1: Geometry and Distance

The arena for multivariable calculus is the two-dimensional **plane** and the three dimensional **space**.

A point in the **plane** has two **coordinates** $P = (x, y)$. A point in space is determined by three coordinates $P = (x, y, z)$. The signs of the coordinates define 4 **quadrants** in the plane and 8 **octants** in space. These regions by intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate axes** $\{y = 0\}$ and $\{x = 0\}$ or **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$.

In two dimensions, the x-coordinate usually directs to the "east" and the y-coordinate points "north". In three dimensions the usual coordinate system has the xy-plane as the "ground" and the z-coordinate axes pointing "up".

1 $P = (2, -3)$ is in the forth quadrant of the plane and $P = (1, 2, 3)$ is in the positive octant of space. The point $(0, 0, -5)$ is on the negative z axis. The point $(1, 2, -3)$ is below the xy-plane.

2 **Problem.** Find the midpoint M of $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$. **Answer.** The midpoint is obtained by taking the average of each coordinate $M = (P + Q)/2 = (-1, 3, 6)$.

3 In computer graphics of photography, the xy-plane contains the retina or film plate. The z coordinate measures the distance towards the viewer. In this **photographic coordinate** system your eyes and mouth are in the plane $z = 0$ and your nose points in the z direction. If the midpoint of your eyes is the origin of the coordinate system and your eyes have the coordinates $(1, 0, 0)$, $(-1, 0, 0)$, then the tip of your nose might have the coordinates $(0, -1, 1)$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}$.

This Euclidean distance is a definition but motivated by **Pythagoras theorem**.

4 **Problem:** Find the distance $d(P, Q)$ between the points $P = (1, 2, 5)$ and $Q = (-3, 4, 7)$ and verify that $d(P, M) + d(Q, M) = d(P, Q)$. **Answer:** The distance is $d(P, Q) = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$. The distance $d(P, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. The distance $d(Q, M)$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. Indeed $d(P, M) + d(M, Q) = d(P, Q)$.

Remarks.
1) Distances can be introduced more abstractly: take any nonnegative function $d(P, Q)$ which satisfies the **triangle inequality** $d(P, Q) + d(Q, R) \geq d(P, R)$ and $d(P, Q) = 0$ if and only if $P = Q$. A set X with such a distance function d is called a **metric space**. Examples of distances are the **Manhatten distance** $d_m(P, Q) = |x - a| + |y - b|$, the **quartic distance** $d_4(P, Q) = ((x - a)^4 + (y - b)^4$ or the **Fermat distance** $d_f(x, y) = d(x, y)$ if $y > 0$ and $d_f(x, y) = 1.33d(x, y)$ if $y < 0$. The constant 1.33 is the **refractive index** and models the upper half plane being filled with air and the lower half plane with water. Shortest paths are bent at the water surface. Each of these distances d, d_m, d_4, d_f make the plane a different metric space.
2) It is **symmetry** which distinguishes the Euclidean distance as the most natural one. The Euclidean distance is determined by $d((1, 0, 0), (0, 0, 0)) = 1$, rotational and translational and scale symmetry $d(\lambda P, \lambda Q) = \lambda d(P, Q)$.
3) We usually work with a **right handed coordinate system**, where the x, y, z axes can be matched with the thumb, pointing and middle finger of the **right hand**. The photographers coordinate system is an example of a **left handed coordinate system**. The x, y, z axes are matched with the thumb and pointing finger and middle finger of the left hand. Nature is not oblivious to parity. Some laws of particle physics are different when they are observed in a mirror. Coordinate systems with different parity can not be rotated into each other.

Points, **curves**, **surfaces** and **solid bodies** are geometric objects which can be described with **functions of several variables**. An example of a curve is a line, an example of a surface is a plane, an example of a solid is the interior of a sphere. We focus in this first lecture on spheres or circles.

A **circle** of radius r centered at $P = (a, b)$ is the collection of points in the plane which have distance r from P.

A **sphere** of radius ρ centered at $P = (a, b, c)$ is the collection of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2+(y-b)^2+(z-c)^2 = \rho^2$.

An **ellipse** is the collection of points P in the plane for which the sum $d(P, A) + d(P, B)$ of the distances to two points A, B is a fixed constant l larger than $d(A, B)$. This allows to draw the ellipse with a string of length l attached at A, B. An algebraic equivalent description is the set of points satisfying an equation $x^2/a^2 + y^2/b^2 = 1$.

5 **Problem:** Is the point $(3, 4, 5)$ outside or inside the sphere $(x-2)^2+(y-6)^2+(z-2)^2 = 16$? **Answer**: The distance of the point to the center of the sphere is $\sqrt{1 + 4 + 9}$ which is smaller than 4 the radius of the sphere. The point is inside.

6 **Problem:** Find an algebraic expression for the set of all points for which the sum of the distances to $A = (1, 0)$ and $B = (-1, 0)$ is equal to 3. **Answer:** Square the equation $\sqrt{(x - 1)^2 + y^2} + \sqrt{(x + 1)^2 + y^2} = 3$, separate the remaining single square root on one side and square again. Simplification gives $20x^2 + 36y^2 = 45$ which is equivalent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b can be computed as follows: because $P = (a, 0)$ satisfies this equation, $d(P, A) + d(P, B) = (a - 1) + (a + 1) = 3$ so that $a = 3/2$. Similarly, the point $Q = (0, b)$ satisfying it gives $d(Q, A) + d(P, B) = 2\sqrt{b^2 + 1} = 3$ or $b = \sqrt{5}/2$.

Here is a verification with the computer algebra system Mathematica. Writing $L = d(P, A)$ and $M = d(P, B)$ we simplify the equation $L^2 + M^2 = 3^2$. The part without square root is $((L + M)^2 + (L - M)^2)/2 - 3^2$. The remaining square root is $((L + M)^2 - (L - M)^2)/2$. Now square both and set them equal to see the equation $20x^2 + 36y^2 = 45$.

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L=Sqrt[(x-1)^2+y^2]; M=Sqrt[(x+1)^2+y^2];
Simplify[(((L+M)^2 + (L-M)^2)/2-3^2)^2 == (((L+M)^2-(L-M)^2)/2)^2]
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The **completion of the square** of an equation $x^2 + bx + c = 0$ is the idea to add $(b/2)^2 - c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives the solution $x = -b/2 \pm \sqrt{(b/2)^2 - c}$.

7 The equation $2x^2 - 10x + 12 = 0$ is equivalent to $x^2 + 5x = -6$. Adding $(5/2)^2$ on both sides gives $(x + 5/2)^2 = 1/4$ so that $x = 2$ or $x = 3$.

8 The equation $x^2 + 5x + y^2 - 2y + z^2 = -1$ is after completion of the square $(x + 5/2)^2 - 25/4 + (y - 1)^2 - 1 + z^2 = -1$ or $(x - 5/2)^2 + (y - 1)^2 + z^2 = (5/2)^2$. We see a sphere **center** $(5/2, 1, 0)$ and **radius** $5/2$.

The method is due to **Al-Khwarizmi** who lived from 780-850 and used it as a method to solve quadratic equations. Even so Al-Khwarizmi worked with numerical examples, it is one of the first important steps of algebra. His work *"Compendium on Calculation by Completion and Reduction"* was dedicated to the Caliph **al Ma'mun**, who had established research center called "House of Wisdom" in Baghdad. [1] In an appendix to "Geometry" of his "Discours de la méthode" which appeared in 1637, **René Descartes** promoted the idea to use algebra to solve geometric problems. Even so Descartes mostly dealt with ruler-and compass constructions, the rectangular coordinate system is now called the **Cartesian coordinate system**. His ideas profoundly changed mathematics. Ideas do not grow in a vacuum. Davis and Hersh write that in its current form, Cartesian geometry is due as much to Descartes own contemporaries and successors as to himself. [2]

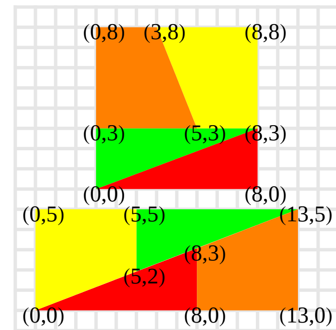
What happens in higher dimensions? A point in four dimensional space for example is labeled with four coordinates (t, x, y, z). In how many hyper chambers is space divided by the coordinate hyperplanes $t = 0, x = 0, y = 0, z = 0$? Answer: There are 16 hyper-regions and each of them contains one of the 16 points (x, y, z, w), where x, y, z, w are either $+1$ or -1.

Homework

1 Describe and sketch the set of points $P = (x, y, z)$ in three dimensional space $\mathbf{R}^3$ represented by
a) $(x - 1)^2 + z^2 = 4$
b) $x - y - z = 2$
c) $xyz = 0$
d) $x^2 = y$

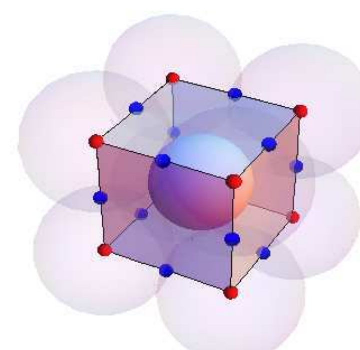
2 a) Find the distances of $P = (3, 4, 0)$ to each of the 3 coordinate axes.
b) Find the distances of $P = (1, 2, 5)$ to each of the 3 coordinate planes.

3 Below you see two rectangles. One has the area $8 \cdot 8 = 64$. The other has the area $65 = 13 \cdot 5$. But these triangles are made up by triangles or trapezoids which match. Measure various distances to see what is going on.



4 Find the center and radius of the sphere $x^2 + 2x + y^2 - 16y + z^2 + 10z + 54 = 0$. Describe the traces of this surface, its intersection with each of the coordinate planes.

5 We place unit spheres on the corners of a unit cube of side length 2 so that adjacent spheres touch. How large is the radius of the sphere in the center of the cube kissing all the 8 spheres?



[1]The book "The mathematics of Egypt, Mesopotamia,China, India and Islam, a Sourcebook, Ed Victor Katz, page 542 contains translations of some of this work.
[2]An entertaining read is "Descartes Secret Notebook" by Amir Aczel.

2: Vectors and Dot Product

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ in space define a **vector** $\vec{v} = \langle x - a, y - b - z - c\rangle$. It points from P to Q and we write also $\vec{v} = \vec{PQ}$. The real numbers numbers p, q, r in a vector $\vec{v} = \langle p, q, r\rangle$ are called the **components** of $\vec{v}$.

