

3: Cross product

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ in the plane is the scalar $v_1w_2 - v_2w_1$.

To remember this, we can write it as a determinant: take the product of the diagonal entries and subtract the product of the side diagonal. $\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle.$$

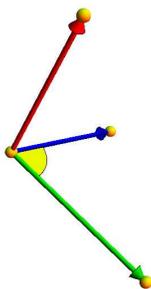
To remember it we write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2w_3 - v_3w_2) - \vec{j}(v_1w_3 - v_3w_1) + \vec{k}(v_1w_2 - v_2w_1)$.

1 The cross product of $\langle 1, 2 \rangle$ and $\langle 4, 5 \rangle$ is $5 - 8 = -3$.

2 The cross product of $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 1 \rangle$ is $\langle -13, 11, -3 \rangle$.



The cross product $\vec{v} \times \vec{w}$ is orthogonal to both \vec{v} and \vec{w} . The product is anti-commutative.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.

The **sin formula**: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof: We verify first the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by \vec{v} and \vec{w} .

Note that this was a definition so that nothing needs to be proven. But we want to make sure that the definition fits with our common intuition we have about area: $|\vec{w}| \sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. We see from the sin-formula that the area does not change if we rotate the vectors around in space because both length and angle stay the same. Area also is linear in each of the vectors v and w . If we make v twice as long, then the area gets twice as large.

$\vec{v} \times \vec{w}$ is zero exactly if \vec{v} and \vec{w} are **parallel**, that is if $\vec{v} = \lambda\vec{w}$ for some real λ .

Proof. This follows immediately from the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are also considered parallel evenso sometimes one calls this anti-parallel.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma)$.

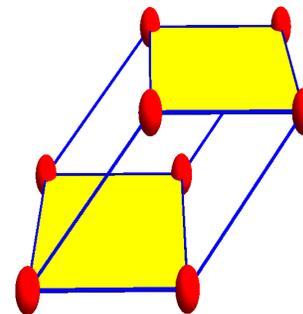
Proof. We express the area of the triangle in three different ways:

$$ab \sin(\gamma) = bc \sin(\alpha) = ac \sin(\beta).$$

Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

3 If $\vec{v} = \langle a, 0, 0 \rangle$ and $\vec{w} = \langle b \cos(\alpha), b \sin(\alpha), 0 \rangle$, then $\vec{v} \times \vec{w} = \langle 0, 0, ab \sin(\alpha) \rangle$ which has length $|ab \sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.



The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$.

The **orientation** of three vectors is the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors form a right handed coordinate system.

Again, we do not have to prove anything since we have just **defined** volume and orientation. Let us still see why this fits with our intuition about volume. The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w}))|$ which is indeed the absolute value of the triple scalar product. The vectors \vec{v}, \vec{w} and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector \vec{v} is your thumb, the second vector \vec{w} is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system.

Since the triple scalar product is linear with respect to each vector we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with a volume twice the volume.

4 Problem: Find the volume of a **cuboid** of width a length b and height c . **Answer:** The cuboid is a parallelepiped spanned by $\langle a, 0, 0 \rangle$, $\langle 0, b, 0 \rangle$ and $\langle 0, 0, c \rangle$. The triple scalar product is abc .

5 Problem Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer:** We first see that it is spanned by the vectors $\vec{u} = \langle 1, 2, 1 \rangle, \vec{v} = \langle 3, 2, 1 \rangle$, and $\vec{w} = \langle 0, 3, 1 \rangle$. We get $\vec{v} \times \vec{w} = \langle -1, -3, 9 \rangle$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

You can skip the following remarks: We have seen that in two dimensions, the cross product is a scalar and in three dimensions, the cross product is a vector. What happens in higher dimensions? There is a generalization called **wedge product** $\vec{v} \wedge \vec{w}$ but the resulting vector is in general in a new "super" space. In four dimensions for example, the wedge product between two vectors \vec{v} and \vec{w} is an object with $6 = 4(4-1)/2$ components. In n dimensions, the product has $n(n-1)/2$ components:

$$\vec{v} \wedge \vec{w} = \langle v_1 w_2 - v_2 w_1, v_1 w_3 - v_3 w_1, v_1 w_4 - v_4 w_1, v_2 w_3 - v_3 w_2, v_2 w_4 - v_4 w_2, v_3 w_4 - v_4 w_3 \rangle.$$

