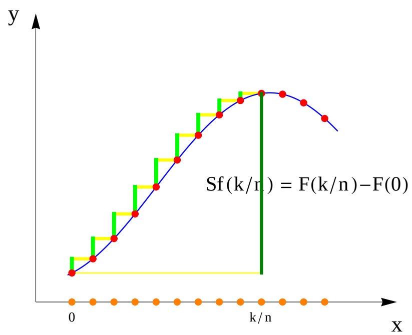
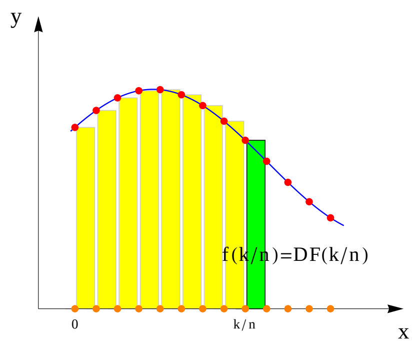
Lecture 15: Double integrals

Here is a one paragraph summary of single variable calculus: if $f(x)$ is a differentiable function, then the **Riemann integral** $\int_a^b f(x)\,dx$ is defined as the limit of the **Riemann sum** $S_n f(x) = \frac{1}{n}\sum_{k/n\in[a,b]} f(k/n)$ for $n \to \infty$. The derivative is the limit of difference quotients $D_n f(x) = n[f(x+1/n) - f(x)]$ as $n \to \infty$. The integral $\int_a^b f(x)\,dx$ is the **signed area** under the graph of f, where "signed" indicates that it can become negative too. The function $F(x) = \int_0^x f(y)\,dy$ is called an **anti-derivative** of f and determined up a constant. The **fundamental theorem of calculus** states

$$F'(x) = f(x), \int_0^x f(x) = F(x) - F(0)\ .$$

This theorem is obtained from the **quantum fundamental theorem** $DF(k/n) = f(k/n) - f(0), Sf(k/n) = F(k/n)$ (which holds for all functions!) in the limit $n \to \infty$ and allows to compute integrals by inverting differentiation. Differentiation rules become integration rules: the product rule becomes integration by parts, the chain rule becomes partial integration.





If $f(x,y)$ is differentiable on a region R, the integral $\int_R f(x,y)\,dxdy$ is defined as the limit of the Riemann sum

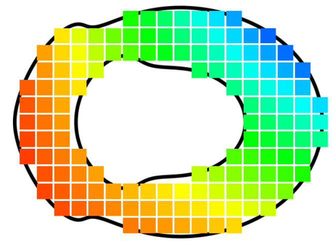
$$\frac{1}{n^2}\sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

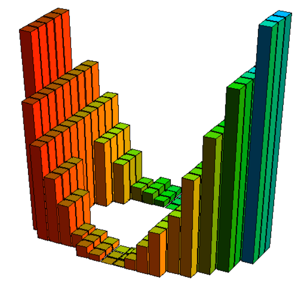
when $n \to \infty$. We write also $\int_R f(x,y)\,dA$ and think of dA as an area element.

1 If we integrate $f(x,y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over y to get $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2 If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside R.

3 The integral $\iint_R f(x,y)\,dA$ divided by the area of R is the **average** value of f on R.





4 One can interpret $\int\int_R f(x,y)\,dydx$ as the **signed volume** of the solid below the graph of f and above R in the $x-y$ plane. As in 1D integration, the volume of the solid below the xy-plane is counted negatively.

Fubini's theorem allows to switch the order of integration over a rectangle if the function f is continuous: $\int_a^b \int_c^d f(x,y)\,dxdy = \int_c^d \int_a^b f(x,y)\,dydx$.

Proof. We have for every n the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]}\sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]}\sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

which holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$.