Vectors can be drawn **everywhere** in space but two vectors with the same components are considered **equal**. Vectors can be translated into each other if and only if their components are the same. If a vector starts at the origin $O = (0, 0, 0)$, then the vector $\vec{v} = \langle p, q, r\rangle$ points to the point (p, q, r). One can therefore identify points $P = (a, b, c)$ with vectors $\vec{v} = \langle a, b, c\rangle$ attached to the origin. To make more clear which objects are vectors, we sometimes draw an arrow on top of it and if $\vec{v} = \vec{PQ}$ then P is the "tail" and Q is the "head" of the vector. To distinguish vectors from points, it is custom to different brackets and write $\langle 2, 3, 4\rangle$ for vectors and $(2, 3, 4)$ for points.

The **sum** of two vectors is $\vec{u} + \vec{v} = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1 + v_1, u_2 + v_2\rangle$. The **scalar multiple** $\lambda\vec{u} = \lambda\langle u_1, u_2\rangle = \langle \lambda u_1, \lambda u_2\rangle$. The difference $\vec{u} - \vec{v}$ can best be seen as the addition of $\vec{u}$ and $(-1) \cdot \vec{v}$.

The vectors $\vec{i} = \langle 1, 0\rangle$, $\vec{j} = \langle 0, 1\rangle$ are called **standard basis vectors** in the plane. In space, one has the basis vectors $\vec{i} = \langle 1, 0, 0\rangle$, $\vec{j} = \langle 0, 1, 0\rangle$, $\vec{k} = \langle 0, 0, 1\rangle$.

Every vector $\vec{v} = \langle p, q\rangle$ in the plane can be written as a combination $\vec{v} = p\vec{i} + q\vec{j}$ of standard basis vectors and every vector $\vec{v} = \langle p, q, r\rangle$ in space can be written as $\vec{v} = p\vec{i} + q\vec{j} + r\vec{k}$. Vectors are abundant in applications. They appear in mechanics: if $\vec{r}(t) = \langle f(t), g(t)\rangle$ is a point in the plane which depends on time t, then $\vec{v} = \langle f'(t), g'(t)\rangle$ will be called the **velocity vector** at $\vec{r}(t)$. Here $f'(t), g'(t)$ are the derivatives. In physics, we often want to determine forces acting on objects. Forces are represented as vectors. In particular, electromagnetic or gravitational fields or velocity fields in fluids are described by vectors. Vectors appear also in computer science: the scalable vector graphics is a standard for the web for describing two-dimensional graphics. In quantum computation, rather than working with bits, one deals with **qbits**, which are vectors. Finally, **color** can be written as a vector $\vec{v} = \langle r, g, b\rangle$, where r is **red**, g is **green** and b is **blue** component of the color vector. An other coordinate system for color is $\vec{v} = \langle c, m, y\rangle = \langle 1 - r, 1 - g, 1 - b\rangle$, where c is **cyan**, m is **magenta** and y is **yellow**. Vectors appear in probability theory and statistics. On a finite probability space, a **random variable** is a vector.

The addition and scalar multiplication of vectors satisfy the laws you know from **arithmetic**. **commutativity** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$, **associativity** $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ and $r * (s * \vec{v}) = (r * s) * \vec{v}$ as well as **distributivity** $(r + s)\vec{v} = \vec{v}(r + s)$ and $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$, where $*$ denotes multiplication with a scalar.

The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is a unit vector.

1 $|\langle 3, 4\rangle| = 5$ and $|\langle 3, 4, 12\rangle| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$ and $\langle 3/5, 4/5\rangle$ and $\langle 3/13, 4/13, 12/13\rangle$. The only vector of length 0 is the zero vector $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = \langle a, b, c\rangle$ and $\vec{w} = \langle p, q, r\rangle$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$.

Remarks.
a) Different notations for the dot product are used in different mathematical fields. while pure mathematicians write $\vec{v} \cdot \vec{w} = (\vec{v}, \vec{w})$, one can see $\langle \vec{v}|\vec{w}\rangle$ in quantum mechanics or $v_i w^i$ or more generally $g_{ij}v^i w^j$ in general relativity. The dot product is also called **scalar product** or **inner product**.
b) Any product $g(v, w)$ which is linear in v and w and satisfies the symmetry $g(v, w) = g(w, v)$ and $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$ can be used as a dot product. An example is $g(v, w) = 3v_1w_1 + 2v_2w_2 + v_3w_3$.

The dot product determines distance and distance determines the dot product.

Proof: Lets write $v = \vec{v}$ in this proof. Using the dot product one can express the length of v as $|v| = \sqrt{v \cdot v}$. On the other hand, from $(v + w) \cdot (v + w) = v \cdot v + w \cdot w + 2(v \cdot w)$ can be solved for $v \cdot w$:

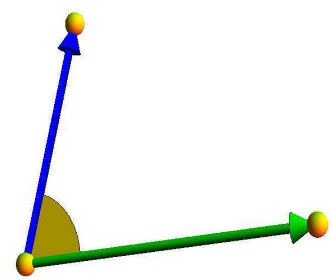
$$v \cdot w = (|v + w|^2 - |v|^2 - |w|^2)/2 .$$

The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. We can assume $|w| = 1$ after scaling the equation. Now plug in $a = v \cdot w$ into the equation $0 \leq (v - aw) \cdot (v - aw)$ to get $0 \leq (v - (v \cdot w)w) \cdot (v - (v \cdot w)w) = |v|^2 + (v \cdot w)^2 - 2(v \cdot w)^2 = |v|^2 - (v \cdot w)^2$ which means $(v \cdot w)^2 \leq |v|^2$.

Having established this, we have a clean definition of what an **angle** is:

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ which satisfies $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$.



Al Kashi's theorem: If a, b, c are the side lengths of a triangle ABC and α is the angle opposite to c, then $a^2 + b^2 = c^2 - 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.

The angle definition works in any space with a dot product. In statistics you have to work with vectors of n components. They are called data or random variables and $\cos(\alpha)$ is called the **correlation** between two random variables $\vec{v}, \vec{w}$ of zero **expectation** $\mathrm{E}[\vec{v}] = (v_1 + \cdots + v_n)/n$. The dot product $v_1 w_1 + \ldots + v_n w_n$ is then the **covariance**, the length $|v|$ is the **standard deviation** and denoted by $\sigma(v)$. The formula $\mathrm{Corr}[v,w] = \mathrm{Cov}[v,w]/(\sigma(v)\sigma(w))$ for the correlation is the familiar angle formula we have seen. It is geometry in n dimensions. We mention this only to convince you that the geometry we do here can be applied to much more. All the computations we have done go through verbatim.

The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

Proof: $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u} \cdot \vec{v}| \leq \vec{u}^2 + \vec{v}^2 + 2|\vec{u}| \cdot |\vec{v}| = (|\vec{u}| + |\vec{v}|)^2$.

Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = \langle 2, 3 \rangle$ is orthogonal to $\vec{w} = \langle -3, 2 \rangle$.

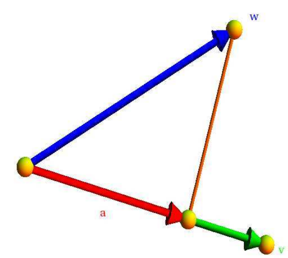
Having given precise definitions of all objects we can now prove **Pythagoras theorem**:

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.

Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$.

Remarks:
1) You have just seen something very powerful: results like Pythagoras (570-495BC) and Al Khashi (1380-1429) theorems were **derived from scratch** on a space V equipped with a dot product. The dot product appeared much later in mathematics (Hamilton 1843, Grassman 1844, Sylvester 1851, Cayley 1858). While we have used geometry as an intuition, the structure was built algebraically without any unjustified assumptions. This is mathematics: if we have a space V in which addition $\vec{v} + \vec{w}$ and scalar multiplication $\lambda \vec{v}$ is given and in which a dot product is defined, then all the just derived results apply. We have **not used** results of Al Khashi or Pythagoras but we have **derived** them and additionally obtained a **clear definition** what an angle is.
2) The derivation you have seen works in any dimension. Why do we care about higher dimensions? As already mentioned a compelling motivation is **statistics**. Given 12 data points like the average monthly temperatures in a year, we deal with a 12 dimensional space. Geometry is useful to describe data. Pythagoras theorem is the property that the variance of two uncorrelated random variables adds up with the formula $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3) A far reaching generalization of the geometry you have just seen is obtained if the dot product $g(v, w)$ is allowed to depend on the place, where the two vectors are attached. This produces **Riemannian geometry** and allows to work with spaces which are intrinsically **curved**. This mathematics is important in **general relativity** which describes gravity in a geometric way and which is one of the pillars of modern physics. But it appears in daily life too. On a hot summer day, if you look close at an object a hot asphalt street, the object can appear distorted or flickers. The dot product depends on the temperature of the air. Light rays no more move on straight lines but gets bent. In extreme cases, when the curvature of light rays is larger than the curvature of the earth, it leads to **Fata morgana** effects: one can see objects which are located beyond the horizon.
4) Why do we not introduce vectors not just as algebraic objects $\langle 1, 2, 3 \rangle$? The reason is that in many applications like physics and even geometry, one wants to work with **affine vectors**, vectors which are attached at points. Forces for example act on points of a body, we will also look at vector fields, where at each point a vector is attached. Considering vectors with the same components as equal gives then the vector space in which we do the algebra. One could define a vector space axiomatically and then build from this affine vectors but it is a bit too abstract and not much is actually gained for the goals we have in mind. An even more modern point of view replaces affine vectors with members of a tangent bundle. But this is only necessary if one deals with spaces which are not flat. Even more general is to allow the space attached at each point to be a more general space like a "group" called fibres. So called "fibre bundles" are the framework of mathematical concepts which describe elementary particles or even space itself. Attaching a circle for example at each point leads to electromagnetism attaching classes of two dimensional matrices leads to the weak force and attaching certain three dimensional matrices leads to the strong force. Allowing this to happen in a curved framework incorporates gravity. One of the main challenges is to include quantum mechanics into that picture. Fundamental physics has become primarily the quest to answer the question "what is space"?

The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$ is a signed length of the vector projection. Its absolute value is the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to the $\vec{w}$-direction.



2 For example, with $\vec{v} = \langle 0, -1, 1 \rangle$, $\vec{w} = \langle 1, -1, 0 \rangle$, $\mathrm{P}(\vec{v}) = \langle 1/2, -1/2, 0 \rangle$. Its length is $1/\sqrt{2}$.

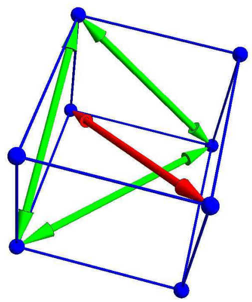
3 Projections are important in physics. For example, if you apply a wind force $\vec{F}$ to a car which drives in the direction $\vec{w}$ and P denotes the projection on $\vec{w}$ then $P(\vec{F})$ is the force which accelerates or slows down the car.

