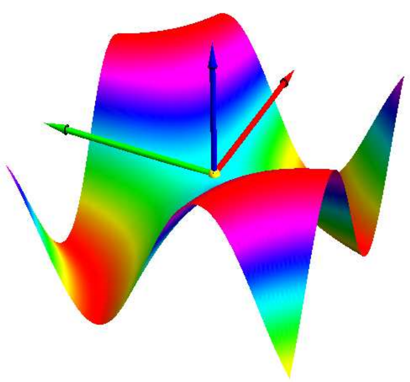
Lecture 5: Functions

A **function of two variables** $f(x,y)$ is a rule which assigns to two numbers x,y a third number $f(x,y)$. For example, the function $f(x,y)=x^2y+2x$ assigns to $(3,2)$ the number $3^2 2+6=24$.

A function is usually defined for all points (x,y) in the plane like in the case $f(x,y)=x^2+\sin(xy)$. In general, we need to restrict the function to a **domain** D in the plane like for $f(x,y)=1/y$, where (x,y) is defined everywhere except on the x-axes $y=0$. The **range** of a function f is the set of values which the function f takes. The function $f(x,y)=1+x^2$ for example takes all values ≥ 1.

The **graph** of $f(x,y)$ is the set $\{(x,y,f(x,y)) \mid (x,y) \in D\}$ in of space. Graphs allow to visualize functions of two variables. It allows us to see the function.



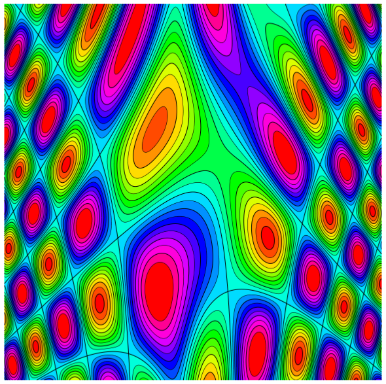
1 The graph of $f(x,y)=\sqrt{1-x^2-y^2}$ on the domain $x^2+y^2<1$ is a half sphere.

2

example function $f(x,y)$	domain D of f	range $= f(D)$ of f
$f(x,y)=\sin(3x+3y)-\log(1-x^2-y^2)$	open unit disc $x^2+y^2<1$	$[-1,\infty)$
$f(x,y)=f(x,y)=x^2+y^3-xy+\cos(xy)$	plane R^2	the real line
$f(x,y)=\sqrt{4-x^2-2y^2}$	$x^2+2y^2\leq 4$	$[0,2]$
$f(x,y)=1/(x^2+y^2-1)$	all except unit circle	the real line
$f(x,y)=1/(x^2+y^2)^2$	all except origin	positive real axis

The set $f(x,y)=c=$ const is called a **contour curve** or **level curve** of f. For example, for $f(x,y)=4x^2+3y^2$, the level curves $f=c$ are ellipses if $c>0$. Drawing several contour curves $\{f(x,y)=c\}$ produces a **contour map** of f.

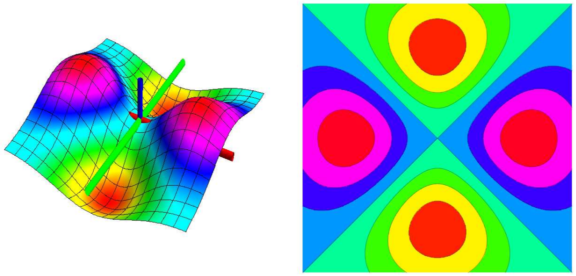
Level curves allow to visualize functions of two variables $f(x,y)$ without leaving the plane. The picture to the right for example shows the level curves of the function $\sin(xy)-\sin(x^2+v)$. Contour curves are encountered every day: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.



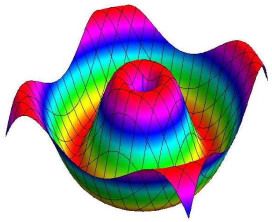
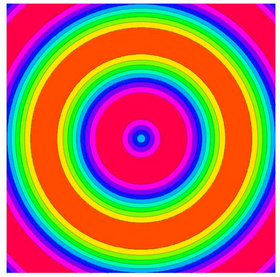
3 For $f(x,y)=x^2-y^2$, the set $x^2-y^2=0$ is the union of the lines $x=y$ and $x=-y$. The set $x^2-y^2=1$ consists of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The set $x^2-y^2=-1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

4 The function $f(x,y)=1-2x^2-y^2$ has contour curves $f(x,y)=1-2x^2+y^2=c$ which are ellipses $2x^2+y^2=1-c$ for $c<1$.

5 For the function $f(x,y)=(x^2-y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2}=c$. We can draw the curves however with the help of a computer:



6 The surface $z=f(x,y)=\sin(\sqrt{x^2+y^2})$ has concentric circles as contour curves.

In applications, we sometimes have to deal with functions which are not continuous. When plotting the rate of change of temperature of water in relation to pressure and volume for example, one experiences **phase transitions**, places where the function value can jump. Mathematicians have tamed discontinuous events with a mathematical field called "catastrophe theory".

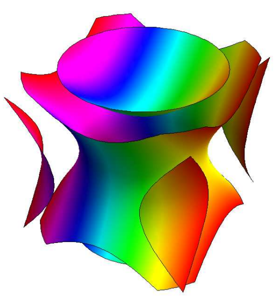
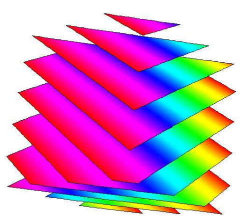
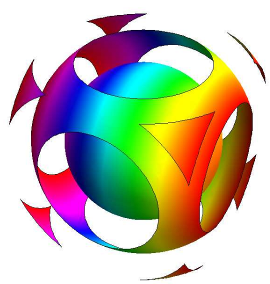
A function $f(x,y)$ is called **continuous** at (a,b) if $f(a,b)$ is finite and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that for any sequence (x_n, y_n) converging to (a,b) we have $f(x_n,y_n) \to f(a,b)$.

