Lecture 21: Greens theorem

Green's theorem is the second integral theorem in the plane. This entire section deals with multivariable calculus in the plane, where we have two integral theorems, the fundamental theorem of line integrals and Greens theorem. First two reminders:

If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a vector field and $C : \vec{r}(t) = \langle x(t), y(t) \rangle, t \in [a,b]$ is a curve, the **line integral**

$$\int_C \vec{F}\ d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\ dt$$

measures the work done by the field $\vec{F}$ along the path.

The **curl** of a two dimensional vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is defined as the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y)\ .$$

The curl(F) measures the **vorticity** of the vector field.

One can write $\nabla \times \vec{F} = \text{curl}(\vec{F})$ because the two dimensional cross product of (∂_x, ∂_y) with $\vec{F} = \langle P, Q \rangle$ is the scalar $Q_x - P_y$.

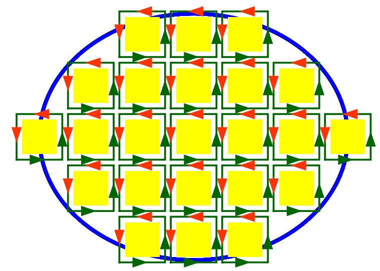
1 For $\vec{F}(x,y) = \langle -y, x \rangle$ we have curl$(F)(x,y) = 2$.

2 If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and $\text{curl}(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairot's theorem.

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a smooth vector field and R is a region for which the boundary C is a curve parametrized so that R is "to the left". Then

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_G \text{curl}(F)\ dxdy\ .$$

Proof. Look first at a small square $G = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $\vec{F} = \langle P, Q \rangle$ along the boundary is $\int_0^\epsilon P(x+t,y)dt + \int_0^\epsilon Q(x+\epsilon, y+t)\ dt - \int_0^\epsilon P(x+t, y+\epsilon)\ dt - \int_0^\epsilon Q(x, y+t)\ dt$. This line integral measures the "circulation" at the place (x,y). Because $Q(x+\epsilon, y) - Q(x,y) \sim Q_x(x,y)\epsilon$ and $P(x, y+\epsilon) - P(x,y) \sim P_y(x,y)\epsilon$, the line integral is $(Q_x - P_y)\epsilon^2 \sim \int_0^\epsilon \int_0^\epsilon \text{curl}(F)\ dxdy$. All identities hold in the limit $\epsilon \to 0$. To prove the statement for a general region G, chop it into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the squares is a Riemann sum approximation of the double integral.



George Green lived from 1793 to 1841. He was a physicist a self-taught mathematician and miller. His work greatly contributed to modern physics.

3 If $\vec{F}$ is a gradient field then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot \vec{dr}$ is zero by the fundamental theorem for line integrals. and $\int \int_G \text{curl}(F) \cdot dA$ is zero because $\text{curl}(F) = \text{curl}(\text{grad}(f)) = 0$.

The already established Clairot identity

$$\text{curl}(\text{grad}(f)) = 0$$

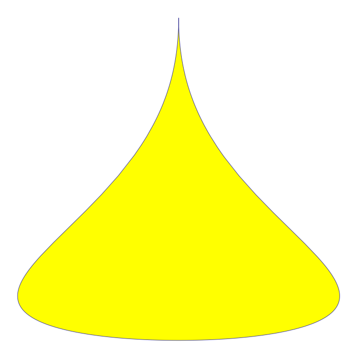
can also checked by writing it as $\nabla \times \nabla f$ and using that the cross product of two identical vectors is 0. Treating ∇ as a vector is called **nabla calculus**.

4 Find the line integral of $\vec{F}(x,y) = \langle x^2 - y^2, 2xy \rangle = \langle P, Q \rangle$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: $\text{curl}(\vec{F}) = Q_x - P_y = 2y - 2y = -4y$ so that $\int_C \vec{F}\ \vec{dr} = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

5 Find the area of the region enclosed by

$$\vec{r}(t) = \langle \frac{\sin(\pi t)^2}{t}, t^2 - 1 \rangle$$

for $-1 \le t \le 1$. To do so, use Greens theorem with the vector field $\vec{F} = \langle 0, x \rangle$.