The projection allows to visualize the dot product. The absolute value of the dot product is the length of the projection. The dot product is positive if v points more towards to w, it is negative if v points away from it. In the next lecture we use the projection to compute distances between various objects.

Homework

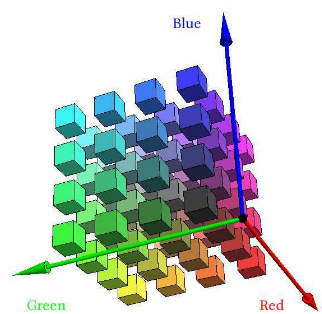
1 Find a unit vector parallel to $\vec{u} - \vec{v}$ if $\vec{u} = \langle 5, 6, 3 \rangle$ and $\vec{v} = \langle 1, 1, 3 \rangle$.

2 An **Euler brick** is a cuboid of dimensions a, b, c such that all face diagonals are integers.
a) Verify that $\vec{v} = \langle a, b, c \rangle = \langle 240, 117, 44 \rangle$ is a vector which leads to an Euler brick.
b) Verify that $\langle a, b, c \rangle = \langle u(4v^2 - w^2), v(4u^2 - w^2), 4uvw \rangle$ leads to an Euler brick if $u^2 + v^2 = w^2$.
If also the space diagonal $\sqrt{a^2 + b^2 + c^2}$ is an integer, an Euler brick is called **perfect**. Nobody has found one, nor proven that it can not exist.



3 **Colors** are encoded by vectors $\vec{v} = \langle$ red , brightgreen , blue $\rangle$. The red, green and blue components of $\vec{v}$ are all real numbers in the interval $[0, 1]$.
a) Determine the angle between the colors yellow and cyan.
b) What is the projection of the mixture $(\vec{v} + \vec{w})/2$ of magenta and orange onto blue?

(0,0,0)	black	(0,0,1)	blue
(1,1,1)	white	(1,1,0)	yellow
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	gray	(1,0,1)	magenta
(1,0,0)	red	(0,1,1)	cyan
(0,1,0)	green	$(1, \frac{1}{2}, 0)$	orange
$(0, 1, \frac{1}{2})$	spring green	$(1, 1, \frac{1}{2})$	khaki
$(1, \frac{1}{2}, \frac{1}{2})$	pink	$(\frac{1}{2}, \frac{1}{4}, 0)$	brown



4 Find the angle between the diagonal of the unit cube and one of the diagonal of one of its faces. Assume that the two diagonals go through the same edge of the cube. You can leave the answer in the form $\cos(\alpha) = ...$

5 Assume $\vec{v} = \langle -4, 2, 2 \rangle$ and $\vec{w} = \langle 3, 0, 4 \rangle$.
a) Find the vector projection of $\vec{v}$ onto $\vec{w}$.
b) Find the scalar component of $\vec{v}$ on $\vec{w}$.

3: Cross product

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ in the plane is the scalar $v_1 w_2 - v_2 w_1$.

To remember this, we can write it as a determinant: take the product of the diagonal entries and subtract the product of the side diagonal. $\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$.

The **cross product** of two vectors $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ in space is defined as the vector

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle \ .$$

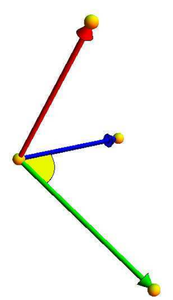
To remember it we write the product as a "determinant":

$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$.

1 The cross product of $\langle 1, 2 \rangle$ and $\langle 4, 5 \rangle$ is $5 - 8 = -3$.

2 The cross product of $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 1 \rangle$ is $\langle -13, 11, -3 \rangle$.



The cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. The product is anti-commutative.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.

The **sin** formula: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof: We verify first the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$.

Note that this was a definition so that nothing needs to be proven. But we want to make sure that the definition fits with our common intuition we have about area: $|\vec{w}| \sin(\alpha)$ is the height of the parallelogram with base length $|\vec{v}|$. We see from the sin-formula that the area does not change if we rotate the vectors around in space because both length and angle stay the same. Area also is linear in each of the vectors v and w. If we make v twice as long, then the area gets twice as large.

$\vec{v} \times \vec{w}$ is zero exactly if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda \vec{w}$ for some real λ.

Proof. This follows immediately from the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are also considered parallel evenso sometimes one calls this anti-parallel.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma$.

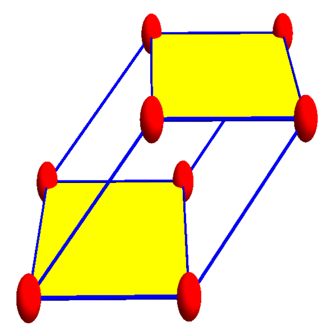
Proof. We express the area of the triangle in three different ways:

$$ab \sin(\gamma) = bc \sin(\alpha) = ac \sin(\beta) \ .$$

Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

3 If $\vec{v} = \langle a, 0, 0 \rangle$ and $\vec{w} = \langle b\cos(\alpha), b\sin(\alpha), 0 \rangle$, then $\vec{v} \times \vec{w} = \langle 0, 0, ab\sin(\alpha) \rangle$ which has length $|ab\sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$.



The absolute value of $[\vec{u}, \vec{v}, \vec{w}]$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$.

The **orientation** of three vectors is the sign of $[\vec{u}, \vec{v}, \vec{w}]$. It is positive if the three vectors form a right handed coordinate system.

Again, we do not have to prove anything since we have just **defined** volume and orientation. Let us still see why this fits with with our intuition about volume. The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is indeed the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand. For example, the vectors $\vec{i}, \vec{j}, \vec{i} \times \vec{j} = \vec{k}$ form a right handed coordinate system.

Since the triple scalar product is linear with respect to each vector we also see that volume is additive. Adding two equal parallelepipeds together for example gives a parallelepiped with a volume twice the volume.

4 **Problem:** Find the volume of a **cuboid** of width a length b and height c. **Answer**. The cuboid is a parallelepiped spanned by $\langle a, 0, 0 \rangle$ $\langle 0, b, 0 \rangle$ and $\langle 0, 0, c \rangle$. The triple scalar product is abc.

5 **Problem** Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that it is spanned by the vectors $\vec{u} = \langle 1, 2, 1 \rangle$, $\vec{v} = \langle 3, 2, 1 \rangle$, and $\vec{w} = \langle 0, 3, 1 \rangle$. We get $\vec{v} \times \vec{w} = \langle -1, -3, 9 \rangle$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

You can skip the following remarks: We have seen that in two dimensions, the cross product is a scalar and in three dimensions, the cross product is a vector. What happens in higher dimensions? There is a generalization called **wedge product** $\vec{v} \wedge \vec{w}$ but the resulting vector is in general in a new "super" space. In four dimensions for example, the wedge product between two vectors $\vec{v}$ and $\vec{w}$ is an object with $6 = 4(4-1)/2$ components. In n dimensions, the product has $n(n-1)/2$ components:

$$\vec{v} \wedge \vec{w} = \langle v_1 w_2 - v_2 w_1, v_1 v_3 - v_3 w_1, v_1 w_4 - v_4 w_1, v_2 w_3 - v_3 w_2, v_2 v_4 - v_4 v_2, v_3 w_4 - v_4 v_3 \rangle .$$

The length of $\vec{v} \wedge \vec{w}$ is the area of the parallelepiped spanned by $\vec{v}$ and $\vec{w}$. It is only in our three dimensional space that we can identify the wedge product between two vectors as a 3-vector again. The wedge product can be extended to an algebra of dimension 2^n. It is called a **superalgebra**. An other approach to generalize the product is to use linear algebra and write a vector v as a 3×3 matrix V, now $v \times w$ corresponds to $VW - WV$. For those of you who take linear algebra:

$$V = \begin{bmatrix} 0 & -v_1 & v_3 \\ v_1 & 0 & -v_2 \\ -v_3 & v_2 & 0 \end{bmatrix}, W = \begin{bmatrix} 0 & -w_1 & w_3 \\ w_1 & 0 & -w_2 \\ -w_3 & w_2 & 0 \end{bmatrix}, VW - WV = \begin{bmatrix} 0 & -(v_2 w_3 - v_3 w_2) & v_1 w_2 - v_2 w_1 \\ v_2 w_3 - v_3 w_2 & 0 & -(v_3 w_1 - v_1 w_3) \\ v_2 w_1 - v_1 w_2 & -v_1 w_3 - w_1 v_3 & 0 \end{bmatrix} .$$

This is what one calls a **Lie algebra**. This can be generalized to $n \times n$ matrices. While for $n = 2$ we got the cross product in two dimensions and for $n = 3$ the cross product in 3 dimensions, we get for $n = 4$ a cross product in six dimensions. Also the triple scalar product has a generalization in n dimensions. Given n vectors, we can write them into a $n \times n$ matrix. The absolute value of the **determinant** of this matrix defines the n-dimensional volume of the parallelepiped spanned by these n vectors.

Historically, the dot product and cross product emerged about at the same time. Determinants were studied already by Gauss in 1801, matrix multiplication in 1812 by Binet. The dot product can be seen as a special case of matrix multiplication. Only in 1844 geometry in n dimensions started to be developed by Grassman. It was Hamilton who described in 1843 first a multiplication $*$ of 4 vectors. It contains intrinsically both dot and cross product because $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-vw, v \times w)$. The cross product can also be realized using matrix multiplication $AB - BA$ of skew symmetric matrices. The proper way to see this now is tensor analysis which started in 1890 with Ricci which contains a tensor product which has an anticommutative version called **exterior algebra**, an example of a superalgebra. More Vector calculus was also developed at the same time by Clifford, Gibbs, Heaviside. This leads to mathematics developed in the 20'th century developed first by Poincare an Elie Cartan about 100 years ago. Motivated heavily by physics, this was generalized to Spinor algebras which are special cases of Clifford algebras. It is fascinating to see that these geometric constructions appear in fundamental physics in the structure of elementary particles. But the dot and cross product appears in down to earth physics: It appears in fundamental equations like the **Lorentz force** $F = q(v \times B)$ which depends on the velocity v of a particle, the charge q and the magnetic field B. An other example is the **angular momentum** $mr \times v$ or **Coriolis force** $--2m\omega \times v$, where ω is a vector in the rotation axes and v is the velocity.

Homework

1 a) Find the volume of the parallelepiped for which the base parallelogram is given by the points $(0, 0, 0), (1, 0, 1, (2, 2, 1), (1, 2, 0)$ and which has an edge connecting $(0, 0, 0)$ with $(3, 4, 5)$.
b) Find the area of the base and use a) to get the height.

