Lecture 9: Partial derivatives

> If $f(x, y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x, y)$ is defined as the derivative of the function $g(x) = f(x, y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

One also uses the short hand notation $f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$. For iterated derivatives, the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f$.

The notation for partial derivatives $\partial_x f, \partial_y f$ were introduced by Carl Gustav Jacobi. Josef Lagrange had used the term "partial differences". Partial derivatives f_x and f_y measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

1 For $f(x, y) = x^4 - 6x^2 y^2 + y^4$, we have $f_x(x, y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x, y) = -12x^2 y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**: it is an equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to more than one variables.

> **Clairot's theorem** If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant h is positive, then $f_x(x, y) = [f(x + h, y) - f(x, y)]/h$. For $h = 0$ we define f_x as before. Compare the two sides for fixed $h > 0$:

$h f_x(x, y) = f(x + h, y) - f(x, y)$ $dy f_y(x, y) = f(x, y + h) - f(x, y)$.
$h^2 f_{xy}(x, y) = f(x + h, y + h) - f(x + h, y + h^2 f_{yx}(x, y) = f(x + h, y + h) - f(x + h, y) - h) - (f(x + h, y) - f(x, y))$ $(f(x, y + h) - f(x, y))$

We have not taken any limits in this proof but established an identity which holds for all $h > 0$, the discrete derivatives f_x, f_y satisfy the relation $f_{xy} = f_{yx}$. We could fancy the identity obtained in the proof as a "quantum Clairot" theorem. If the classical derivatives f_{xy}, f_{yx} are both continuous, we can take the limit $h \to 0$ to get the classical Clairot's theorem as a "classical limit". Note that the quantum Clairot theorem shown first in this proof holds for **any** functions $f(x, y)$ of two variables. We do not even need the functions to be continuous.

2 Find $f_{xxxxxyxxxxx}$ for $f(x) = \sin(x) + x^6 y^{10} \cos(y)$. Answer: Do not compute, but think.

3 The continuity assumption for f_{xy} is necessary. The example

$$f(x, y) = \frac{x^3 y - xy^3}{x^2 + y^2}$$

contradicts Clairaut's theorem:

$f_x(x, y) = (3x^2 y - y^3)/(x^2 + y^2) - 2x(x^3 y - xy^3)/(x^2 + y^2)^2$, $f_x(0, y) = -y$, $f_{xy}(0, 0) = -1$, $f_y(x, y) = (x^3 - 3xy^2)/(x^2 + y^2) - 2y(x^3 y - xy^3)/(x^2 + y^2)^2$, $f_y(x, 0) = x$, $f_{y,x}(0, 0) = 1$.

> An equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

Here are some examples of partial differential equations. You should know the first 4 well.

4 The **wave equation** $f_{tt}(t, x) = f_{xx}(t, x)$ governs the motion of light or sound. The function $f(t, x) = \sin(x - t) + \sin(x + t)$ satisfies the wave equation.

5 The **heat equation** $f_t(t, x) = f_{xx}(t, x)$ describes diffusion of heat or spread of an epidemic. The function $f(t, x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

6 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $f(x, y) = x^3 - 3xy^2$ is an example satisfying the Laplace equation.

7 The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $f(t, x) = e^{-(x+t)^2}$ satisfy the advection equation.

8 The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $f(x, y) = \cos(x) + \sin(y)$ satisfies the eiconal equation.

9 The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break. The function $f(t, x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1 + \sqrt{\frac{1}{t}} e^{-x^2/(4t)}}$ satisfies the Burgers equation.

10 The **KdV equation** $f_t + 6 f f_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $f(t, x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2 t))$ satisfies the KdV equation.

