

## Lecture 20: Theorem of lineintegrals

If  $\vec{F}$  is a vector field in the plane or in space and  $C : t \mapsto \vec{r}(t)$  is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

is called the **line integral** of  $\vec{F}$  along the curve  $C$ .

The short-hand notation  $\int_C \vec{F} \cdot d\vec{r}$  is also used. In physics, if  $\vec{F}(x, y, z)$  is a force field, then  $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$  is called **power** and the line integral  $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$  is called **work**. In electrodynamics, if  $\vec{F}(x, y, z)$  is an electric field, then the line integral  $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$  gives the **electric potential**.

- 1 Let  $C : t \mapsto \vec{r}(t) = \langle \cos(t), \sin(t) \rangle$  be a circle parametrized by  $t \in [0, 2\pi]$  and let  $\vec{F}(x, y) = \langle -y, x \rangle$ . Calculate the line integral  $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ .

**Solution:** We have  $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = 2\pi$

- 2 Let  $\vec{r}(t)$  be a curve given in polar coordinates as  $\vec{r}(t) = \cos(t), \phi(t) = t$  defined on  $[0, \pi]$ . Let  $\vec{F}$  be the vector field  $\vec{F}(x, y) = \langle -xy, 0 \rangle$ . Calculate the line integral  $\int_C \vec{F} \cdot d\vec{r}$ . **Solution:** In Cartesian coordinates, the curve is  $r(t) = (\cos^2(t), \cos(t) \sin(t))$ . The velocity vector is then  $r'(t) = \langle -2\sin(t) \cos(t), -\sin^2(t) + \cos^2(t) \rangle = \langle x(t), y(t) \rangle$ . The line integral is

$$\begin{aligned} \int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt &= \int_0^\pi \langle \cos^3(t) \sin(t), 0 \rangle \cdot \langle -2\sin(t) \cos(t), -\sin^2(t) + \cos^2(t) \rangle dt \\ &= -2 \int_0^\pi \sin^2(t) \cos^4(t) dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8. \end{aligned}$$

Here is the first generalization of the fundamental theorem of calculus to higher dimensions. It is called the **fundamental theorem of line integrals**.

**Fundamental theorem of line integrals:** If  $\vec{F} = \nabla f$ , then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

In other words, the line integral is the potential difference between the end points  $\vec{r}(b)$  and  $\vec{r}(a)$ , if  $\vec{F}$  is a gradient field.

- 3 Let  $f(x, y, z)$  be the temperature distribution in a room and let  $\vec{r}(t)$  the path of a fly in the room, then  $f(\vec{r}(t))$  is the temperature, the fly experiences at the point  $\vec{r}(t)$  at time  $t$ . The change of temperature for the fly is  $\frac{d}{dt} f(\vec{r}(t))$ . The line-integral of the temperature gradient  $\nabla f$  along the path of the fly coincides with the temperature difference between the end point and initial point.

- 4 If  $\vec{r}(t)$  is parallel to the level curve of  $f$ , then  $d/dt f(\vec{r}(t)) = 0$  and  $\vec{r}'(t)$  orthogonal to  $\nabla f(\vec{r}(t))$ .

- 5 If  $\vec{r}(t)$  is orthogonal to the level curve, then  $|d/dt f(\vec{r}(t))| = |\nabla f| |\vec{r}'(t)|$  and  $\vec{r}'(t)$  is parallel to  $\nabla f(\vec{r}(t))$ .

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \frac{d}{dt} f(\vec{r}(t)) dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

For a gradient field, the line-integral along any closed curve is zero.

When is a vector field a gradient field?  $\vec{F}(x, y) = \nabla f(x, y)$  implies  $P_y(x, y) = Q_x(x, y)$ . If this does not hold at some point,  $\vec{F} = \langle P, Q \rangle$  is no gradient field. This is called the **component test** or Clairot test. We will see later that the condition  $\text{curl}(\vec{F}) = Q_x - P_y = 0$  implies that the field is conservative, if the region satisfies a certain property.