2 a) Assume $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. Verify that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$.
b) Find $(\vec{u} + \vec{v}) \cdot (\vec{v} \times \vec{w})$ if $\vec{u}, \vec{v}, \vec{w}$ are unit vectors which are orthogonal to each other and $\vec{u} \times \vec{v} = \vec{w}$.

3 To find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$, we find a vector $\langle a, b, c \rangle$ normal to the plane and fix d so that P is in the plane.

4 Verify the Lagrange formula

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

for general vectors $\vec{a}, \vec{b}, \vec{c}$ in space. The formula can be remembered as "BAC minus CAB".

5 Assume you know that the triple scalar product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ between $\vec{u}, \vec{v}, \vec{w}$ is equal to 4. Find the values of $[\vec{v}, \vec{u}, \vec{w}]$ and $[\vec{u} + \vec{v}, \vec{v}, \vec{w}]$.

4: Lines and Planes

A point $P = (p, q, r)$ and a vector $\vec{v} = \langle a, b, c \rangle$ define the **line**

$$L = \{ \langle p, q, r \rangle + t\langle a, b, c \rangle, t \in \mathbf{R} \} .$$

The line consists of all points obtained by adding a multiple of the vector $\vec{v}$ to the vector $\vec{OP} = \langle p, q, r \rangle$. The line contains the point P as well as a suitably translated copy of $\vec{v}$. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter t as "time". At time $t = 0$, we are at the point P, whereas at time $t = 1$ we are at $\vec{OP} + \vec{v}$.

If t is restricted to values in a **parameter interval** $[s, u]$, then $L = \{ \langle p, q, r \rangle + t\langle a, b, c \rangle, s \leq t \leq u \}$ is a **line segment** which connects $\vec{r}(s)$ with $\vec{r}(u)$.

1 Problem. Get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, we form the vector $\vec{v} = \vec{PQ} = \langle 1, 3, 4 \rangle$ and get $L = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 1, 3, 4 \rangle; \}$. This can be written also as $\vec{r}(t) = \langle 1 + t, 1 + 3t, 2 + 4t \rangle$. If we write $\langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 1, 3, 4 \rangle$ as a collection of equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ and solve the first equation for t:

$$L = \{ (x, y, z) \mid (x - 1)/2 = (y - 1)/3 = (z - 2)/4 \} .$$

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = \langle a, b, c \rangle$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

Proof. Each of these expressions is equal to t. These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$.

2 Find the symmetric equations for the line through the two points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$, we first form the parametric equations $\langle x, y, z \rangle = \langle 0, 1, 1 \rangle + t\langle 2, 2, 3 \rangle$ or $x = 2t, y = 1+2t, z = 1+3t$. Solving each equation for t gives the symmetric equation $x/2 = (y-1)/2 = (z - 1)/3$.

3 **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a, b, c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{ \vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.

4 An example is $\Sigma = \{ \langle x, y, z \rangle = \langle 1, 1, 2 \rangle + t\langle 2, 4, 6 \rangle + s\langle 1, 0, -1 \rangle \}$. This is called the **parametric description** of a plane.

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = \langle a, b, c \rangle$, this means $ax + by + cz = ax_0 by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have just shown

The equation for a plane containing $\vec{v}$ and $\vec{w}$ and a point P is

$$ax + by + cz = d ,$$

where $\langle a, b, c \rangle = \vec{v} \times \vec{w}$ and d is obtained by plugging in P.

5 **Problem:** Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = \langle 1, 2, 0 \rangle$ and $\vec{w} = \langle 2, 2, 2 \rangle$. We have $\vec{n} = \langle 4, -2, -2 \rangle$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

The **angle between the two planes** $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two vectors $\vec{n} = \langle a, b, c \rangle$ and $\vec{m} = \langle e, f, g \rangle$.

6 Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. Answer: find the angle between $\vec{n} = \langle 1, 1, 0 \rangle$ and $\vec{m} = \langle 1, 1, 1 \rangle$. It is $\arccos(2/\sqrt{6})$.

Finally, lets look at some distance functions.

1) If P is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between P and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{|\langle -7, 0, -3 \rangle \cdot \langle 2, 4, 5 \rangle|}{|\langle 2, 4, 5 \rangle|} = \frac{29}{\sqrt{45}} .$$

2) If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from $P = (2,3,1)$ to the line $\vec{r}(t) = (1,1,2) + t(5,0,1)$, compute

$$d(P,L) = \frac{|\langle -1,-2,1\rangle \times \langle 5,0,1\rangle|}{\langle 5,0,1\rangle} = \frac{|\langle -2,6,10\rangle|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}} .$$

3) If L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines. For example, to compute the distance between $\vec{r}(t) = (2,1,4) + t(-1,1,0)$ and M is the line $\vec{s}(t) = (-1,0,2) + t(5,1,2)$ form the cross product of $\langle -1,1,0\rangle$ and $\langle 5,1,2\rangle$ is $\langle 2,2,-6\rangle$. The distance between these two lines is

$$d(L,M) = \frac{|(3,1,2) \cdot (2,2,-6)|}{|\langle 2,2,-6\rangle|} = \frac{4}{\sqrt{44}} .$$

4) To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma,\Pi) = \frac{|e-d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

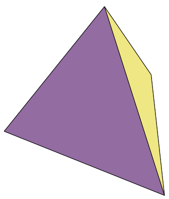
$$\frac{|8-1|}{|\langle 5,4,3\rangle|} = \frac{7}{\sqrt{50}} .$$

To finish this chapter, lets mention a distance problem which has a great deal of application and motivates the material of the upcoming week: The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites. Each GPS satellite sends periodically signals which are triggered by an atomic clock. While the distance to each satellite is not known, the difference from the distances to two satellites can be determined from the time delay of the two signals. This clever trick has the consequence that the receiver does not need to contain an atomic clock itself. To understand this better, we need to know about functions of three variables and surfaces.

Homework

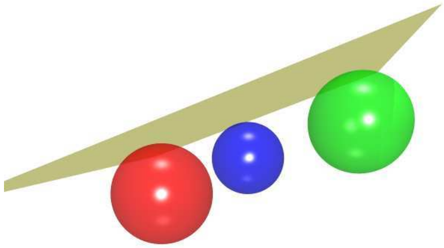
1 Find the parametric and symmetric equation for the line which passes through the points $P = (1,2,3)$ and $Q = (3,4,5)$.

2 A regular tetrahedron has vertices at the points $P_1 = (0,0,3)$,$P_2 = (0,\sqrt{8},-1)$, $P_3 = (-\sqrt{6},-\sqrt{2},-1)$ and $P_4 = (\sqrt{6},-\sqrt{2},-1)$. Find the distance between two edges which do not intersect.



3 Find a parametric equation for the line through the point $P = (3,1,2)$ that is perpendicular to the line $L : x = 1+t, y = 1-t, z = 2t$ and intersects this line in a point Q.

4 Given three spheres of radius 1 centered at $A = (1,2,0), B = (4,5,0), C = (1,3,2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side. **Hint.** There are two such planes. You want to consider the plane through the three points A, B, C first. You only need to find one of the two possible planes touching all the spheres on the same side.



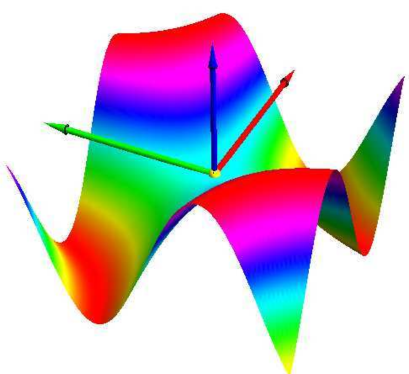
5 a) Find the distance between the point $P = (3,3,4)$ and the line $x = y = z$.
b) Parametrize the line $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ in $a)$ and find the minimum of the function $f(t) = d(P,\vec{r}(t))^2$. Verify that the the minimal value agrees with a).

Lecture 5: Functions

A **function of two variables** $f(x,y)$ is a rule which assigns to two numbers x,y a third number $f(x,y)$. For example, the function $f(x,y)=x^2y+2x$ assigns to $(3,2)$ the number $3^2 2+6=24$.

A function is usually defined for all points (x,y) in the plane like in the case $f(x,y)=x^2+\sin(xy)$. In general, we need to restrict the function to a **domain** D in the plane like for $f(x,y)=1/y$, where (x,y) is defined everywhere except on the x-axes $y=0$. The **range** of a function f is the set of values which the function f takes. The function $f(x,y)=1+x^2$ for example takes all values ≥ 1.

The **graph** of $f(x,y)$ is the set $\{(x,y,f(x,y)) \mid (x,y) \in D\ \}$ in of space. Graphs allow to visualize functions of two variables. It allows us to see the function.



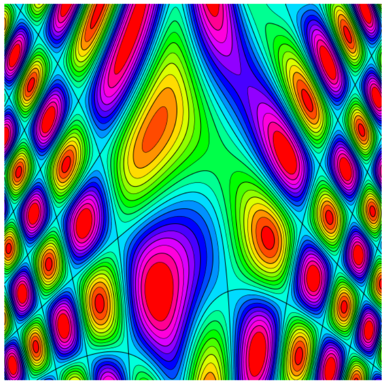
1 The graph of $f(x,y)=\sqrt{1-x^2-y^2}$ on the domain $x^2+y^2<1$ is a half sphere.

2

example function $f(x,y)$	domain D of f	range $= f(D)$ of f
$f(x,y)=\sin(3x+3y)-\log(1-x^2-y^2)$	open unit disc $x^2+y^2<1$	$[-1,\infty)$
$f(x,y)=f(x,y)=x^2+y^3-xy+\cos(xy)$	plane R^2	the real line
$f(x,y)=\sqrt{4-x^2-2y^2}$	$x^2+2y^2\leq 4$	$[0,2]$
$f(x,y)=1/(x^2+y^2-1)$	all except unit circle	the real line
$f(x,y)=1/(x^2+y^2)^2$	all except origin	positive real axis

The set $f(x,y)=c=$ const is called a **contour curve** or **level curve** of f. For example, for $f(x,y)=4x^2+3y^2$, the level curves $f=c$ are ellipses if $c>0$. Drawing several contour curves $\{f(x,y)=c\ \}$ produces a **contour map** of f.

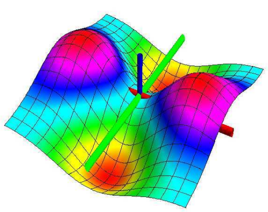
Level curves allow to visualize functions of two variables $f(x,y)$ without leaving the plane. The picture to the right for example shows the level curves of the function $\sin(xy)-\sin(x^2+v)$. Contour curves are encountered every day: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.

