Math S21a: Multivariable calculus **Oliver Knill, Summer 2011**

1: Geometry and Distance

The arena for multivariable calculus is the two-dimensional **plane** and the three dimensional **space**.

> A point in the **plane** has two **coordinates** $P = (x, y)$. A point in space is determined by three coordinates $P = (x, y, z)$. The signs of the coordinates define 4 **quadrants** in the plane and 8 **octants** in space. These regions by intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

In two dimensions, the x-coordinate usually directs to the "east" and the y-coordinate points "north". In three dimensions the usual coordinate system has the xy-plane as the "ground" and the z-coordinate axes pointing "up".

1 $P = (2, -3)$ is in the forth quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -5)$ is on the negative z axis. The point $(1, 2, -3)$ is below the xy-plane.

2 **Problem.** Find the midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$. **Answer.** The midpoint is obtained by taking the average of each coordinate $M = (P + Q)/2 = (-1, 3, 6)$.

3 In computer graphics of photography, the xy-plane contains the retina or film plate. The z coordinate measures the distance towards the viewer. In this **photographic coordinate** system your eyes and mouth are in the plane $z = 0$ and your nose points in the z direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0)$, $(-1, 0, 0)$, then the tip of your nose might have the coordinates $(0, -1, 1)$.

> The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

This Euclidean distance is a definition but motivated by **Pythagoras theorem**.

4 **Problem:** Find the distance $d(P, Q)$ between the points $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

Remarks.
1) Distances can be introduced more abstractly: take any nonnegative function $d(P, Q)$ which satisfies the **triangle inequality** $d(P, Q) + d(Q, R) \geq d(P, R)$ and $d(P, Q) = 0$ if and only if $P = Q$. A set X with such a distance function d is called a **metric space**. Examples of distances are the **Manhatten distance** $d_m(P, Q) = |x - a| + |y - b|$, the **quartic distance** $d_4(P, Q) = ((x - a)^4 + (y - b)^4$ or the **Fermat distance** $d_f(x, y) = d(x, y)$ if $y > 0$ and $d_f(x, y) = 1.33d(x, y)$ if $y < 0$. The constant 1.33 is the **refractive index** and models the upper half plane being filled with air and the lower half plane with water. Shortest paths are bent at the water surface. Each of these distances d, d_m, d_4, d_f make the plane a different metric space.
2) It is **symmetry** which distinguishes the Euclidean distance as the most natural one. The Euclidean distance is determined by $d((1, 0, 0), (0, 0, 0)) = 1$, rotational and translational and scale symmetry $d(\lambda P, \lambda Q) = \lambda d(P, Q)$.
3) We usually work with a **right handed coordinate system**, where the x, y, z axes can be matched with the thumb, pointing and middle finger of the **right hand**. The photographers coordinate system is an example of a **left handed coordinate system**. The x, y, z axes are matched with the thumb and pointing finger and middle finger of the left hand. Nature is not oblivious to parity. Some laws of particle physics are different when they are observed in a mirror. Coordinate systems with different parity can not be rotated into each other.

Points, **curves**, **surfaces** and **solid bodies** are geometric objects which can be described with **functions of several variables**. An example of a curve is a line, an example of a surface is a plane, an example of a solid is the interior of a sphere. We focus in this first lecture on spheres or circles.

> A **circle** of radius r centered at $P = (a, b)$ is the collection of points in the plane which have distance r from P.

> A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2+(y-b)^2+(z-c)^2 = \rho^2$.

> An **ellipse** is the collection of points P in the plane for which the sum $d(P, A) + d(P, B)$ of the distances to two points A, B is a fixed constant l larger than $d(A, B)$. This allows to draw the ellipse with a string of length l attached at A, B. An algebraic equivalent description is the set of points satisfying an equation $x^2/a^2 + y^2/b^2 = 1$.

5 **Problem:** Is the point $(3, 4, 5)$ outside or inside the sphere $(x-2)^2+(y-6)^2+(z-2)^2 = 16$? **Answer**: The distance of the point to the center of the sphere is $\sqrt{1 + 4 + 9}$ which is smaller than 4 the radius of the sphere. The point is inside.

