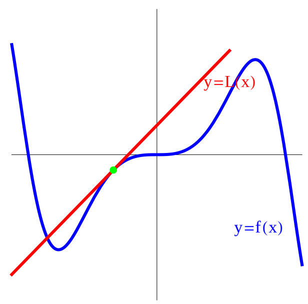
Lecture 10: Linearization

In single variable calculus, you have seen the following definition:

> The **linear approximation** of $f(x)$ at a point a is the linear function
> $$L(x) = f(a) + f'(a)(x-a) .$$



The graph of the function L is close to the graph of f at a. We generalize this now to higher dimensions:

> The **linear approximation** of $f(x,y)$ at (a,b) is the linear function
> $$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) .$$
> The **linear approximation** of a function $f(x,y,z)$ at (a,b,c) is
> $$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) .$$

Using the **gradient**

$$\nabla f(x,y) = \langle f_x, f_y \rangle, \qquad \nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle ,$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .$$

How do we justify the linearization? If the second variable $y = b$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x,b) = f(a,b) + f_x(a,b)(x-a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0,y) = f(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

1 What is the linear approximation of the function $f(x,y) = \sin(\pi x y^2)$ at the point $(1,1)$? We have $(f_x(x,y), y_f(x,y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = \langle \pi \cos(\pi), 2\pi \cos(\pi) \rangle = \langle -\pi, 2\pi \rangle$.

2 Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942 .$$

3 Here is an example in three dimensions: find the linear approximation to $f(x,y,z) = xy + yz + zx$ at the point $(1,1,1)$. Since $f(1,1,1) = 3$, and $\nabla f(x,y,z) = (y+z, x+z, y+x), \nabla f(1,1,1) = (2,2,2)$. we have $L(x,y,z) = f(1,1,1) + (2,2,2) \cdot (x-1, y-1, z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

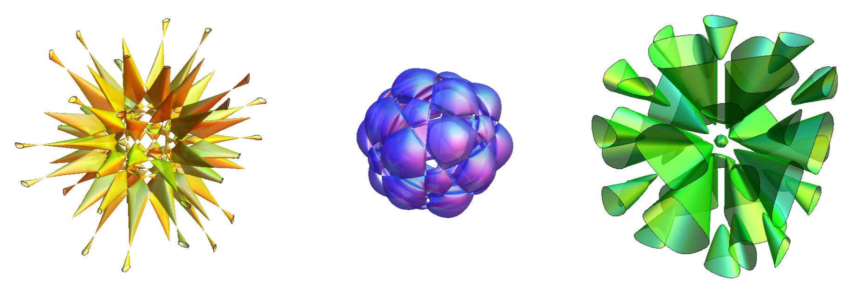
4 Estimate $f(0.01, 24.8, 1.02)$ for $f(x,y,z) = e^x \sqrt{y} z$.
Solution: take $(x_0,y_0,z_0) = (0,25,1)$, where $f(x_0,y_0,z_0) = 5$. The gradient is $\nabla f(x,y,z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0,y_0,z_0) = (0,25,1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x-x_0, y-y_0, z-z_0) = 5 + (5, 1/10, 5)(x-0, y-25, z-1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

5 Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2,4)$.
Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $\langle a,b \rangle = \nabla f(2,4) = \langle -g'(2), 1 \rangle = \langle -4, 1 \rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

6 The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x,y,z) &= (3+5t)(-1+x^2+y^2+z^2)^2(-2+t+x^2+y^2+z^2)^2 \\ &+ 8(x^2-t^4y^2)(-(t^4x^2)+z^2)(y^2-t^4z^2)(x^4-2x^2y^2+y^4-2x^2z^2-2y^2z^2+z^4) , \end{aligned}$$

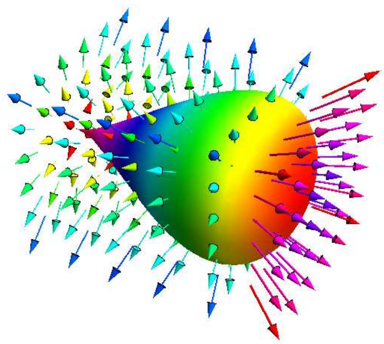
where $t = (\sqrt{5}+1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5}-1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x,y,z) = 8$. Find the tangent plane of the later surface at the point $(1,1,0)$. **Answer:** We have $\nabla f(1,1,0) = \langle 64, 64, 0 \rangle$. The surface is $x + y = d$ for some constant d. By plugging in $(1,1,0)$ we see that $x + y = 2$.



7 The quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2,2,2)$ of this pair shaped surface? We get $\langle a,b,c \rangle = \langle 20,4,4 \rangle$ and so the equation of the plane $20x + 4y + 4z = 56$, where we have obtained the constant to the right by plugging in the point $(x,y,z) = (2,2,2)$.



Remark: some books use **differentials** etc to describe linearizations. This is 19 century notation and terminology and should be avoided by all means. For us, the linearlization of a function at a point is a linear function in the same number of variables. 20th century mathematics has invented the notion of **differential forms** which is a valuable mathematical notion, but it is a concept which becomes only useful in follow-up courses which build on multivariable calculus like Riemannian geometry. The notion of "differentials" comes from a time when calculus was still foggy in some areas. Unfortunately it has survived and appears even in some calculus books.

Homework

1 If $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x,y)$ at the point $(1,1,2)$. Extimate $f(1.01, 0.98)$.

2 Estimate $1000^{1/5}$ using linear approximation

3 Find $f(0.01, 0.999)$ for $f(x,y) = \cos(\pi xy)y + \sin(x + \pi y)$.

4 Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

5 Sketch a contour map of the function

$$f(x,y) = x^2 + 9y^2$$

find the **gradient vector** $\nabla f = \langle f_x, f_y \rangle$ of f at the point $(1,1)$. Draw it together with the tangent line $ax + by = d$ to the curve at $(1,1)$.