

Lecture 23: Stokes Theorem

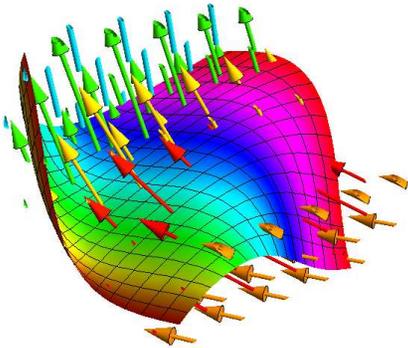
Assume a surface S is parametrized as $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ over a domain G in the uv -plane.

The **flux integral** of \vec{F} through S is defined as the double integral

$$\int \int_G \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dudv .$$

With the short hand notation $d\vec{S} = (\vec{r}_u \times \vec{r}_v) \, dudv$ representing an infinitesimal normal vector to the surface, this can be written as $\int \int_S \vec{F} \cdot d\vec{S}$. The interpretation is that if \vec{F} = fluid velocity field, then $\int \int_S \vec{F} \cdot d\vec{S}$ is the amount of fluid passing through S in unit time.

Because $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$ is a unit vector normal to the surface and on the surface, $\vec{F} \cdot \vec{n}$ is the normal component of the vector field with respect to the surface. One could write therefore also $\int \int_S \vec{F} \cdot d\vec{S} = \int \int \vec{F} \cdot \vec{n} \, dS$ where dS is the surface element we know from when we computed surface area. The function $\vec{F} \cdot \vec{n}$ is the scalar projection of \vec{F} in the normal direction. Whereas the formula $\int \int 1 \, dS$ gave the area of the surface with $dS = |\vec{r}_u \times \vec{r}_v| \, dudv$, the flux integral weights each area element dS with the normal component of the vector field with $\vec{F}(\vec{r}(u, v)) \cdot \vec{n}(\vec{r}(u, v))$. We do not use this formula for computations because computing \vec{n} gives additional work. We just determine the vectors $\vec{F}(\vec{r}(u, v))$ and $\vec{r}_u \times \vec{r}_v$ and integrate its dot product over the domain.



- 1 Compute the flux of $\vec{F}(x, y, z) = \langle 0, 1, z^2 \rangle$ through the upper half sphere S parametrized by

$$\vec{r}(u, v) = \langle \cos(u) \sin(v), \sin(u) \sin(v), \cos(v) \rangle .$$

Solution. We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u, v)) = \langle 0, 1, \cos^2(v) \rangle$ so that

$$\int_0^{2\pi} \int_0^\pi -\langle 0, 1, \cos^2(v) \rangle \cdot \langle \cos(u) \sin^2(v), \sin(u) \sin^2(v), \cos(v) \sin(v) \rangle \, dudv .$$

The flux integral is $\int_0^{2\pi} \int_{\pi/2}^\pi -\sin^2(v) \sin(u) - \cos^3(v) \sin(v) \, dudv$ which is $-\int_{\pi/2}^\pi \cos^3 v \sin(v) \, dv = \cos^4(v)/4 \Big|_0^{\pi/2} = -1/4$.

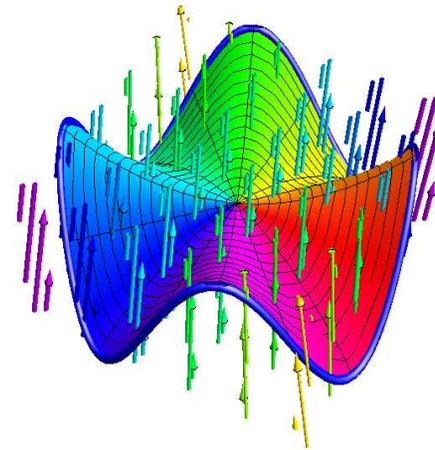
- 2 Calculate the flux of the vector field $\vec{F}(x, y, z) = \langle 1, 2, 4z \rangle$ through the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 \leq 1$. **Solution:** We can parametrize the surface as $\vec{r}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), r^2 \rangle$ where $\vec{r}_r \times \vec{r}_\theta = \langle -2r^2 \cos(\theta), -2r^2 \sin(\theta), r \rangle$ and $\vec{F}(\vec{r}(u, v)) = \langle 1, 2, 4r^2 \rangle$. We get $\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2 \cos(v) - 4r^2 \sin(v) + 4r^3) \, drd\theta = 2\pi$.
- 3 Compute the flux of $\vec{F}(x, y, z) = \langle 2, 3, 1 \rangle$ through the torus parameterized as $\vec{r}(u, v) = \langle (2 + \cos(v)) \cos(u), (2 + \cos(v)) \sin(u), \sin(v) \rangle$, where both u and v range from 0 to 2π . **Solution.** There is no computation is needed. Think about what the flux means.

The following theorem is the second fundamental theorem of calculus in three dimensions. Since we integrate over two and one dimensional objects many concepts of multivariable calculus come together.

