8: Arc length and curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then $L = \int_a^b |\vec{r}\,'(t)|\,dt$ is called the **arc length of the curve**.

In space, we have $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$.

1 The arc length of the **circle** of radius R given by $\vec{r}(t) = \langle R\cos(t), R\sin(t)\rangle$ parameterized by $0 \le t \le 2\pi$ is 2π because the speed $|\vec{r}'(t)|$ is constant and equal to R. The answer is $2\pi R$.

2 The **helix** $\vec{r}(t) = (\cos(t), \sin(t), t)$ has velocity $\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$ and constant speed $|\vec{r}\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.

3 What is the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2\rangle$$

for $1 \le t \le 2$? Answer: Because $\vec{r}'(t) = \langle 1, 1/t, t\rangle$, we have $\vec{r}'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t\,dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, we an call it "opportunity".

4 Find the arc length of the curve $\vec{r}(t) = \langle 3t^2, 6t, t^3\rangle$ from $t = 1$ to $t = 3$.

5 What is the arc length of the curve $\vec{r}(t) = \langle \cos^3(t), \sin^3(t)\rangle$? Answer: We have $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi} (3/2)\sin(2t)\,dt = 6$.

6 Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3\rangle$ for $-1 \le t \le 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2}\,dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^1 |x|\sqrt{1+x^2}\,dx = 2\int_0^1 x\sqrt{1+x^2}\,dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

7 The arc length of an **epicycle** $\vec{r}(t) = \langle t + \sin(t), \cos(t)\rangle$ parameterized by $0 \le t \le 2\pi$. We have $|\vec{r}'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)}\,dt$. A **substitution** $t = 2u$ gives $L = \int_0^\pi \sqrt{2 + 2\cos(2u)}\,2du = \int_0^\pi \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\,2du = \int_0^\pi \sqrt{4\cos^2(u)}\,2du = 4\int_0^\pi |\cos(u)|\,du = 8$.

8 The arc length of the **catenary** $\vec{r}(t) = \langle t, \cosh(t)\rangle$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1\ ,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by $\vec{r}(t) = \langle \cos(t^2), \sin(t^2)\rangle$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\,dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

10 Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = \langle \cos(3t), \sin(5t)\rangle$ leads to the arc length integral $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\,dt$ which can only be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ **unit tangent vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

The curvature does not depend on the parametrization.

Proof. Let $s(t)$ be an other parametrization, then by the chain rule $d/dt T'(s(t)) = T'(s(t))s'(t)$ and $d/dt r(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.

Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

11 The curve $\vec{r}(t) = \langle t, f(t)\rangle$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

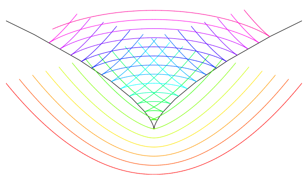
For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by N and $B$is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because B is automatically normal to T and N, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

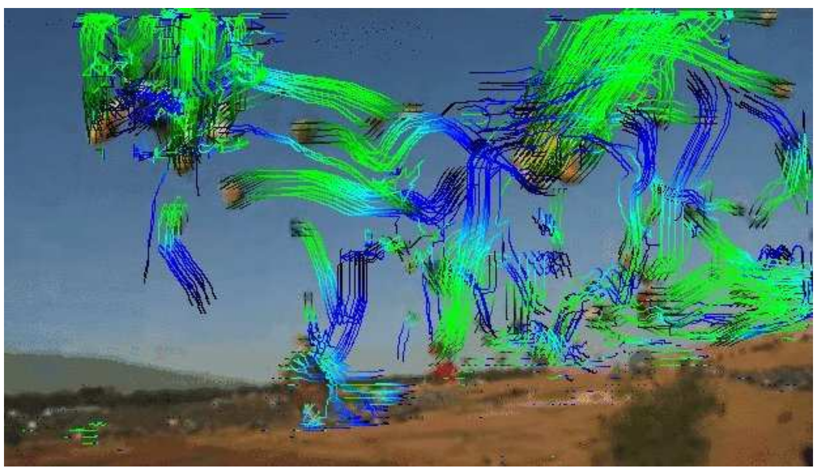
Here is an application of curvature: If a curve $\vec{r}(t)$ represents a **wave front** and $\vec{n}(t)$ is a **unit vector normal** to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.



A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We prove this in class. Finally, lets mention that curvature is important also in **computer vision**. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to **corners** in the picture. This is useful when **tracking** or **identifying** objects.



Tracking balloons in a movie taken at a balloon festival in Albuquerque. The program computes curvature in order to identify interesting points, then tracks them over time.

Homework

1 Find the arc length of the curve

$$\vec{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle ,$$

where the time parameter satisfies $0 \leq t \leq \pi$.

2 Find the curvature of $\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), t\rangle$ at the point $(1, 0, 0)$.

3 Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = \langle t^2, t^3, 0\rangle$ for $t = 2$. Do the vectors depend continuously on t for all t?

4 Let $\vec{r}(t) = \langle t, t^2\rangle$. Find the equation for the **caustic**

$$\vec{s}(t) = \vec{r}(t) + \frac{\vec{N}(t)}{\kappa(t)}$$

which is known also as the **evolute** of the curve.

5 If $\vec{r}(t) = \langle -\sin(t), \cos(t)\rangle$ is the boundary of a coffee cup and light enters in the direction $\langle -1, 0\rangle$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.