6 Green's theorem allows to express the coordinates of the **centroid** = center of mass

$$(\int \int_G x \, dA/A, \int \int_G y \, dA/A)$$

using line integrals. With the vector field $\vec{F} = \langle 0, x^2 \rangle$ we have

$$\int \int_G x \, dA = \int_C \vec{F} \, d\vec{r} \, .$$

7 An important application of Green is to **compute area**. With the vector fields $\vec{F}(x,y) = \langle P, Q \rangle = \langle -y, 0 \rangle$ or $\vec{F}(x,y) = \langle 0, x \rangle$ have vorticity $\text{curl}(\vec{F})(x,y) = 1$. For $\vec{F}(x,y) = \langle 0, x \rangle$, the right hand side in Green's theorem is the **area** of G:

$$\text{Area}(G) = \int_C x(t)\dot{y}(t) \, dt \, .$$

8 Let G be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of G is 0 from $(a,0)$ to $(b,0)$ because $\vec{F}(x,y) = \langle 0,0 \rangle$ there. The line integral is also zero from $(b,0)$ to $(b,f(b))$ and $(a,f(a))$ to $(a,0)$ because $N = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b \langle -y(t), 0 \rangle \cdot \langle 1, f'(t) \rangle \, dt = \int_a^b f(t) \, dt$. Green's theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If $\vec{F}$ is a gradient field then $\text{curl}(F) = 0$ everywhere.

Is the converse true? Here is the answer:

A region R is called **simply connected** if every closed loop in R can be pulled together to a point in R.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\int_C \vec{F} \, d\vec{r} = 0$. So $\vec{F}$ has the closed loop property in G and is therefore a gradient field there.

In the homework, you look at an example of a not simply connected region where the $\text{curl}(\vec{F}) = 0$ does not imply that $\vec{F}$ is a gradient field.

An engineering application of Greens theorem is the **planimeter**, a mechanical device for measuring areas. We will demonstrate it in class. Historically it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field $\vec{F}$ associated to a planimeter which is obtained by placing a unit vector perpendicular to the arm).
One can prove that $\vec{F}$ has vorticity 1. The planimeter calculates the line integral of $\vec{F}$ along a given curve. Green's theorem assures it is the area.

Homework

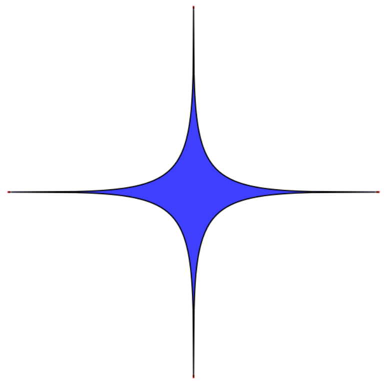
1 Calculate the line integral $\int_C \vec{F} \, d\vec{r}$ with $\vec{F} = \langle 2y + x\sin(y), x^2\cos(y) - 3y^{200\sin(y)} \rangle$ along a triangle C with edges $(0,0)$, $(\pi/2, 0)$ and $(\pi/2, \pi/2)$.

2 Evaluate the line integral of the vector field $\vec{F}(x,y) = \langle xy^2, x^2 \rangle$ along the rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$.

3 Find the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$$

using Green's theorem. The curve is parameterized by $t \in [0, 2\pi]$.



4 Let G be the region $x^6 + y^6 \leq 1$. Compute the line integral of the vector field $\vec{F}(x,y) = \langle x^6, y^6 \rangle$ along the boundary.

5 Let $\vec{F}(x,y) = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$. Let $C : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle, t \in [0, 2\pi]$.
a) Compute $\int_C \vec{F} \cdot d\vec{r}$.
b) Show that $\text{curl}(\vec{F}) = 0$ everywhere for $(x,y) \neq (0,0)$.
c) Let $f(x,y) = \arctan(y/x)$. Verify that $\nabla f = \vec{F}$.
d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does *a*) and *b*) not contradict Green's theorem?