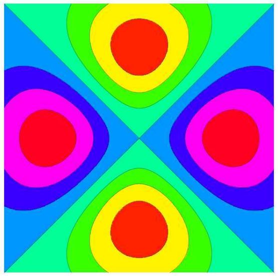


3 For $f(x,y)=x^2-y^2$, the set $x^2-y^2=0$ is the union of the lines $x=y$ and $x=-y$. The set $x^2-y^2=1$ consists of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The set $x^2-y^2=-1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

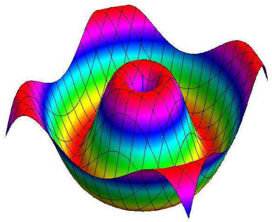
4 The function $f(x,y)=1-2x^2-y^2$ has contour curves $f(x,y)=1-2x^2+y^2=c$ which are ellipses $2x^2+y^2=1-c$ for $c<1$.

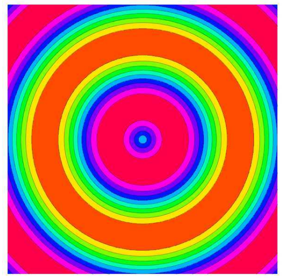
5 For the function $f(x,y)=(x^2-y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2}=c$. We can draw the curves however with the help of a computer:





6 The surface $z=f(x,y)=\sin(\sqrt{x^2+y^2})$ has concentric circles as contour curves.





In applications, we sometimes have to deal with functions which are not continuous. When plotting the rate of change of temperature of water in relation to pressure and volume for example, one experiences **phase transitions**, places where the function value can jump. Mathematicians have tamed discontinuous events with a mathematical field called "catastrophe theory".

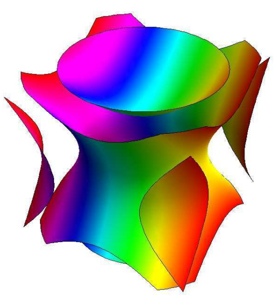
A function $f(x,y)$ is called **continuous** at (a,b) if $f(a,b)$ is finite and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that for any sequence (x_n, y_n) converging to (a,b) we have $f(x_n, y_n) \to f(a,b)$.

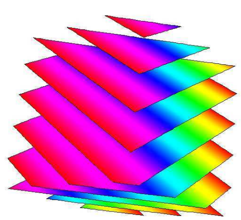
Continuity means that if (x,y) is close to (a,b), then $f(x,y)$ is close to $f(a,b)$. Continuity for functions of more than two variables is defined in the same way. Continuity is not always easy to check but fortunately, we do not have to worry about it most of the time. Lets look at some examples:

7 **Example:** For $f(x,y) = (xy)/(x^2+y^2)$, we have $\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} x^2/(2x^2) = 1/2$ and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous.

8 For $f(x,y) = (x^2y)/(x^2+y^2)$, it is better to describe the function using polar coordinates: $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly if $r \to 0$. The function is continuous.

A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. We can visualize it by **contour surfaces** $g(x,y,z) = c$, where c is constant. To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x=0, y=0$ or $z=0$.





Many surfaces can be described as level surfaces. If this is the case, we call this an **implicit description** of a surface. Here are some examples we know already:

9 The function $g(x,y,z) = 2 + \sin(xyz)$ is an example. It could define the temperature distribution in space. We can no more draw a graph of g because that would be an object in 4 dimensions.

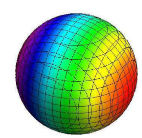
10 The level surfaces of $g(x,y,z) = x^2+y^2+z^2$ are spheres. The level surfaces of $g(x,y,z) = 2x^2+y^2+3z^2$ are ellipsoids.

11 For $g(x,y,z) = z - f(x,y)$, the level surface $g = 0$ which is the graph $z = f(x,y)$ of a function of two variables. For example, for $g(x,y,z) = z - x^2 - y^2 = 0$, we have the graph $z = x^2+y^2$ of the function $f(x,y) = x^2+y^2$ which is a paraboloid. Note however that most surfaces of the form $g(x,y,z) = c$ can not be written as graphs. The sphere is an example, where we need two graphs to cover it.

12 The equation $ax+by+cz = d$ is a plane. With $\vec{n} = \langle a,b,c\rangle$ and $\vec{x} = \langle x,y,z\rangle$, we can rewrite the equation $\vec{n}\cdot\vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n}\cdot\vec{x}_0 = d$. so that $\vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$. This means that every vector $\vec{x}-\vec{x}_0$ in the plane is orthogonal to $\vec{n}$.

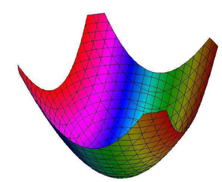
13 If the function depends only quadratically on variables, that is if $f(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+m$ then the surface $f(x,y,z) = 0$ is called a **quadric**. Lets look at a few of them.

Sphere



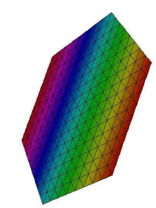
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

Paraboloid



$(x-a)^2+(y-b)^2-c = z$

Plane

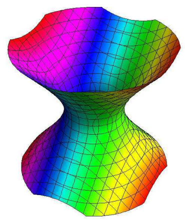


$ax+by+cz = d$

One sheeted Hyperboloid

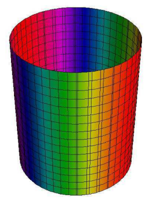
Cylinder

Two sheeted Hyperboloid



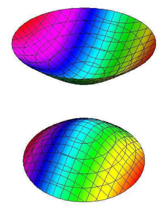
$(x-a)^2+(y-b)^2-(z-c)^2 = r^2$

Ellipsoid



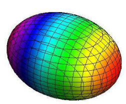
$(x-a)^2+(y-b)^2 = r^2$

Hyperbolic paraboloid

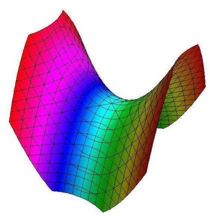


$(x-a)^2+(y-b)^2-(z-c)^2 = -r^2$

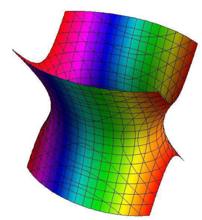
Elliptic hyperboloid



$x^2/a^2+y^2/b^2+z^2/c^2 = 1$

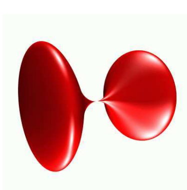


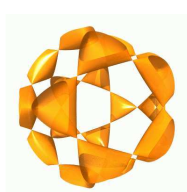
$x^2-y^2+z = 1$

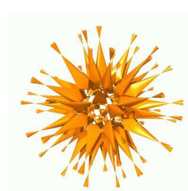


$x^2/a^2+y^2/b^2-z^2/c^2 = 1$

14 Higher order polynomial surfaces can be intriguingly beautiful. If the function involves only multiplications of variables x, y, z and $x \to f(x, x, x)$ has degree d, then it is called a **degree** d **polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.







Homework

1 Draw the graph of the function $f(x, y) = (x^2-1)/(y^2+1)$. It can also be seen as the level surface $g(x, y, z) = z - f(x, y) = 0$. Find the equations for the three traces of the surface S and sketch the contour map.

2 Consider the surface $z^2-4z+x^2-2x-y = 0$. Draw the three traces. What surface is it?

3 Draw the Fermat surface $x^4+y^4 = z^4$ and its traces.

4 a) Sketch the graph and contour map of the function $f(x, y) = \sin(x^2+y^2)/(1+x^2+y^2)$.
b) Sketch the graph and contour map of the function $g(x, y) = |x| - |y|$.

5 Verify that the line $\vec{r}(t) = \langle 1, 3, 2\rangle + t\langle 1, 2, 1\rangle$ is contained in the surface $z^2-x^2-y = 0$.

6: Parametrized surfaces

There is a second, fundamentally different way to describe a surface, the **parametrization**.

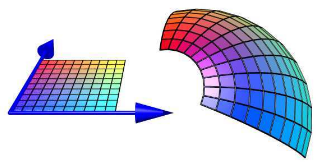
A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle ,$$

where $x(u,v), y(u,v), z(u,v)$ are three functions of two variables.

Because two parameters u and v are involved, the map $\vec{r}$ is also called uv-**map**.

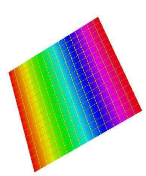
A **parametrized surface** is the image of the uv-map. The domain of the uv-map is called the **parameter domain**.



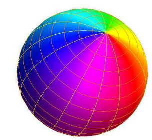
If we keep the first parameter u constant, then $v \mapsto \vec{r}(u,v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u,v)$ traces a curve the surface. These curves are called **grid curves**.

A computer draws surfaces using grid curves. The world of parametric surfaces is intriguing and complex. You can explore this world with the help of the computer algebra system Mathematica. You can survive the parametrization of surfaces topic by keeping in mind 4 important examples. They are *really important* because they: are cases we can understand well and which consequently will return again and again.

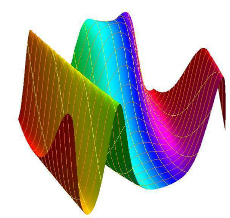
1 **Planes**. Parametric: $\vec{r}(s,t) = \vec{OP} + s\vec{v} + t\vec{w}$
Implicit: $ax + by + cz = d$. Parametric to Implicit: find the normal vector $\vec{n} = \vec{v} \times \vec{w}$.
Implicit to Parametric: find two vectors $\vec{v}, \vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and forming $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.



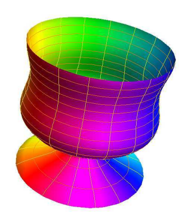
2 **Spheres**: Parametric: $\vec{r}(u,v) = \langle a,b,c \rangle + \langle \rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v) \rangle$.
Implicit: $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.
Parametric to implicit: reading off the radius.
Implicit to parametric: find the center (a,b,c) and the radius r possibly by completing the square.



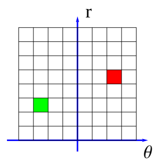
3 **Graphs**:
Parametric: $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$
Implicit: $z - f(x,y) = 0$. Parametric to Implicit: think about $z = f(x,y)$
Implicit to Parametric: use x and y as the parameterizations.

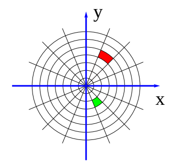


4 **Surfaces of revolution**:
Parametric: $r(u,v) = (g(v)\cos(u), g(v)\sin(u), v)$
Implicit: $\sqrt{x^2+y^2} = r = g(z)$ can be written as $x^2 + y^2 = g(z)^2$.
Parametric to Implicit: read off the function $g(z)$ the distance to the z-axis.
Implicit to Parametric: use the function g.



A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \text{arctg}(y/x)$. We have $(x, y) = (r\cos(\theta), r\sin(\theta))$.





