# Lecture 16: Surface area

A **polar region** is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

In Cartesian coordinates the parametrization of the boundary curve is $\vec{r}(t) = \langle r(t)\cos(\theta(t)), r(t)\sin(\theta(t)) \rangle$. We are especially interested in regions which are bound by **polar graphs**, where $\theta(t) = t$.

1 The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**.

2 The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

3 The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**. It looks like an infinity sign.
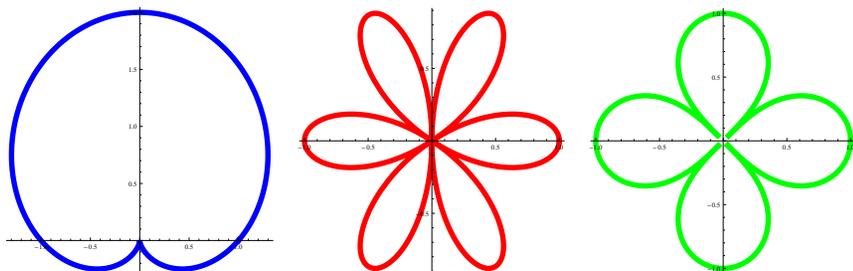
To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x,y)\,dxdy = \iint_R f(r\cos(\theta), r\sin(\theta)) r\,drd\theta$$

4 Integrate
$$f(x,y) = x^2 + x^2 + xy \,,$$
over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\,dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta)) r\,d\theta dr = 2\pi/4$.

5 We have earlier computed area of the disc $\{x^2 + y^2 \le 1\}$ using substitution. It is more elegant to do this integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r\,drd\theta = 2\pi r^2/2\big|_0^1 = \pi \,.$$

Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has the area $r\,d\theta dr$. If you have seen some linear algebra, note that the Jacobean matrix dT has the determinant r.
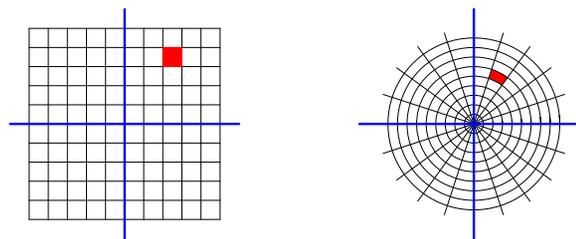
We can now integrate over type I or type II regions in the (θ, r) plane. like **flowers**: $\{(\theta, r) \,|\, 0 \le r \le f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.
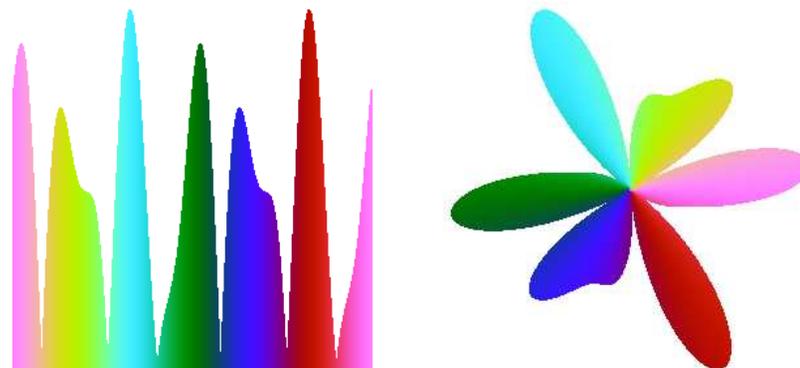
A polar region shown in polar coordinates. It is a type I region.

The same region in the xy coordinate system is not type I or II.

6 Integrate the function $f(x,y) = 1$ $\{(\theta, r(\theta)) \,|\, r(\theta) \le |\cos(3\theta)|\}$.

$$\int \int_R 1\,dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r\,dr\,d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\,d\theta = \pi/2 \,.$$

7 Integrate $f(x,y) = y\sqrt{x^2 + y^2}$ over the region $R = \{(x,y) \,|\, 1 < x^2 + y^2 < 4, y > 0\}$.

$$\int_1^2 \int_0^{\pi} r\sin(\theta) r\, r\,d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta)\,d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^{\pi} \sin(\theta)\,d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

8 The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b} \right)^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]
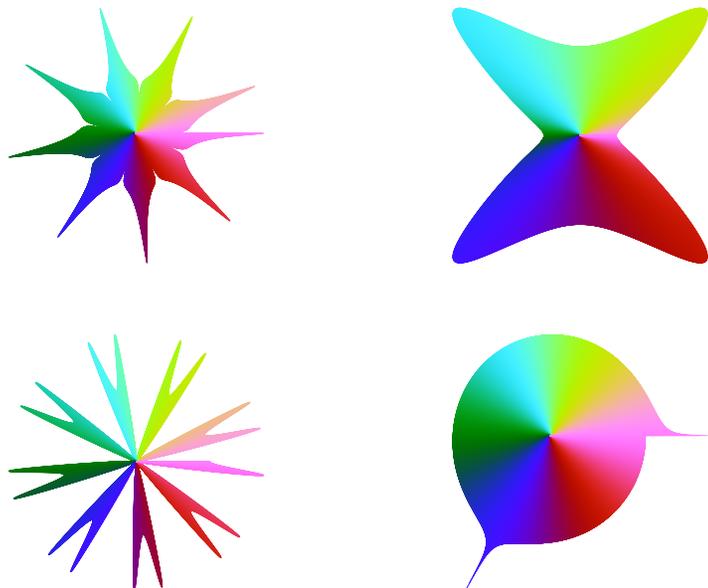
A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int \int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \, .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$, the two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u+du] \times [v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv$.

9 The parametrized surface $\vec{r}(u,v) = \langle 2u, 3v, 0 \rangle$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = \langle 0, 0, 6 \rangle$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.
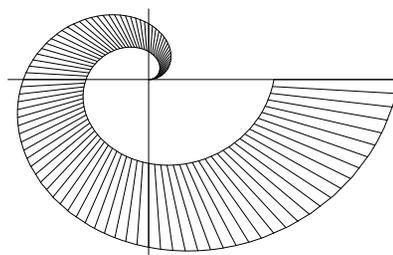
[1]"Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

10 The map $\vec{r}(u,v) = \langle L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v) \rangle$ maps the rectangle $G = [0, 2\pi] \times [0, \pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int \int_R 1 \, dS = \int_0^{2\pi} \int_0^\pi L^2 \sin(v) \, dvdu = 4\pi L^2$

11 For graphs $(u,v) \mapsto \langle u, v, f(u,v) \rangle$, we have $\vec{r}_u = (1, 0, f_u(u,v))$ and $\vec{r}_v = (0, 1, f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region G is $\int \int_G \sqrt{1 + f_u^2 + f_v^2} \, dudv$.

12 Lets take a surface of revolution $\vec{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u) \rangle$ on $R = [0, 2\pi] \times [a, b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int \int |\vec{r}_u \times \vec{r}_v| \, dudv = 2\pi \int_a^b |f(v)| \sqrt{1 + f'(v)^2} \, dv$.

Homework

1 Integrate $f(x,y) = x^2$ over the unit disc $\{x^2 + y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

2 Find $\int \int_R (x^2 + y^2)^{10} \, dA$, where R is the part of the unit disc $\{x^2 + y^2 \leq 1\}$ for which $y > x$.

3 What is the area of the region which is bounded by the following three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x-axis?

4 Find the average value of $f(x,y) = x^2 + y^2$ on the annular region $R : 1 \leq |(x,y)| \leq 2$. The average is $\int_R f \, dxdy / \int_R 1 \, dxdy$.

5 Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \leq 9$.