6: Parametrized surfaces

There is a second, fundamentally different way to describe a surface, the **parametrization**.
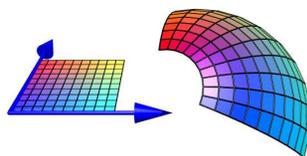
> A **parametrization** of a surface is a vector-valued function
> $$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \, ,$$
> where $x(u,v), y(u,v), z(u,v)$ are three functions of two variables.

Because two parameters u and v are involved, the map $\vec{r}$ is also called uv-**map**.

> A **parametrized surface** is the image of the uv-map. The domain of the uv-map is called the **parameter domain**.

> If we keep the first parameter u constant, then $v \mapsto \vec{r}(u,v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u,v)$ traces a curve the surface. These curves are called **grid curves**.

A computer draws surfaces using grid curves. The world of parametric surfaces is intriguing and complex. You can explore this world with the help of the computer algebra system Mathematica. You can survive the parametrization of surfaces topic by keeping in mind 4 important examples. They are *really important* because they: are cases we can understand well and which consequently will return again and again.

1 **Planes**. Parametric: $\vec{r}(s,t) = \vec{OP} + s\vec{v} + t\vec{w}$
Implicit: $ax + by + cz = d$. Parametric to Implicit: find the normal vector $\vec{n} = \vec{v} \times \vec{w}$. Implicit to Parametric: find two vectors $\vec{v}, \vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and forming $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.
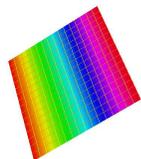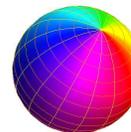
2 **Spheres**: Parametric: $\vec{r}(u,v) = \langle a, b, c \rangle + \langle \rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v) \rangle$.
Implicit: $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.
Parametric to implicit: reading off the radius.
Implicit to parametric: find the center (a,b,c) and the radius r possibly by completing the square.
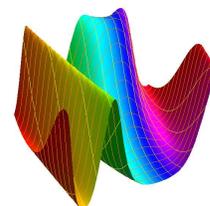
3 **Graphs**:
Parametric: $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$
Implicit: $z - f(x,y) = 0$. Parametric to Implicit: think about $z = f(x,y)$
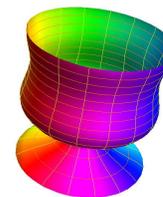Implicit to Parametric: use x and y as the parameterizations.

4 **Surfaces of revolution**:
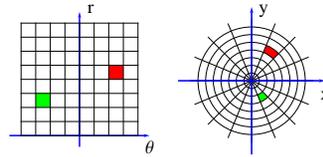Parametric: $r(u,v) = (g(v)\cos(u), g(v)\sin(u), v)$
Implicit: $\sqrt{x^2 + y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.
Parametric to Implicit: read off the function $g(z)$ the distance to the z-axis.
Implicit to Parametric: use the function g.

A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(y/x)$. We have $(x, y) = (r\cos(\theta), r\sin(\theta))$.

The formula $\theta = \text{arctg}(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ have the same θ value. In order to get the correct θ, one can take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the limit when $x \to 0^+$, then add π if $x < 0$ or $x = 0, y < 0$.

If we represent points in space as

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

we speak of **cylindrical coordinates**.

Here are some level surfaces in cylindrical coordinates:

5 $r = 1$ is a **cylinder**, $r = |z|$ is a **double cone** $\theta = 0$ is a **half plane** $r = \theta$ is a **rolled sheet of paper**

6 $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

Spherical coordinates use the distance ρ to the origin as well as two angles θ and ϕ. The first angle θ is the polar angle in polar coordinates of the first two coordinates and ϕ is the angle between the vector $\vec{OP}$ and the z-axis. A point has the **spherical coordinate**

$$(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)).$$

There are two important figures. The distance to the z axes $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can be seen in the right picture.

$x = \rho\cos(\theta)\sin(\phi),$
$y = \rho\sin(\theta)\sin(\phi),$
$z = \rho\cos(\phi)$
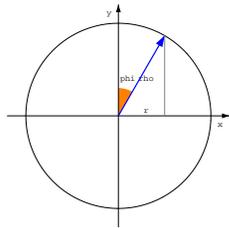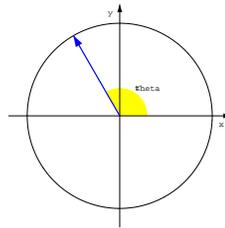
Here are some level surfaces described in spherical coordinates:

7 $\rho = 1$ is a **sphere**, the surface $\phi = \pi/2$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta)\sin(\phi)$ is an example of a **bumpy sphere**.

Homework

1 Plot the surface with the parametrization

$$\vec{r}(u, v) = \langle v^5\cos(u), v^5\sin(u), v \rangle,$$

where $u \in [0, 2\pi]$ and $v \in \mathbf{R}$.

2 Find a parametrization for the plane which contains the three points $P = (3, 7, 1), Q = (1, 2, 1)$ and $R = (0, 3, 4)$.

3 a) Find a parametrisations of the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1, z < 0$ by using that the surface is a graph $z = f(x, y)$.
b) Find a second parametrization but use angles ϕ, θ similarly as for the sphere.

4 Find a parametrisation of the **torus** which is obtained as the set of points which have distance 1 from the circle $(2\cos(\theta), 2\sin(\theta), 0)$, where θ is the angle occuring in cylindrical and spherical coordinates.

Hint: Keep $u = t$ as one of the parameters and let r the distance of a point on the torus to the z-axis. This distance is $r = 2 + \cos(\phi)$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = \sin(\phi)$. To finish the parametrization problem, you have to translate back from cylindrical coordinates $(r, \theta, z) = (2 + \cos(\phi), \theta, \sin(\phi))$ to Cartesian coordinates (x, y, z). Write down your result in the form $\vec{r}(\theta, \phi) = \langle x(\theta, \phi), y(\theta, \phi), z(\theta, \phi) \rangle$.
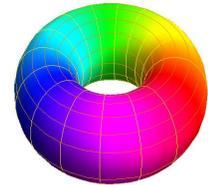
5 a) What is the equation for the surface $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(3\phi)|$ in spherical coordinates (ρ, θ, ϕ).