

4: Lines and Planes

A point $P = (p, q, r)$ and a vector $\vec{v} = \langle a, b, c \rangle$ define the **line**

$$L = \{ \langle p, q, r \rangle + t \langle a, b, c \rangle, t \in \mathbf{R} \} .$$

The line consists of all points obtained by adding a multiple of the vector \vec{v} to the vector $\vec{OP} = \langle p, q, r \rangle$. The line contains the point P as well as a suitably translated copy of \vec{v} . Every vector contained in the line is necessarily parallel to \vec{v} . We think about the parameter t as "time". At time $t = 0$, we are at the point P , whereas at time $t = 1$ we are at $\vec{OP} + \vec{v}$.

If t is restricted to values in a **parameter interval** $[s, u]$, then $L = \{ \langle p, q, r \rangle + t \langle a, b, c \rangle, s \leq t \leq u \}$ is a **line segment** which connects $\vec{r}(s)$ with $\vec{r}(u)$.

- 1** Problem. Get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, we form the vector $\vec{v} = \vec{PQ} = \langle 1, 3, 4 \rangle$ and get $L = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t \langle 1, 3, 4 \rangle \}$. This can be written also as $\vec{r}(t) = \langle 1 + t, 1 + 3t, 2 + 4t \rangle$. If we write $\langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t \langle 1, 3, 4 \rangle$ as a collection of equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ and solve the first equation for t :

$$L = \{ \langle x, y, z \rangle \mid (x - 1)/2 = (y - 1)/3 = (z - 2)/4 \} .$$

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = \langle a, b, c \rangle$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

Proof. Each of these expressions is equal to t . These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$.

- 2** Find the symmetric equations for the line through the two points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$, we first form the parametric equations $\langle x, y, z \rangle = \langle 0, 1, 1 \rangle + t \langle 2, 2, 3 \rangle$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$. Solving each equation for t gives the symmetric equation $x/2 = (y - 1)/2 = (z - 1)/3$.
- 3** **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a, b, c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors \vec{v}, \vec{w} define a **plane** $\Sigma = \{ \vec{OP} + t\vec{v} + s\vec{w}, \text{ where } t, s \text{ are real numbers} \}$.

- 4** An example is $\Sigma = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t \langle 2, 4, 6 \rangle + s \langle 1, 0, -1 \rangle \}$. This is called the **parametric description** of a plane.

If a plane contains the two vectors \vec{v} and \vec{w} , then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both \vec{v} and \vec{w} . Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to \vec{n} , we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0), P = (x, y, z)$, and $\vec{n} = \langle a, b, c \rangle$, this means $ax + by + cz = ax_0 + by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have just shown

The equation for a plane containing \vec{v} and \vec{w} and a point P is

$$ax + by + cz = d ,$$

where $\langle a, b, c \rangle = \vec{v} \times \vec{w}$ and d is obtained by plugging in P .

- 5** **Problem:** Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.

Answer: The plane contains the two vectors $\vec{v} = \langle 1, 2, 0 \rangle$ and $\vec{w} = \langle 2, 2, 2 \rangle$. We have $\vec{n} = \langle 4, -2, -2 \rangle$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

The **angle between the two planes** $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two vectors $\vec{n} = \langle a, b, c \rangle$ and $\vec{m} = \langle e, f, g \rangle$.

- 6** Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. Answer: find the angle between $\vec{n} = \langle 1, 1, 0 \rangle$ and $\vec{m} = \langle 1, 1, 1 \rangle$. It is $\arccos(2/\sqrt{6})$.

Finally, lets look at some distance functions.

- 1) If P is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q , then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between P and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{| \langle -7, 0, -3 \rangle \cdot \langle 2, 4, 5 \rangle |}{|\langle 2, 4, 5 \rangle|} = \frac{29}{\sqrt{45}} .$$

- 2) If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between P and the line L . Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from $P = (2, 3, 1)$ to the line $\vec{r}(t) = (1, 1, 2) + t(5, 0, 1)$, compute

$$d(P, L) = \frac{|(-1, -2, 1) \times \langle 5, 0, 1 \rangle|}{|\langle 5, 0, 1 \rangle|} = \frac{|(-2, 6, 10)|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}.$$

3) If L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L, M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines L and M . Proof: the distance is the length of the vector projection of \vec{PQ} onto $\vec{u} \times \vec{v}$ which is normal to both lines. For example, to compute the distance between $\vec{r}(t) = (2, 1, 4) + t(-1, 1, 0)$ and M is the line $\vec{s}(t) = (-1, 0, 2) + t(5, 1, 2)$ form the cross product of $\langle -1, 1, 0 \rangle$ and $\langle 5, 1, 2 \rangle$ is $\langle 2, 2, -6 \rangle$. The distance between these two lines is

$$d(L, M) = \frac{|(3, 1, 2) \cdot \langle 2, 2, -6 \rangle|}{|\langle 2, 2, -6 \rangle|} = \frac{4}{\sqrt{44}}.$$

4) To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma, \Pi) = \frac{|e - d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

$$\frac{|8 - 1|}{|\langle 5, 4, 3 \rangle|} = \frac{7}{\sqrt{50}}.$$

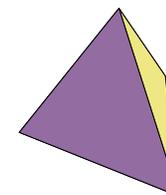
To finish this chapter, let's mention a distance problem which has a great deal of application and motivates the material of the upcoming week: The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites. Each GPS satellite sends periodically signals which are triggered by an atomic clock. While the distance to each satellite is not known, the difference from the distances to two satellites can be determined from the time delay of the two signals. This clever trick has the consequence that the receiver does not need to contain an atomic clock itself. To understand this better, we need to know about functions of three variables and surfaces.



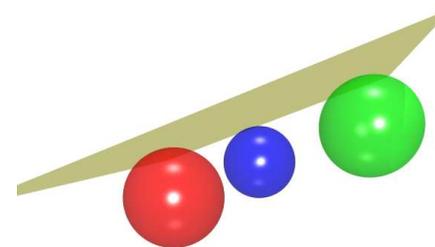
Homework

- 1 Find the parametric and symmetric equation for the line which passes through the points $P = (1, 2, 3)$ and $Q = (3, 4, 5)$.

- 2 A regular tetrahedron has vertices at the points $P_1 = (0, 0, 3), P_2 = (0, \sqrt{8}, -1), P_3 = (-\sqrt{6}, -\sqrt{2}, -1)$ and $P_4 = (\sqrt{6}, -\sqrt{2}, -1)$. Find the distance between two edges which do not intersect.



- 3 Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1 + t, y = 1 - t, z = 2t$ and intersects this line in a point Q .
- 4 Given three spheres of radius 1 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side. **Hint.** There are two such planes. You want to consider the plane through the three points A, B, C first. You only need to find one of the two possible planes touching all the spheres on the same side.



- 5 a) Find the distance between the point $P = (3, 3, 4)$ and the line $x = y = z$.
 b) Parametrize the line $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ in a) and find the minimum of the function $f(t) = d(P, \vec{r}(t))^2$. Verify that the minimal value agrees with a).