The formula $\theta = \text{arctg}(y/x)$ defines the angle θ only up to an addition of π. The points (x, y) and $(-x, -y)$ have the same θ value. In order to get the correct θ, one can take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the limit when $x \to 0^+$, then add π if $x < 0$ or $x = 0, y < 0$.

If we represent points in space as

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

we speak of **cylindrical coordinates**.

Here are some level surfaces in cylindrical coordinates:

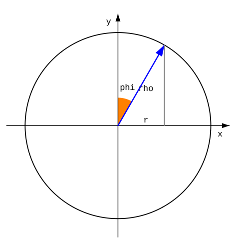
5 $r = 1$ is a **cylinder**, $r = |z|$ is a **double cone** $\theta = 0$ is a **half plane** $r = \theta$ is a **rolled sheet of paper**

6 $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

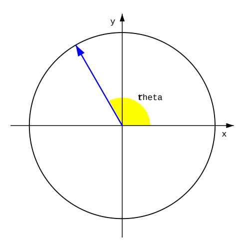
Spherical coordinates use the distance ρ to the origin as well as two angles θ and ϕ. The first angle θ is the polar angle in polar coordinates of the first two coordinates and ϕ is the angle between the vector $\vec{OP}$ and the z-axis. A point has the **spherical coordinate**

$$(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) .$$

There are two important figures. The distance to the z axes $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can be seen in the right picture.



$$x = \rho\cos(\theta)\sin(\phi),$$
$$y = \rho\sin(\theta)\sin(\phi),$$
$$z = \rho\cos(\phi)$$



Here are some level surfaces described in spherical coordinates:

7 $\rho = 1$ is a **sphere**, the surface $\phi = \pi/2$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta)\sin(\phi)$ is an example of a **bumpy sphere**.

Homework

1 Plot the surface with the parametrization

$$\vec{r}(u, v) = \langle v^5\cos(u), v^5\sin(u), v\rangle ,$$

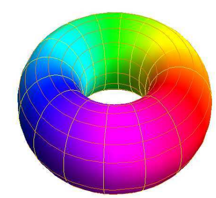
where $u \in [0, 2\pi]$ and $v \in \mathbf{R}$.

2 Find a parametrization for the plane which contains the three points $P = (3, 7, 1), Q = (1, 2, 1)$ and $R = (0, 3, 4)$.

3 a) Find a parametrisations of the lower half of the ellipsoid $2x^2 + 4y^2 + z^2 = 1, z < 0$ by using that the surface is a graph $z = f(x, y)$.
b) Find a second parametrization but use angles ϕ, θ similarly as for the sphere.

4 Find a parametrisation of the **torus** which is obtained as the set of points which have distance 1 from the circle $(2\cos(\theta), 2\sin(\theta), 0)$, where θ is the angle occuring in cylindrical and spherical coordinates.

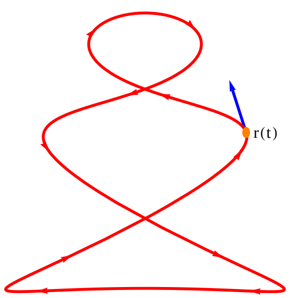
Hint: Keep $u = t$ as one of the parameters and let r the distance of a point on the torus to the z-axis. This distance is $r = 2 + \cos(\phi)$ if ϕ is the angle you see on Figure 1. You can read off from the same picture also $z = \sin(\phi)$. To finish the parametrization problem, you have to translate back from cylindrical coordinates $(r, \theta, z) = (2 + \cos(\phi), \theta, \sin(\phi))$ to Cartesian coordinates (x, y, z). Write down your result in the form $\vec{r}(\theta, \phi) = \langle x(\theta, \phi), y(\theta, \phi\rangle, z(\theta, \phi))$.

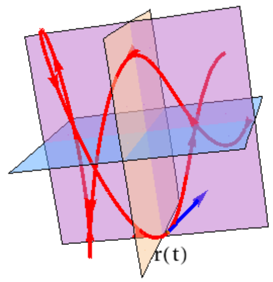


5 a) What is the equation for the surface $x^2 + y^2 - 5x = z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(3\phi)|$ in spherical coordinates (ρ, θ, ϕ).

7: Parametrized curves

A **parametrization** of a curve is a map $\vec{r}(t) = \langle x(t), y(t) \rangle$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. In three dimensions, the parametrization is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and the image of r is a **parametrized curve** in space.





We always think of the **parameter** t as **time**. For a fixed time t, we have a vector $\langle x(t), y(t), z(t) \rangle$ in space. As t varies, the end point of this vector moves along the curve. The parametrization contains more information about the curve then the curve alone. It tells for example, how fast we go along the curve.

1 1) The parametrization $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ describes a curve in the plane. It is an example of a **Lissajous curve**.

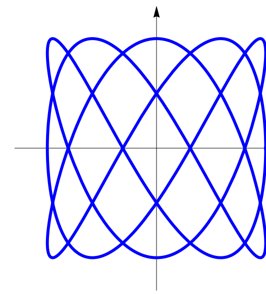
2 2) If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = \langle t, f(t) \rangle$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola.

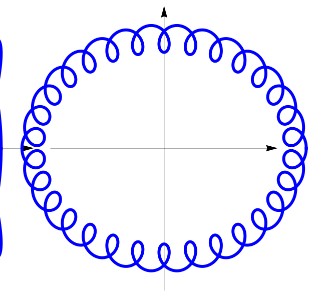
3 3) With $x(t) = 2\cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows an **ellipse**. We can see this from $x(t)^2/4 + y(t)^2 = 1$. We can can overlay an other circular motion to get an epicycle $\vec{r}(t) = \langle \cos(t) + \cos(31t)/4, 4\sin(t) + \sin(31t)/4 \rangle$.

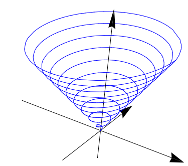
4 4) With $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$ we get the parametrization of a **space curve** $\vec{r}(t) = \langle t\cos(t), t\sin(t), t \rangle$. It traces a **helix** which has a radius changing linearly.

5 If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in the previous example but the curve is traversed **faster**. The **parameterization** of the curve has changed.

6 If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve again but we traverse it in the **opposite direction**.







7 If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a + t(u - a), b + t(v - b), c + t(w - c) \rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q. For example, $\vec{r}(t) = \langle 1 + t, 1 - t, 2 + 3t \rangle$ connects the points $P = (1, 1, 2)$ with $Q = (2, 0, 1)$.

Sometimes it is possible to eliminate the time parameter t and write the curve using equations. We need one equation to do so in two dimensions and two equations in three dimensions.

Some curves can be written as the intersection of two surfaces. The pair of equations $f(x, y, z) = 0, g(x, y, z) = 0$ is called an implicit description of a curve.

8 The symmetric equations describing a line $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ can be seen as the intersection of two surfaces.

9 If f and g are polynomials, the set of points satisfying $f(x, y, z) = 0, g(x, y, z) = 0$ is an example of an **algebraic variety**. An example is the set of points in space satisfying $x^2 - y^2 + z^3 = 0, x^5 - y + z^5 + xy = 3$. An other example is the set of points satisfying $x^2 + 4y^2 = 5, x + y + z = 1$ which is an ellipse in space.

10 For $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$, then $x = t\cos(z), y = t\sin(z)$ and we can see that $x^2 + y^2 = z^2$. The curve is located on a cone. We also have $x/y = \tan(z)$ so that we could see the curve as an intersection of two surfaces. Detecting relations between x, y, z can help to understand the curve.

11 Curves describe the paths of particles, celestial bodies, or quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time.. Here are some more places, where curves appear:

Strings or knots	are closed curves in space.
Large Molecules	like RNA or proteins can be modeled as curves.
Computer graphics:	surfaces are represented by mesh of curves.
Typography:	fonts represented by Bezier curves.
Space time:	curve in space-time describes the motion of an object
Topology:	space filling curves, boundaries of surfaces or knots.

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is called the **velocity** at time t. Its length $|\vec{r}\,'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**.

Any vector parallel to the velocity vector $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

Here are where velocities, acceleration and jerk are computed:

Position	$\vec{r}\,(t)$	$= \langle \cos(3t), \sin(2t), 2\sin(t) \rangle$
Velocity	$\vec{r}\,'(t)$	$= \langle -3\sin(3t), 2\cos(2t), 2\cos(t) \rangle$
Acceleration	$\vec{r}\,''(t)$	$= \langle -9\cos(3t), -4\sin(2t), -2\sin(t) \rangle$
Jerk	$\vec{r}\,'''(t)$	$= \langle 27\sin(3t), 8\cos(2t), -2\cos(t) \rangle$

Lets look at some examples of velocities and accelerations:

Signals in nerves:	40 m/s	Train:	0.1-0.3 m/s^2
Plane:	70-900 m/s	Car:	3-8 m/s^2
Sound in air:	Mach 1=340 m/s	Free fall:	1G = 9.81 m/s^2
Speed of bullet:	1200-1500 m/s	Space shuttle:	$3G = 30m/s^2$
Earth around the sun:	30'000 m/s	Combat plane F16:	9G m/s^2
Sun around galaxy center:	200'000 m/s	Ejection from F16:	14G m/s^2.
Light in vacuum:	300'000'000 m/s	Electron in vacuum tube:	10^{15} m/s^2

The **addition rule** in one dimension $(f+g)' = f'+g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ and the **chain rule** $(f(g))' = f'(g)g'$ generalize to vector-valued functions because in each component, we have the single variable rule.

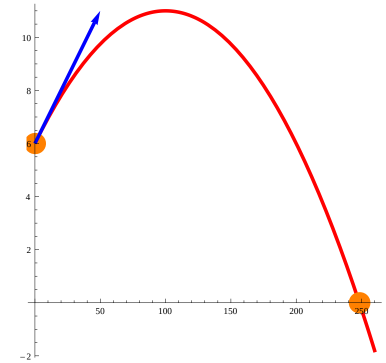
$(\vec{v}+\vec{w})' = \vec{v}\,' + \vec{w}\,'$, $(c\vec{v})' = c\vec{v}\,'$, $(\vec{v}\cdot\vec{w})' = \vec{v}\,'\cdot\vec{w} + \vec{v}\cdot\vec{w}\,'$ $(\vec{v}\times\vec{w})' = \vec{v}\,'\times\vec{w} + \vec{v}\times\vec{w}\,'$ $(\vec{v}(f(t)))' = \vec{v}\,'(f(t))f'(t)$.

The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}\,'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}\,''(t)$ at all times as well as initial velocity and position $\vec{r}\,'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

The **free fall** is the case when acceleration is constant. The direction of the constant force defines what is "down". If $\vec{r}\,''(t) = \langle 0,0,-10\rangle$, $\vec{r}\,'(0) = \langle 0,1000,2\rangle$, $\vec{r}(0) = \langle 0,0,h\rangle$, then $\vec{r}(t) = \langle 0, 1000t, h+2t-10t^2/2\rangle$.