11 The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $f(t, x) = e^{i(kx - \frac{\hbar}{2m} k^2 t)}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

Here are the graphs of the solutions of the equations. Can you match them with the PDE's?
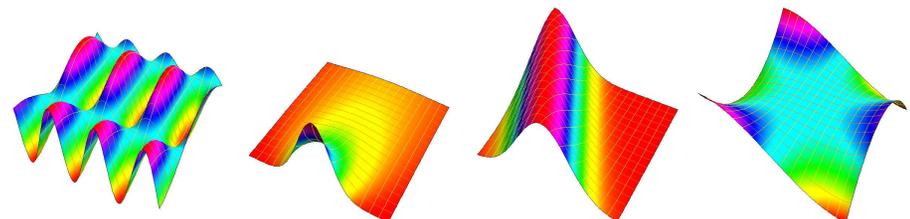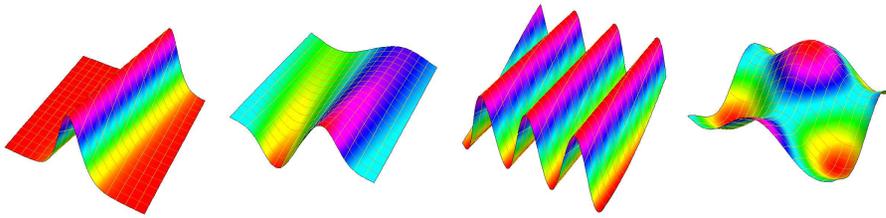
Notice that in all these examples, we have just given one possible solution to the partial differential equation. There are in general many solutions and only additional conditions like initial or boundary conditions determine the solution uniquely. If we know $f(0, x)$ for the Burgers equation, then the solution $f(t, x)$ is determined. A course on partial differential equations would show you how to get the solution.

Paul Dirac once said: "A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed. We will not learn here to find solutions to partial differential equations. But you should be able to verify that a given function is a solution of the equation.

Homework

1 Verify that $f(t, x) = \sin(\cos(t + x))$ is a solution of the transport equation $f_t(t, x) = f_x(t, x)$.

2 Verify that $f(x, y) = 3y^2 + x^3$ satisfies the **Euler-Tricomi** partial differential equation $u_{xx} = xu_{yy}$. This PDE is useful in describing **transonic flow**. Can you find an other solution which is not a multiple of the solution given in this problem?

3 Verify that $f(x, t) = e^{-rt} \sin(x + ct)$ satisfies the driven transport equation $f_t(x, t) = cf_x(x, t) - rf(x, t)$ It is sometimes also called the **advection equation**.

4 The partial differential equation $f_{xx} + f_{yy} = f_{tt}$ is called the wave equation in two dimensions. It describes waves in a pool for example.
a) Show that if $f(x, y, t) = \sin(nx + my) \sin(\sqrt{n^2 + m^2}t)$ satisfies the wave equation. It describes waves in a square where $x \in [0, \pi]$ and $y \in [0, \pi]$. The waves are zero at the boundary of the pool.
b) Verify that if we have two such solutions with different n, m then also the sum is a solution.
c) For which k is $f(x, y, t) = \sin(nx) \cos(nt) + \sin(mx) \cos(mt) + \sin(nx + my) \cos(kt)$ a solution of the wave equation? Verify that the wave is periodic in time $f(x, y, t + 2\pi) = f(x, y, t)$ if $m^2 + n^2 = k^2$ is a **Pythagorean triple**.
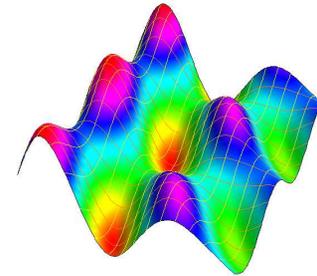
5 The partial differential equation $f_t + ff_x = f_{xx}$ is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the Navier Stokes equation which are used to describe the weather. Verify that the function

$$f(t, x) \frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}e^{-\frac{x^2}{4t}} + 1}}$$

is a solution of the Burgers equation.

Remark. This calculation might need a bit perseverance, when done by hand. You are welcome to use technology if you should get stuck. Here is an example on how to check that a function is a solution of a partial differential equation in Mathematica:

```
f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[ D[f[t,x],t] == D[f[t,x],{x,2}]]
```