $\boxed{3\sqrt{14} = \sqrt{126}}$.

Problem 9) (10 points)

Complete in the following table with missing parameterizations, implicit descriptions or graphs. Each line is a specific surface.

parametric $\vec{r}(u,v) =$	implicit	graph	name
		$z = x^2 + y^2$	
$\langle \sin(v)\cos(u), \sin(v)\sin(u), \cos(v) \rangle$			upper half sphere
	$x^2 - y^2 + z = 0$		
$\langle 1,1,1 \rangle + u\langle 1,0,1 \rangle + v\langle 0,1,1 \rangle$			

Solution:

parametric $\vec{r}(u,v) =$	implicit	graph	name
$\langle u, v, u^2 + v^2 \rangle$	$z - x^2 - y^2$	$z = x^2 + y^2$	elliptic paraboloid
$\langle \sin(v)\cos(u), \sin(v)\sin(u), \cos(v) \rangle$	$z - \sqrt{1 - x^2 - y^2}$	$z = \sqrt{1 - x^2 - y^2}$	upper half sphere
$\langle u, v, u^2 - v^2 \rangle$	$x^2 - y^2 + z = 0$	$z = y^2 - x^2$	hyperbolic paraboloi
$\langle 1,1,1 \rangle + u\langle 1,0,1 \rangle + v\langle 0,1,1 \rangle$	$x + y - z = 1$	$z = x + y - 1$	plane

Problem 10) (10 points)

What is the distance between the two cylinders $x^2 + y^2 = 1$ and $(z-2)^2 + (x-5)^2 = 4$?

Hint. Find the distance between their central axes first.

Solution:
The parametrization of the first axis is $\vec{r}(t) = \langle 0,0,0\rangle + t\langle 0,0,1\rangle$. The parametrization of the second axis is $\vec{r}(t) = \langle 5,0,2\rangle + t\langle 0,1,0\rangle$. The distance is $5,\vec{0},2 \cdot (\langle 0,0,1\rangle \times \langle 0,1,0\rangle/|\langle 0,1,0\rangle| = \langle 5,0,2\rangle \cdot \langle 1,0,0\rangle = 5$. The distance between the cylinders is $5-1-2=2$. The final answer is $\boxed{2}$.

Problem 11) (10 points)

Find the symmetric equation

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

of the intersection of the two planes

$$2x+y+z=4$$

and

$$x-y+2z=5 .$$

Solution:
We can find the direction of the intersection by taking the cross product of $\langle 2,1,1\rangle$ with $\langle 1,-1,2\rangle$ which is $\langle 3,-3,-3\rangle$. One point of intersection is $(3,-2,0)$. We have therefore the parametrization $\vec{r}(t) = \langle 3,-2,0\rangle + t\langle 3,-3,-3\rangle = \langle 3+3t,-2-3t,-3t\rangle$. The symmetric equation is

$$\frac{x-3}{3} = \frac{y+2}{3} = \frac{z}{-3} .$$

An other solution can be found by finding two intersection points P,Q and getting the direction of the intersection as $\vec{PQ}$. Note that there are many solutions to this problem, depending on which point on the line was.