6 **Problem:** Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1, 0)$ and $B = (-1, 0)$ is equal to 3. **Answer:** Square the equation $\sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2 + 36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b can be computed as follows: because $P = (a, 0)$ satisfies this equation, $d(P, A) + d(P, B) = (a - 1) + (a + 1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0, b)$ satisfying it gives $d(Q, A) + d(P, B) = 2\sqrt{b^2 + 1} = 3$ or $b = \sqrt{5}/2$.

Here is a verification with the computer algebra system Mathematica. Writing $L = d(P, A)$ and $M = d(P, B)$ we simplify the equation $L^2 + M^2 = 3^2$. The part without square root is $((L + M)^2 + (L - M)^2)/2 - 3^2$. The remaining square root is $((L + M)^2 - (L - M)^2)/2$. Now square both and set them equal to see the equation $20x^2 + 36y^2 = 45$.

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L=Sqrt[(x-1)^2+y^2]; M=Sqrt[(x+1)^2+y^2];
Simplify[(((L+M)^2 + (L-M)^2)/2-3^2)^2 == (((L+M)^2-(L-M)^2)/2)^2]
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The **completion of the square** of an equation $x^2 + bx + c = 0$ is the idea to add $(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives the solution $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.

7 The equation $2x^2 - 10x + 12 = 0$ is equivalent to $x^2 + 5x = -6$. Adding $(5/2)^2$ on both sides gives $(x + 5/2)^2 = 1/4$ so that $x = 2$ or $x = 3$.

8 The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after completion of the square $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(5/2, 1, 0)$ and **radius** $5/2$.

The method is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work *"Compendium on Calculation by Completion and Reduction"* was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad. [1] In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. Ideas do not grow in a vacuum. Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. [2]

What happens in higher dimensions? A point in four dimensional space for example is labeled with four coordinates (t, x, y, z). In how many hyper chambers is space divided by the coordinate hyperplanes $t = 0, x = 0, y = 0, z = 0$? Answer: There are 16 hyper-regions and each of them contains one of the 16 points (x, y, z, w), where x, y, z, w are either $+1$ or -1.

Homework

1 Describe and sketch the set of points $P = (x, y, z)$ in three dimensional space $\mathbf{R}^3$ represented by

a) $(x - 1)^2 + z^2 = 4$

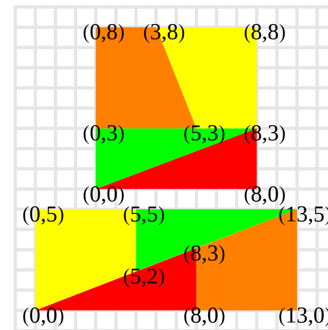
b) $x - y - z = 2$

c) $xyz = 0$

d) $x^2 = y$

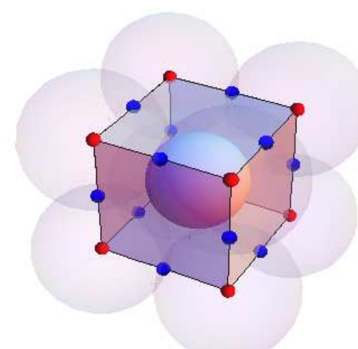
2 a) Find the distances of $P = (3, 4, 0)$ to each of the 3 coordinate axes.
b) Find the distances of $P = (1, 2, 5)$ to each of the 3 coordinate planes.

3 Below you see two rectangles. One has the area $8 \cdot 8 = 64$. The other has the area $65 = 13 \cdot 5$. But these triangles are made up by triangles or trapezoids which match. Measure various distances to see what is going on.



4 Find the center and radius of the sphere $x^2 + 2x + y^2 - 16y + z^2 + 10z + 54 = 0$. Describe the traces of this surface, its intersection with each of the coordinate planes.

5 We place unit spheres on the corners of a unit cube of side length 2 so that adjacent spheres touch. How large is the radius of the sphere in the center of the cube kissing all the 8 spheres?



[1]The book "The mathematics of Egypt, Mesopotamia,China, India and Islam, a Sourcebook, Ed Victor Katz, page 542 contains translations of some of this work.

[2]An entertaining read is "Descartes Secret Notebook" by Amir Aczel.