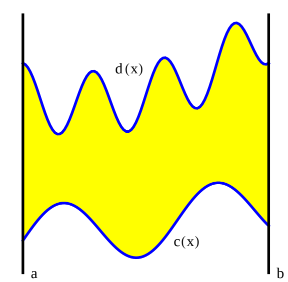
Fubini's theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

A **type I region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \}\ .$$

An integral over such a region is called a **type I integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx\ .$$

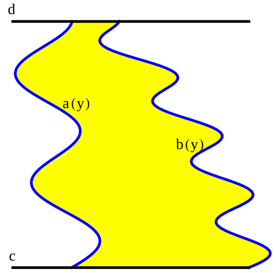


A **type II region** is of the form

$$R = \{(x,y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\ \} .$$

An integral over such a region is called a **type II integral**

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$



5 Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \leq x \leq \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

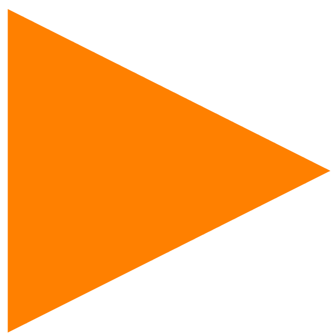
$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx$$

We have now an integral, which we can solve by substitution

$$= -\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3} .$$

6 Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of y: they are $x = y-1$ and $x = 1-y$

$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(\frac{1}{4}-\frac{1}{3}) = \frac{1}{10} .$$



7 Let R be the triangle $1 \geq x \geq 0, 0 \leq y \leq x$. What is

$$\int\int_R e^{-x^2}\ dxdy\ ?$$

The type II integral $\int_0^1[\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions.
The type I integral $\int_0^1[\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$

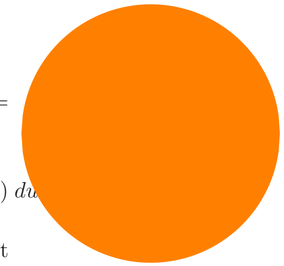
8 The area of a disc of radius R is

$$\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\ dydx = \int_{-R}^{R} 2\sqrt{R^2-x^2}\ dx .$$

This integral can be solved with the substitution $x = R\sin(u), dx = R\cos(u)$

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)}R\cos(u)\ du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u)\ du .$$

Using a double angle formula we get $R^2\int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u))}{2}\ du = R^2\pi$. We will now see how to do that better in polar coordinates.



Remark: The Riemann integral just defined works well for continuous functions. In other branches of mathematics like probability theory, a better integral is needed. The **Lebesgue integral** fits the bill. Its definition is close to the Riemann integral which we have given as the limit $n^{-2}\int_{(x_k,y_l)\in R} f(x_k,y_l)$ where $x_k = k/n, y_l = l/n$. The Lebesgue integral replaces the regularly spaced (x_k,y_l) grid with random points x_k, y_l and uses the same formula. The following Mathematica code computes the integral $\int_0^1\int_0^1 x^2y$ using this **Monte Carlo definition** of the Lebesgue integral.

```
M=10000;  R:=Random[];  f[x_,y_]:=x^2 y;  Sum[f[R,R],{M}]/M
```

It is as elegant than the numerical Riemann sum computation

```
M=100;  f[x_,y_]:=x^2 y;  Sum[f[k/M,l/M],{k,M},{l,M}]/M^2
```

but the Lebesgue integral is usually closer to the actual answer 1/6 than the Riemann integral. Note that for all continuous functions, the Lebesgue integral gives the same results than the Riemann integral. It does not change calculus. But it is useful for example to compute nasty integrals like the area of the Mandelbrot set.

Homework

1 Find the double integral $\int_1^4\int_0^2(3x-\sqrt{y})\ dxdy$.

2 Find the area of the region

$$R = \{(x,y) \mid 0 \leq x \leq 2\pi, \sin(x)-1 \leq y \leq \cos(x)+2\}$$

and use it to compute the average value $\int\int_R f(x,y)\ dxdy/area(R)$ of $f(x,y) = y$ over that region.

3 Find the volume of the solid lying under the paraboloid $z = x^2+y^2$ and above the rectangle $R = [-2,2]\times[-3,3] = \{(x,y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\ \}$.

4 Calculate the iterated integral $\int_0^1\int_x^{2-x}(x^2-y)\ dydx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

5 Evaluate the double integral

$$\int_0^2\int_{x^2}^4 \frac{x}{e^{y^2}}\ dydx .$$