If $r''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) - \vec{F}t^2/2$.



Homework

1 Sketch the plane curve $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t^3, t^2\rangle$ for $t \in [-1,1]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as its acceleration $\vec{r}\,''(t)$ at the point $t = 2$.

2 A device in a car measures the acceleration $\vec{r}\,''(t) = \langle \cos(t), -\cos(3t)\rangle$ at time t. Assume that the car is at the origin $(0,0)$ at time $t = 0$ and has zero speed at $t = 0$, what is its position $\vec{r}(t)$ at time t?

3 Verify that the curve $\vec{r}(t) = \langle t\cos(t), 2t\sin(t), t^2\rangle$ is located on the **elliptical paraboloid**

$$z = x^2 + \frac{y^2}{4} \; .$$

Use this fact to sketch the curve.

4 Find the parameterization $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ of the curve obtained by intersecting the elliptical cylinder $x^2/9 + y^2/4 = 1$ with the surface $z = xy$. Find the velocity vector $\vec{r}\,'(t)$ at the time $t = \pi/2$.

5 Consider the curve

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle t^2, 1+t, 1+t^3\rangle \; .$$

Check that it passes through the point $(1,0,0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.

8: Arc length and curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then $L = \int_a^b |\vec{r}\,'(t)|\,dt$ is called the **arc length of the curve**.

In space, we have $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$.

1 The arc length of the **circle** of radius R given by $\vec{r}(t) = \langle R\cos(t), R\sin(t)\rangle$ parameterized by $0 \leq t \leq 2\pi$ is 2π because the speed $|\vec{r}'(t)|$ is constant and equal to R. The answer is $2\pi R$.

2 The **helix** $\vec{r}(t) = (\cos(t), \sin(t), t)$ has velocity $\vec{r}\,'(t) = (-\sin(t), \cos(t), 1)$ and constant speed $|\vec{r}\,'(t)| = (-\sin(t), \cos(t), 1) = \sqrt{2}$.

3 What is the arc length of the curve

$$\vec{r}(t) = \langle t, \log(t), t^2/2\rangle$$

for $1 \leq t \leq 2$? Answer: Because $\vec{r}'(t) = \langle 1, 1/t, t\rangle$, we have $\vec{r}'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t\,dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, we an call it "opportunity".

4 Find the arc length of the curve $\vec{r}(t) = \langle 3t^2, 6t, t^3\rangle$ from $t = 1$ to $t = 3$.

5 What is the arc length of the curve $\vec{r}(t) = \langle \cos^3(t), \sin^3(t)\rangle$? Answer: We have $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi}(3/2)\sin(2t)\,dt = 6$.

6 Find the arc length of $\vec{r}(t) = \langle t^2/2, t^3/3\rangle$ for $-1 \leq t \leq 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2}\,dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^1 |x|\sqrt{1+x^2}\,dx = 2\int_0^1 x\sqrt{1+x^2}\,dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

7 The arc length of an **epicycle** $\vec{r}(t) = \langle t + \sin(t), \cos(t)\rangle$ parameterized by $0 \leq t \leq 2\pi$. We have $|\vec{r}'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi}\sqrt{2+2\cos(t)}\,dt$. A **substitution** $t = 2u$ gives $L = \int_0^\pi \sqrt{2+2\cos(2u)}\,2du = \int_0^\pi \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\,2du = \int_0^\pi \sqrt{4\cos^2(u)}\,2du = 4\int_0^\pi |\cos(u)|\,du = 8$.

8 The arc length of the **catenary** $\vec{r}(t) = \langle t, \cosh(t)\rangle$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1\ ,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by $\vec{r}(t) = \langle \cos(t^2), \sin(t^2)\rangle$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\,dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

10 Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = \langle \cos(3t), \sin(5t)\rangle$ leads to the arc length integral $\int_0^{2\pi}\sqrt{9\sin^2(3t) + 25\cos^2(5t)}\,dt$ which can only be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)/|\vec{r}\,'(t)|$ **unit tangent vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

The curvature is the length of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

The curvature does not depend on the parametrization.

Proof. Let $s(t)$ be an other parametrization, then by the chain rule $d/dtT'(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.

Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

11 The curve $\vec{r}(t) = \langle t, f(t)\rangle$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1+f'(t)^2}^3$$

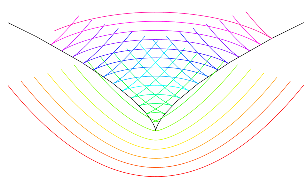
For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by N and $B$is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because B is automatically normal to T and N, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

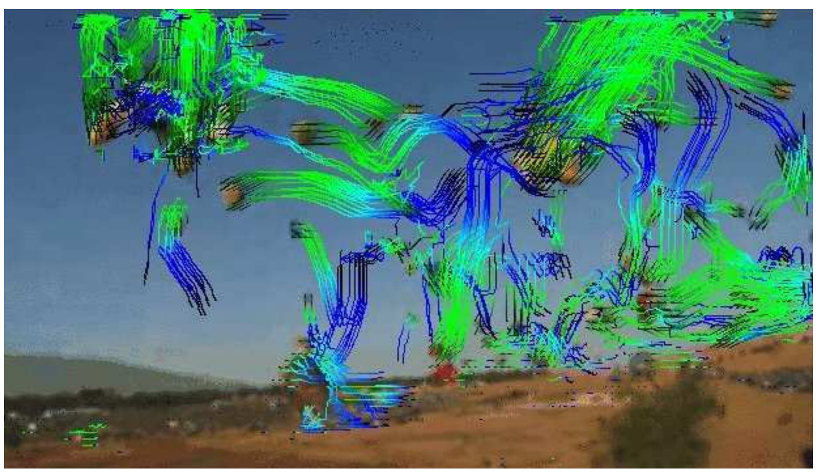
Here is an application of curvature: If a curve $\vec{r}(t)$ represents a **wave front** and $\vec{n}(t)$ is a **unit vector normal** to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call that curve the **evolute** of the original curve.



A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We prove this in class. Finally, lets mention that curvature is important also in **computer vision**. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to **corners** in the picture. This is useful when **tracking** or **identifying** objects.



Tracking balloons in a movie taken at a balloon festival in Albuquerque. The program computes curvature in order to identify interesting points, then tracks them over time.

Homework

1 Find the arc length of the curve

$$\vec{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle ,$$

where the time parameter satisfies $0 \leq t \leq \pi$.

2 Find the curvature of $\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), t\rangle$ at the point $(1, 0, 0)$.

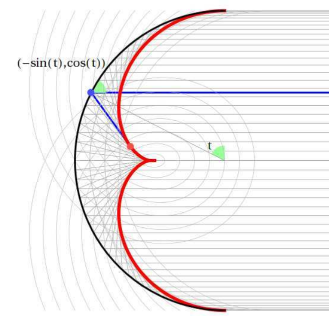
3 Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = \langle t^2, t^3, 0\rangle$ for $t = 2$. Do the vectors depend continuously on t for all t?

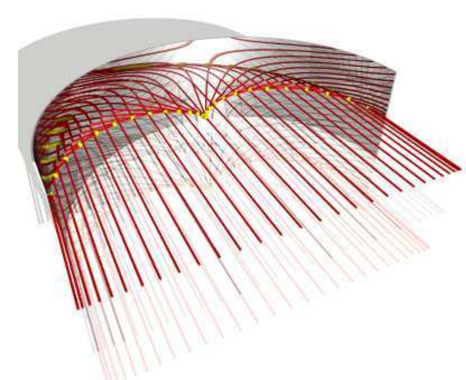
4 Let $\vec{r}(t) = \langle t, t^2\rangle$. Find the equation for the **caustic**

$$\vec{s}(t) = \vec{r}(t) + \frac{\vec{N}(t)}{\kappa(t)}$$

which is known also as the **evolute** of the curve.

5 If $\vec{r}(t) = \langle -\sin(t), \cos(t)\rangle$ is the boundary of a coffee cup and light enters in the direction $\langle -1, 0\rangle$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.





Lecture 9: Partial derivatives

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

One also uses the short hand notation $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. For iterated derivatives, the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$.

The notation for partial derivatives $\partial_x f, \partial_y f$ were introduced by Carl Gustav Jacobi. Josef Lagrange had used the term "partial differences". Partial derivatives f_x and f_y measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

1 For $f(x,y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x,y) = 4x^3 - 12xy^2, f_{xx} = 12x^2 - 12y^2, f_y(x,y) = -12x^2y + 4y^3, f_{yy} = -12x^2 + 12y^2$ and see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**: it is an equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to more than one variables.

Clairot's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant h is positive, then $f_x(x,y) = [f(x+h,y) - f(x,y)]/h$. For $h = 0$ we define f_x as before. Compare the two sides for fixed $h > 0$:

$hf_x(x,y) = f(x+h,y) - f(x,y)$
$h^2 f_{xy}(x,y) = f(x+h,y+h) - f(x+h,y+h) - (f(x+h,y) - f(x,y))$

$dyf_y(x,y) = f(x,y+h) - f(x,y)$.
$h^2 f_{yx}(x,y) = f(x+h,y+h) - f(x+h,y) - (f(x,y+h) - f(x,y))$

We have not taken any limits in this proof but established an identity which holds for all $h > 0$, the discrete derivatives f_x, f_y satisfy the relation $f_{xy} = f_{yx}$. We could fancy the identity obtained in the proof as a "quantum Clairot" theorem. If the classical derivatives f_{xy}, f_{yx} are both continuous, we can take the limit $h \to 0$ to get the classical Clairot's theorem as a "classical limit". Note that the quantum Clairot theorem shown first in this proof holds for **any** functions $f(x,y)$ of two variables. We do not even need the functions to be continuous.

2 Find $f_{xxxxxyxxxxx}$ for $f(x) = \sin(x) + x^6 y^{10} \cos(y)$. Answer: Do not compute, but think.

3 The continuity assumption for f_{xy} is necessary. The example

$$f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$$

contradicts Clairaut's theorem:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,

$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

Here are some examples of partial differential equations. You should know the first 4 well.

4 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $\boxed{f(t,x) = \sin(x-t) + \sin(x+t)}$ satisfies the wave equation.

5 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $\boxed{f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}}$ satisfies the heat equation.

6 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $\boxed{f(x,y) = x^3 - 3xy^2}$ is an example satisfying the Laplace equation.

7 The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $\boxed{f(t,x) = e^{-(x+t)^2}}$ satisfy the advection equation.

