Lecture 17: Triple integrals

If $f(x,y,z)$ is a function of three variables and E is a **solid region** in space, then $\int\int\int_E f(x,y,z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(i/n,j/n,k/n)\in E} f(\frac{i}{n},\frac{j}{n},\frac{k}{n}) \ .$$

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1 Assume E is the box $[0,1]\times[0,1]\times[0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

To compute the integral we start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the outer layer: $\int_0^1 3x^2 dx$=1. When we calculate the most inner integral, we fix x and y. The integral is integrating up $f(x,y,z)$ along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these two dimensional sections.

There are two important methods to compute volume: the "washer method" and the "sandwich method". The washer method from single variable calculus reduces the problem directly to a one dimensional integral. The new sandwich method reduces the problem to a two dimensional integration problem.

The **washer method** slices the solid along a line. If $g(z)$ is the double integral along the two dimensional slice, then $\int_a^b g(z)\ dz$. The **sandwich method** sees the solid sandwiched between the graphs of two functions $g(x,y)$ and $h(x,y)$ over a common two dimensional region R. The integral becomes $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz] dA$.

2 An important special case of the sandwich method is the volume

$$\int_R \int_0^{f(x,y)} 1\ dzdxdy \ .$$

under the graph of a function $f(x,y)$ and above a region R. It is the integral $\int\int_R f(x,y)\ dA$. What we actually have computed is a triple integral

3 Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz] dA \ .$$

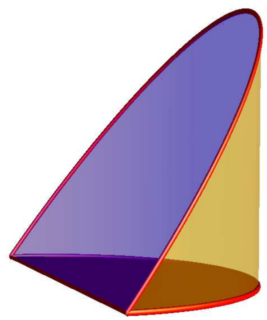
which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2}\ dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}r\ drd\theta = 4\pi/3$.

With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2)\ dz = 4\pi/3$.

4 The mass of a body with density $\rho(x,y,z)$ is defined as $\int\int\int_R \rho(x,y,z)\ dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4-x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1. **Solution:**

$$\begin{aligned} &\int_0^2 \int_0^6 \int_0^{4-x^2} dz\ dy\ dx = \int_0^2 \int_0^6 (4-x^2)\ dydx \\ = \ & 6\int_0^2 (4-x^2)\ dx = 6(4x - x^3/3)|_0^2 = 32 \end{aligned}$$

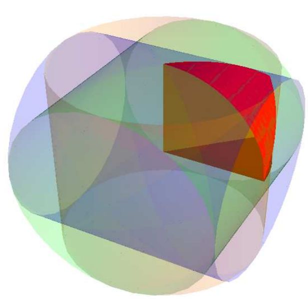
5 The solid region bound by $x^2+y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed his Riemann sum integration technique. It appears in every calculus text book. Find the volume. **Solution**. Look from the situation from above and picture it in the $x-y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\ dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos(\theta)\ drd\theta = 2/3$.



6 Finding the volume of the solid region bound by the three cylinders $x^2+y^2 = 1$, $x^2+z^2 = 1$ and $y^2+z^2 = 1$ is one of the most famous volume integration problems. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2+z^2 = 1$ matters. The polar integration problem

$$16\int_0^{\pi/4}\int_0^1 \sqrt{1-r^2\cos^2(\theta)}r\ drd\theta$$

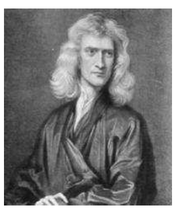
has an inner r-integral of $(16/3)(1-\sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1+\sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x)+\sec(x)+\tan(x)$. The result is $16-8\sqrt{2}$.



The problem of computing volumes has been tackled early in mathematics:

Archimedes (287-212 BC) designed an integration method which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of **exhaustion** is close to the numerical method of integration by Riemann sum. In our terminology, Archimedes used the **washer method** to reduce the problem to a single variable problem. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. His **displacement method** later would morph into Cavalieri principle and modern rearrangement techniques in modern analysis. Heureka!

Cavalieri (1598-1647) would build on Archimedes ideas and determine area and volume using tricks like the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibniz(1646-1716): Newton and Leibniz, developed calculus independently. The new tool made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools as we do here.

Remarks which can as usual be skipped. 1) Here is an other way to compute integrals: the Lebesgue integral is more powerful than the Riemann integral: suppose we want to calculate the volume of some solid body R which we assumed to be contained inside the unit cube $[0,1] \times [0,1] \times [0,1]$. The **Monte Carlo method** shoots randomly n times onto the unit cube and count the number k of times, we hit the solid. The result k/n approximates the volume. Here is a Mathematica example with one eights of the unit ball:

$$\mathrm{R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of n=10000 times. The volume so measured is 0.5277. The actual volume of 1/8'th of the sphere is $\pi/6 = 0.524$. For $n \to \infty$ the Monte Carlo computation gives the actual volume. The Monte-Carlo integral is stronger than the Riemann integral. It is equivalent to the **Lebesgue integral** and allows to measure much more sets than solids with piecewise smooth boundaries.
2) Is there an "integral' which is able to measure **every solid** in space and which has the property that the volume of a rotated or translated body remains the same? No! Most sets one can define are "crazy" in the sense that one can not measure their volume. An example is the **paradox of Banach and Tarski** which tells that one can slice up the unit ball $x^2 + y^2 + z^2 \leq 1$ into 5 pieces A, B, C, D, E, rotate and translate them in space so that the pieces A, B, C fit together to be a unit ball and D, E again form an other unit ball. Since the volume has obviously doubled and volume should be additive in the sense that the volume of two disjoint sets should be the sum of the volumes, some of the sets A, B, C, D, E have no defined volume.

Homework

1 Evaluate the triple integral

$$\int_0^1 \int_0^z \int_0^{2y} z e^{-y^2} \, dx dy dz .$$

2 Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 9 - (x^2 + y^2)$ and satisfying $x \geq 0$.

3 Find the **moment of inertia** $\int\int\int_E (x^2 + y^2) \, dV$ of a cone

$$E = \{x^2 + y^2 \leq z^2 \; 0 \leq z \leq 1 \} ,$$

which has the z-axis as its center of symmetry.

4 Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 3)$.

5 What is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$?