The **boundary** of a surface is a curve oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector v , a vector w pointing towards the surface and the normal vector n to the surface form a right handed coordinate system.

Stokes theorem: Let S be a surface bounded by a curve C and \vec{F} be a vector field. Then

$$\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$



The boundary curve C is oriented such that if you walk along the surface. If your head points into the direction of the normal $\vec{r}_u \times \vec{r}_v$, then the surface is to your left.

One can prove this similarly as Greens theorem. Chop up a as a union of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along these boundaries adds up to the line integral of the boundary of S . Stokes theorem for a small triangle can be reduced to Greens theorem because with a coordinate system such that the triangle is in the $x - y$ plane, the flux of the field is the double integral $Q_x - P_y$.

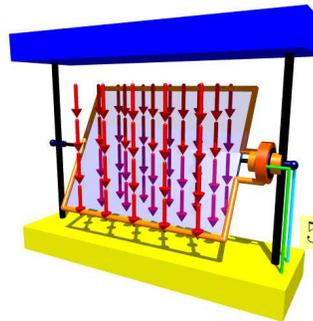
4 Let $\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$ and let S be the upper semi hemisphere, then $\text{curl}(\vec{F})(x, y, z) = \langle 0, 0, 2 \rangle$. The surface is parameterized by $\vec{r}(u, v) = \langle \cos(u) \sin(v), \sin(u) \sin(v), \cos(v) \rangle$ on $G = [0, 2\pi] \times [0, \pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}'(u, v)$ so that $\text{curl}(\vec{F})(x, y, z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v) \sin(v) 2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v) dv du = 2\pi$. The boundary C of S is parameterized by $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ so that $d\vec{r} = \vec{r}'(t)dt = \langle -\sin(t), \cos(t), 0 \rangle dt$ and $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F} \cdot d\vec{r}$ along the boundary is 2π .

5 If S is a surface in the xy -plane and $\vec{F} = \langle P, Q, 0 \rangle$ has zero z component, then $\text{curl}(\vec{F}) = \langle 0, 0, Q_x - P_y \rangle$ and $\text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y dx dy$. We see that for a surface which is flat, Stokes theorem is a consequence of Green's theorem. If we put the coordinate axis so that the surface is in the xy -plane, then the vector field F induces a vector field on the surface such that its 2D curl is the normal component of $\text{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\text{curl}(\vec{F}) \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ is the two dimensional curl: $\vec{F}(\vec{r}(u, v)) \cdot \langle 0, 0, 1 \rangle = Q_x - P_y$. If C is the boundary of the surface, then $\int_S \vec{F}(\vec{r}(u, v)) \cdot \langle 0, 0, 1 \rangle dudv = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$.

6 Calculate the flux of the curl of $\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$ through the surface parameterized by $\vec{r}(u, v) = \langle \cos(u) \cos(v), \sin(u) \cos(v), \cos^2(v) + \cos(v) \sin^2(u + \pi/2) \rangle$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

7 For every surface bounded by a curve C , the flux of $\text{curl}(\vec{F})$ through the surface is the same. Proof. The flux of the curl of a vector field through a surface S depends only on the boundary of S . Compare this with the earlier statement that for every curve between two points A, B the line integral of $\text{grad}(f)$ along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C .

The electric field E and the magnetic field B are linked by the **Maxwell equation** $\text{curl}(\vec{E}) = -\dot{\vec{B}}$. Take a closed wire C which bounds a surface S and consider $\int_S \vec{B} \cdot d\vec{S}$, the flux of the magnetic field through S . Its change can be related with a voltage using Stokes theorem: $d/dt \int_S \vec{B} \cdot d\vec{S} = \int_S \dot{\vec{B}} \cdot d\vec{S} = \int_S -c \text{curl}(\vec{E}) \cdot d\vec{S} = -c \int_C \vec{E} \cdot d\vec{r} = U$, where U is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator like an alternator in a car. Stokes theorem explains why we can generate electricity from motion.



Stokes theorem was found by Ampère in 1825. George Gabriel Stokes (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840.



George Gabriel Stokes



André Marie Ampère

Homework

1 Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \langle x^2y, x^3/3, xy \rangle$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

2 If S is the surface $x^6 + y^6 + z^6 = 1$ and assume \vec{F} is a smooth vector field. Explain why $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

3 Evaluate the flux integral

$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S},$$

where $\vec{F}(x, y, z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x \rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z + 1)^2 = 2, z > 0$ oriented so that the normal vector points upwards.

4 Find the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the circle of radius 3 in the xz -plane oriented counter clockwise when looking from the point $(0, 1, 0)$ onto the plane and where \vec{F} is the vector field

$$\vec{F}(x, y, z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z)) \rangle.$$

Use a convenient surface S which has C as a boundary.

Find the flux integral $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where $\vec{F}(x, y, z) = \langle 2 \cos(\pi y)e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y)e^{2x}, 2xz \rangle$ and S is the **thorn** surface parametrized by

$$\vec{r}(s, t) = \langle (1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s \rangle$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.