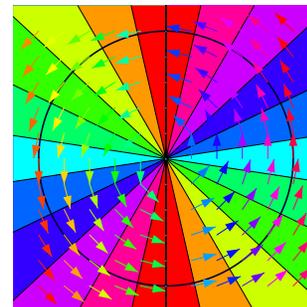
- 6 Let  $\vec{F}(x, y) = \langle 2xy^2 + 3x^2, 2yx^2 \rangle$ . Find a potential  $f$  of  $\vec{F} = \langle P, Q \rangle$ .  
Solution: The potential function  $f(x, y)$  satisfies  $f_x(x, y) = 2xy^2 + 3x^2$  and  $f_y(x, y) = 2yx^2$ . Integrating the second equation gives  $f(x, y) = x^2y^2 + h(x)$ . Partial differentiation with respect to  $x$  gives  $f_x(x, y) = 2xy^2 + h'(x)$  which should be  $2xy^2 + 3x^2$  so that we can take  $h(x) = x^3$ . The potential function is  $f(x, y) = x^2y^2 + x^3$ . Find  $g, h$  from  $f(x, y) = \int_0^x P(x, y) dx + h(y)$  and  $f_y(x, y) = g(x, y)$ .

- 7 Let  $\vec{F}(x, y) = \langle P, Q \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$ . It is a gradient field because  $f(x, y) = \arctan(y/x)$  has the property that  $f_x = (-y/x^2)/(1+y^2/x^2) = P$ ,  $f_y = (1/x)/(1+y^2/x^2) = Q$ . However, the line integral  $\int_\gamma \vec{F} \cdot d\vec{r}$ , where  $\gamma$  is the unit circle is

$$\int_0^{2\pi} \left\langle \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)} \right\rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$$

which is  $\int_0^{2\pi} 1 dt = 2\pi$ . What is wrong?

**Solution:** note that the potential  $f$  as well as the vector-field  $F$  are not differentiable everywhere. The curl of  $F$  is zero except at  $(0, 0)$ , where it is not defined.

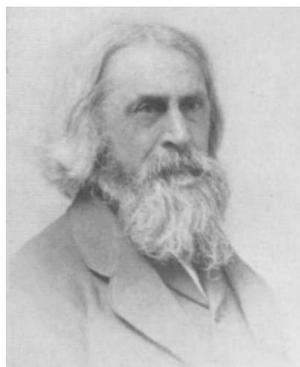


Remarks: The fundamental theorem of line integrals works in any dimension. You can formulate and check it yourself. The reason is that curves, vector fields, chain rule and integration along curves are easy to generalize in any dimensions. We will see later that if  $R$  is a simply connected region then  $\vec{F}$  is a gradient field if and only if  $\text{curl}(\vec{F}) = 0$  everywhere in  $R$ . A region  $R$  is called **simply connected**, if every curve in  $R$  can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annular region is an example which is not. Any region with a hole is not simply connected. For simply connected regions, the existence of a gradient field is equivalent to the field having curl zero everywhere.

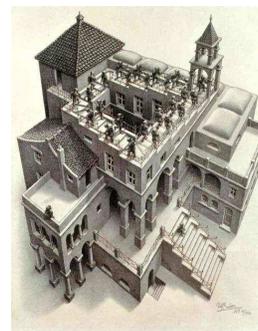
A device which implements a non gradient force field is called a **perpetual motion machine**. Mathematically, it realizes a force field for which along some closed loops the energy gain is nonnegative. By possibly changing the direction, the energy change is positive. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up with and to see why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.



Why does this "perpetual motion machine" not work? The former Harvard professor Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the "antropic principle". "Such a series of motions would receive the technical name of a "perpetual motion" by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".



Nonconservative fields can also be generated by **optical illusion** as **M.C. Escher** did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"?



## Homework

- 1 Let  $C$  be the space curve  $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$  for  $t \in [0, 1]$  and let  $\vec{F}(x, y, z) = \langle y, x, 5 \rangle$ . Calculate the line integral  $\int_C \vec{F} \cdot d\vec{r}$ .
- 2 Find the work done by the force field  $F(x, y) = (x \sin(y), y)$  on a particle that moves along the parabola  $y = x^2$  from  $(-1, 1)$  to  $(2, 4)$ .
- 3 Let  $\vec{F}$  be the vector field  $\vec{F}(x, y) = \langle -y, x \rangle / 2$ . Compute the line integral of  $F$  along an ellipse  $\vec{r}(t) = \langle a \cos(t), b \sin(t) \rangle$  with width  $2a$  and height  $2b$ . The result should depend on  $a$  and  $b$ .
- 4 After this summer school, you relax in a Jacuzzi and move along curve  $C$  which is given by part of the curve  $x^{10} + y^{10} = 1$  in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity  $\vec{F}(x, y) = \langle x, y^4 \rangle$ . Calculate the line integral  $\int_C \vec{F} \cdot d\vec{r}$ , the energy you gain from the fluid force.



- 5 Find a closed curve  $C : \vec{r}(t)$  for which the vector field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x^2 \rangle$$

satisfies  $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \neq 0$ .