The length of $\vec{v} \wedge \vec{w}$ is the area of the parallelepiped spanned by \vec{v} and \vec{w} . It is only in our three dimensional space that we can identify the wedge product between two vectors as a 3-vector again. The wedge product can be extended to an algebra of dimension 2^n . It is called a **superalgebra**. An other approach to generalize the product is to use linear algebra and write a vector v as a 3×3 matrix V , now $v \times w$ corresponds to $VW - WV$. For those of you who take linear algebra:

$$V = \begin{bmatrix} 0 & -v_1 & v_3 \\ v_1 & 0 & -v_2 \\ -v_3 & v_2 & 0 \end{bmatrix}, W = \begin{bmatrix} 0 & -w_1 & w_3 \\ w_1 & 0 & -w_2 \\ -w_3 & w_2 & 0 \end{bmatrix}, VW - WV = \begin{bmatrix} 0 & -(v_2 w_3 - v_3 w_2) & v_1 w_2 - v_2 w_1 \\ v_2 w_3 - v_3 w_2 & 0 & -(v_3 w_1 - v_1 w_3) \\ v_1 w_2 - v_2 w_1 & v_3 w_1 - w_1 w_3 & 0 \end{bmatrix}.$$

This is what one calls a **Lie algebra**. This can be generalized to $n \times n$ matrices. While for $n = 2$ we got the cross product in two dimensions and for $n = 3$ the cross product in 3 dimensions, we get for $n = 4$ a cross product in six dimensions. Also the triple scalar product has a generalization in n dimensions. Given n vectors, we can write them into a $n \times n$ matrix. The absolute value of the **determinant** of this matrix defines the n -dimensional volume of the parallelepiped spanned by these n vectors.

Historically, the dot product and cross product emerged about at the same time. Determinants were studied already by Gauss in 1801, matrix multiplication in 1812 by Binet. The dot product can be seen as a special case of matrix multiplication. Only in 1844 geometry in n dimensions started to be developed by Grassman. It was Hamilton who described in 1843 first a multiplication $*$ of 4 vectors. It contains intrinsically both dot and cross product because $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-vw, v \times w)$. The cross product can also be realized using matrix multiplication $AB - BA$ of skew symmetric matrices. The proper way to see this now is tensor analysis which started in 1890 with Ricci which contains a tensor product which has an anticommutative version called **exterior algebra**, an example of a superalgebra. More Vector calculus was also developed at the same time by Clifford, Gibbs, Heaviside. This leads to mathematics developed in the 20'th century developed first by Poincare and Elie Cartan about 100 years ago. Motivated heavily by physics, this was generalized to Spinor algebras which are special cases of Clifford algebras. It is fascinating to see that these geometric constructions appear in fundamental physics in the structure of elementary particles. But the dot and cross product appears in down to earth physics: It appears in fundamental equations like the **Lorentz force** $F = q(v \times B)$ which depends on the velocity v of a particle, the charge q and the magnetic field B . An other example is the **angular momentum** $m\vec{r} \times \vec{v}$ or **Coriolis force** $-2m\vec{\omega} \times \vec{v}$, where $\vec{\omega}$ is a vector in the rotation axes and \vec{v} is the velocity.

Homework

- 1 a) Find the volume of the parallelepiped for which the base parallelogram is given by the points $(0, 0, 0), (1, 0, 1), (2, 2, 1), (1, 2, 0)$ and which has an edge connecting $(0, 0, 0)$ with $(3, 4, 5)$.
b) Find the area of the base and use a) to get the height.
- 2 a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.
- 3 To find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$, we find a vector $\langle a, b, c \rangle$ normal to the plane and fix d so that P is in the plane.

4 Verify the Lagrange formula

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

for general vectors $\vec{a}, \vec{b}, \vec{c}$ in space. The formula can be remembered as "BAC minus CAB".

5 Assume you know that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ between $\vec{u}, \vec{v}, \vec{w}$ is equal to 4. Find the values of $[\vec{v}, \vec{u}, \vec{w}]$ and $[\vec{u} + \vec{v}, \vec{v}, \vec{w}]$.