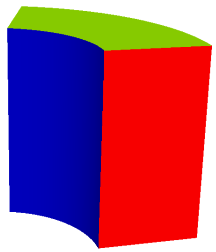
Lecture 18: Spherical Coordinates

Remember that **cylindrical coordinates** are coordinates in space in which polar coordinates are chosen in the xy plane and the z-coordinate is left untouched. A surface of revolution can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r,\theta,z) = (r\cos(\theta), r\sin(\theta), z)$, produces the same integration factor r as in polar coordinates.

$$\int\int_{T(R)} f(x,y,z)\ dxdydz = \int\int_R g(r,\theta,z)\ \boxed{r}\ drd\theta dz$$

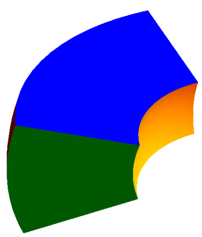


Remember also that spherical coordinates use ρ, the distance to the origin as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \ .$$

The integration factor can be seen by measuring the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

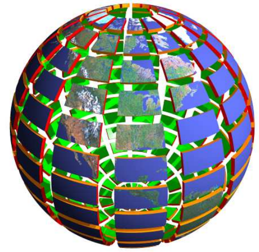
$$\int\int_{T(R)} f(x,y,z)\ dxdydz = \int\int_R g(\rho,\theta,z)\ \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d$$



1 A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\ d\phi d\theta d\rho \ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.



The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x,y,z)^2\ dzdydx$, where $r(x,y,z) = R\sin(\phi)$ is the distance from the axis L.

2 For a sphere of radius R we obtain with respect to the z-axis:

$$\begin{aligned} I &= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi)\ d\phi d\theta d\rho \\ &= (\int_0^\pi \sin^3(\phi)\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2\pi}\ d\theta) \\ &= (\int_0^\pi \sin(\phi)(1-\cos^2(\phi))\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2pi}\ d\theta) \\ &= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15} \ . \end{aligned}$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8\cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8\cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned} \overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\cos(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\ \overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\sin(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\ \overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\cos(\phi)\sin(\phi)\ d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V} \end{aligned}$$

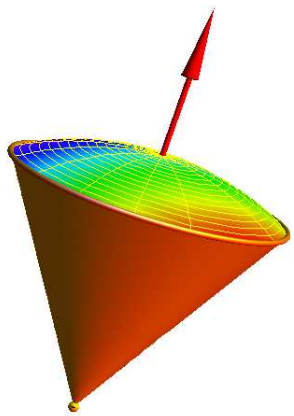
4 Find $\int\int\int_R z^2\ dV$ for the solid obtained by intersecting $\{1 \le x^2+y^2+z^2 \le 4\ \}$ with the double cone $\{z^2 \ge x^2+y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with

2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi)\, d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5})2\pi(\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi\frac{31}{5}(1 - 2^{-3/2}) .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.



Remarks: There are other coordinate systems besides Euclidean, cylindrical and spherical. One of them are **toral coordinates**, where $T(r, \phi, \theta) = (1+r\cos(\phi))\cos(\theta), (1+r\cos(\phi)\sin(\theta), r\sin(\phi))$, a coordinate system which works inside the solid torus $r \leq 1$.
Are there spherical coordinates in higher dimensions? Yes, of course. They are called **hyperspherical coordinates**. In four dimensions for example we would have a third angle ψ and get

$$(x, y, z, w) = (\rho\sin(\psi)\sin(\phi)\sin(\theta), \rho\sin(\psi)\sin(\phi)\cos(\theta), \rho\sin(\psi)\cos(\phi), \rho\cos(\psi)) .$$

The four dimensional case is especially interesting because one can write the sphere S^3 in four dimensions as the set of pairs of complex numbers z, w with $|z|^2 + |w|^2 = 1$. The 3 sphere is special because it is equal to the group $SU(2)$ of all unitary 2×2 matrices of determinant 1. It is also the set of all quaternions of length 1. The quaternions are historically interesting for multivariable calculus because they predated vector calculus we teach today and incorporate both the dot and cross product.

Homework

1 The density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1.$ is given by the forth power of the distance to the z-axes: $\sigma(x, y, z) = (x^2 + y^2)^2$. Find its mass

$$M = \int\int\int_E (x^2 + y^2)^2 \, dxdydz .$$

2 Find the moment of inertia $\int\int\int_E (x^2 + y^2)\, dV$ of the body E whose volume is given by the integral

$$\text{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi)\, d\rho d\theta d\phi .$$

3 A solid is described in spherical coordinates by the inequality $\rho \leq \sin(\phi)$. Find its volume.

4 Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

5 Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 1$.