Continuity means that if (x,y) is close to (a,b), then $f(x,y)$ is close to $f(a,b)$. Continuity for functions of more than two variables is defined in the same way. Continuity is not always easy to check but fortunately, we do not have to worry about it most of the time. Lets look at some examples:

7 **Example:** For $f(x,y) = (xy)/(x^2+y^2)$, we have $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous.

8 For $f(x,y) = (x^2y)/(x^2+y^2)$, it is better to describe the function using polar coordinates: $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous.

A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. We can visualize it by **contour surfaces** $g(x,y,z) = c$, where c is constant. To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x=0, y=0$ or $z=0$.

Many surfaces can be described as level surfaces. If this is the case, we call this an **implicit description** of a surface. Here are some examples we know already:

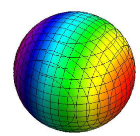
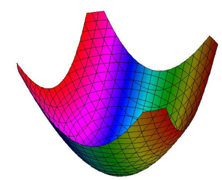
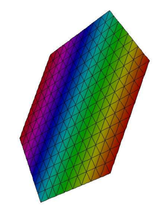
9 The function $g(x,y,z) = 2+\sin(xyz)$ is an example. It could define the temperature distribution in space. We can no more draw a graph of g because that would be an object in 4 dimensions.

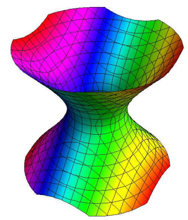
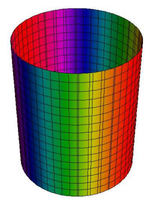
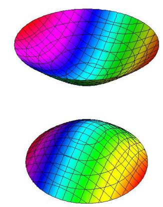
10 The level surfaces of $g(x,y,z) = x^2+y^2+z^2$ are spheres. The level surfaces of $g(x,y,z) = 2x^2+y^2+3z^2$ are ellipsoids.

11 For $g(x,y,z) = z - f(x,y)$, the level surface $g = 0$ which is the graph $z = f(x,y)$ of a function of two variables. For example, for $g(x,y,z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2+y^2$ of the function $f(x,y) = x^2+y^2$ which is a paraboloid. Note however that most surfaces of the form $g(x,y,z) = c$ can not be written as graphs. The sphere is an example, where we need two graphs to cover it.

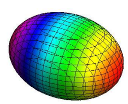
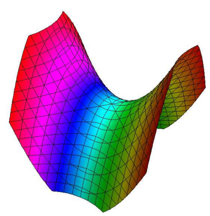
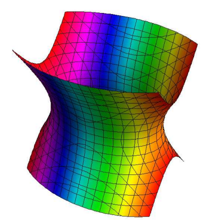
12 The equation $ax+by+cz = d$ is a plane. With $\vec{n} = \langle a,b,c\rangle$ and $\vec{x} = \langle x,y,z\rangle$, we can rewrite the equation $\vec{n}\cdot\vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n}\cdot\vec{x}_0 = d$. so that $\vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$. This means that every vector $\vec{x}-\vec{x}_0$ in the plane is orthogonal to $\vec{n}$.

13 If the function depends only quadratically on variables, that is if $f(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+m$ then the surface $f(x,y,z) = 0$ is called a **quadric**. Lets look at a few of them.

Sphere	Paraboloid	Plane
		
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$	$(x-a)^2+(y-b)^2-c = z$	$ax+by+cz = d$
One sheeted Hyperboloid	Cylinder	Two sheeted Hyperboloid

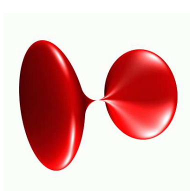
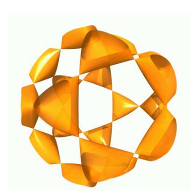
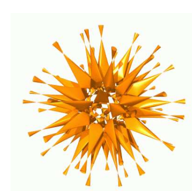




$(x-a)^2+(y-b)^2-(z-c)^2=r^2$	$(x-a)^2+(y-b)^2=r^2$	$(x-a)^2+(y-b)^2-(z-c)^2=-r^2$
Ellipsoid	Hyperbolic paraboloid	Elliptic hyperboloid

$x^2/a^2+y^2/b^2+z^2/c^2=1$	$x^2-y^2+z=1$	$x^2/a^2+y^2/b^2-z^2/c^2=1$

14 Higher order polynomial surfaces can be intriguingly beautiful. If the function involves only multiplications of variables x, y, z and $x \to f(x,x,x)$ has degree d, then it is called a **degree** d **polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.

Homework

1 Draw the graph of the function $f(x,y)=(x^2-1)/(y^2+1)$. It can also be seen as the level surface $g(x,y,z)=z-f(x,y)=0$. Find the equations for the three traces of the surface S and sketch the contour map.

2 Consider the surface $z^2-4z+x^2-2x-y=0$. Draw the three traces. What surface is it?

3 Draw the Fermat surface $x^4+y^4=z^4$ and its traces.

4 a) Sketch the graph and contour map of the function $f(x,y)=\sin(x^2+y^2)/(1+x^2+y^2)$.
b) Sketch the graph and contour map of the function $g(x,y)=|x|-|y|$.

5 Verify that the line $\vec{r}(t)=\langle 1,3,2\rangle+t\langle 1,2,1\rangle$ is contained in the surface $z^2-x^2-y=0$.