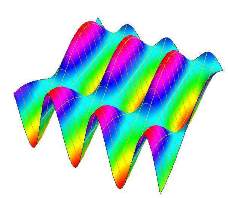
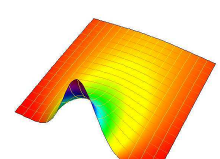
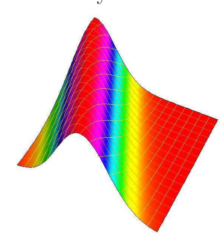
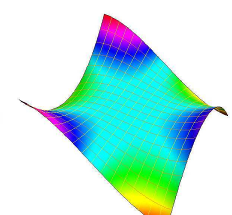
8 The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $\boxed{f(x,y) = \cos(x) + \sin(y)}$ satisfies the eiconal equation.

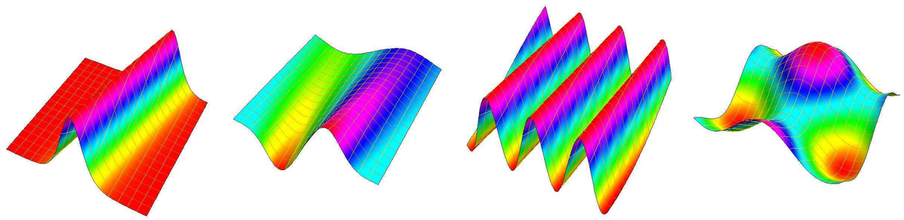
9 The **Burgers equation** $f_t + ff_x = f_{xx}$ describes waves at the beach which break. The function $\boxed{f(t,x) = \frac{x}{t}\frac{\sqrt{\frac{1}{t}}e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}}e^{-x^2/(4t)}}}$ satisfies the Burgers equation.

10 The **KdV equation** $f_t + 6ff_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $\boxed{f(t,x) = \frac{a^2}{2}\cosh^{-2}(\frac{a}{2}(x - a^2t))}$ satisfies the KdV equation.

11 The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $\boxed{f(t,x) = e^{i(kx - \frac{\hbar}{2m}k^2 t)}}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

Here are the graphs of the solutions of the equations. Can you match them with the PDE's?



Notice that in all these examples, we have just given one possible solution to the partial differential equation. There are in general many solutions and only additional conditions like initial or boundary conditions determine the solution uniquely. If we know $f(0,x)$ for the Burgers equation, then the solution $f(t,x)$ is determined. A course on partial differential equations would show you how to get the solution.

Paul Dirac once said: "A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpreted the later as an **antiparticle**: the existence of antiparticles was later confirmed. We will not learn here to find solutions to partial differential equations. But you should be able to verify that a given function is a solution of the equation.

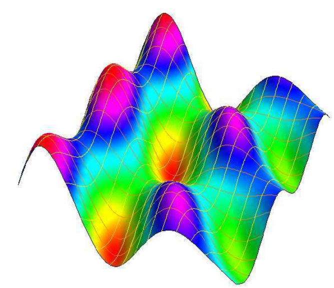
Homework

1 Verify that $f(t,x) = \sin(\cos(t+x))$ is a solution of the transport equation $f_t(t,x) = f_x(t,x)$.

2 Verify that $f(x,y) = 3y^2 + x^3$ satisfies the **Euler-Tricomi** partial differential equation $u_{xx} = xu_{yy}$. This PDE is useful in describing **transonic flow**. Can you find an other solution which is not a multiple of the solution given in this problem?

3 Verify that $f(x,t) = e^{-rt}\sin(x+ct)$ satisfies the driven transport equation $f_t(x,t) = cf_x(x,t) - rf(x,t)$ It is sometimes also called the **advection equation**.

4 The partial differential equation $f_{xx} + f_{yy} = f_{tt}$ is called the wave equation in two dimensions. It describes waves in a pool for example.
a) Show that if $f(x,y,t) = \sin(nx+my)\sin(\sqrt{n^2+m^2}t)$ satisfies the wave equation. It describes waves in a square where $x \in [0,\pi]$ and $y \in [0,\pi]$. The waves are zero at the boundary of the pool.
b) Verify that if we have two such solutions with different n,m then also the sum is a solution.
c) For which k is $f(x,y,t) = \sin(nx)\cos(nt) + \sin(mx)\cos(mt) + \sin(nx+my)\cos(kt)$ a solution of the wave equation? Verify that the wave is periodic in time $f(x,y,t+2\pi) = f(x,y,t)$ if $m^2+n^2 = k^2$ is a **Pythagorean triple**.



5 The partial differential equation $f_t + ff_x = f_{xx}$ is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the Navier Stokes equation which are used to describe the weather. Verify that the function

$$f(t,x)\frac{\left(\frac{1}{t}\right)^{3/2} x e^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}}e^{-\frac{x^2}{4t}} + 1}$$

is a solution of the Burgers equation.

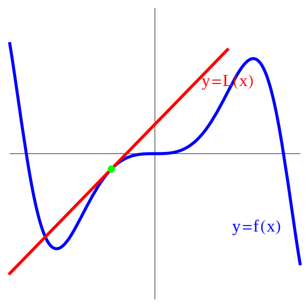
Remark. This calculation might need a bit perseverance, when done by hand. You are welcome to use technology if you should get stuck. Here is an example on how to check that a function is a solution of a partial differential equation in Mathematica:

```
f[t_,x_]:=(1/Sqrt[t])*Exp[-x^2/(4t)];
Simplify[ D[f[t,x],t] == D[f[t,x],{x,2}]]
```

Lecture 10: Linearization

In single variable calculus, you have seen the following definition:

> The **linear approximation** of $f(x)$ at a point a is the linear function
> $$L(x) = f(a) + f'(a)(x - a) .$$



The graph of the function L is close to the graph of f at a. We generalize this now to higher dimensions:

> The **linear approximation** of $f(x, y)$ at (a, b) is the linear function
> $$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) .$$
> The **linear approximation** of a function $f(x, y, z)$ at (a, b, c) is
> $$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) .$$

Using the **gradient**

$$\nabla f(x, y) = \langle f_x, f_y\rangle, \qquad \nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle ,$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .$$

How do we justify the linearization? If the second variable $y = b$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x, b) = f(a, b) + f_x(a, b)(x - a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0, y) = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

1 What is the linear approximation of the function $f(x, y) = \sin(\pi x y^2)$ at the point $(1, 1)$? We have $(f_x(x, y), y_f(x, y) = (\pi y^2 \cos(\pi x y^2), 2y\pi \cos(\pi x y^2))$ which is at the point $(1, 1)$ equal to $\nabla f(1, 1) = \langle \pi \cos(\pi), 2\pi \cos(\pi)\rangle = \langle -\pi, 2\pi\rangle$.

2 Linearization can be used to estimate functions near a point. In the previous example,

$$-0.00943 = f(1+0.01, 1+0.01) \sim L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 + 3\pi = -0.00942 .$$

3 Here is an example in three dimensions: find the linear approximation to $f(x, y, z) = xy + yz + zx$ at the point $(1, 1, 1)$. Since $f(1, 1, 1) = 3$, and $\nabla f(x, y, z) = (y + z, x + z, y + x), \nabla f(1, 1, 1) = (2, 2, 2)$. we have $L(x, y, z) = f(1, 1, 1) + (2, 2, 2) \cdot (x - 1, y - 1, z - 1) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = 2x + 2y + 2z - 3$.

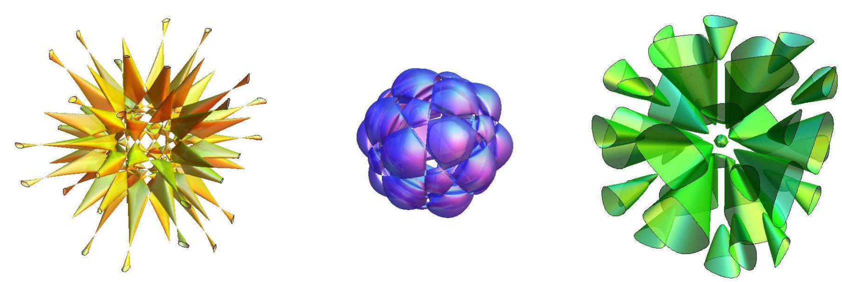
4 Estimate $f(0.01, 24.8, 1.02)$ for $f(x, y, z) = e^x \sqrt{y} z$.
Solution: take $(x_0, y_0, z_0) = (0, 25, 1)$, where $f(x_0, y_0, z_0) = 5$. The gradient is $\nabla f(x, y, z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0, y_0, z_0) = (0, 25, 1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x, y, z) = f(x_0, y_0, z_0) + \nabla f(x_0, y_0, z_0)(x - x_0, y - y_0, z - z_0) = 5 + (5, 1/10, 5)(x - 0, y - 25, z - 1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

5 Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2, 4)$.
Solution: the level curve $f(x, y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2, 4)$ is obtained by computing the gradient $\langle a, b\rangle = \nabla f(2, 4) = \langle -g'(2), 1\rangle = \langle -4, 1\rangle$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

6 The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x, y, z) &= (3 + 5t)(-1 + x^2 + y^2 + z^2)^2(-2 + t + x^2 + y^2 + z^2)^2 \\ &+ 8(x^2 - t^4 y^2)(-(t^4 x^2) + z^2)(y^2 - t^4 z^2)(x^4 - 2x^2 y^2 + y^4 - 2x^2 z^2 - 2y^2 z^2 + z^4) , \end{aligned}$$

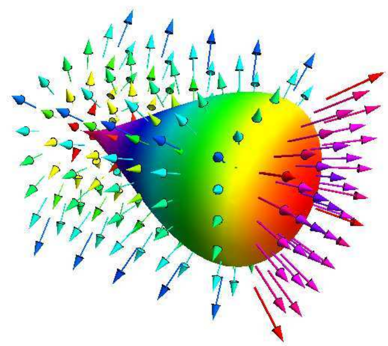
where $t = (\sqrt{5} + 1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5} - 1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x, y, z) = 8$. Find the tangent plane of the later surface at the point $(1, 1, 0)$. **Answer:** We have $\nabla f(1, 1, 0) = \langle 64, 64, 0\rangle$. The surface is $x + y = d$ for some constant d. By plugging in $(1, 1, 0)$ we see that $x + y = 2$.



7 The quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2,2,2)$ of this pair shaped surface? We get $\langle a,b,c\rangle = \langle 20,4,4\rangle$ and so the equation of the plane $20x+4y+4z = 56$, where we have obtained the constant to the right by plugging in the point $(x,y,z) = (2,2,2)$.



Remark: some books use **differentials** etc to describe linearizations. This is 19 century notation and terminology and should be avoided by all means. For us, the linearlization of a function at a point is a linear function in the same number of variables. 20th century mathematics has invented the notion of **differential forms** which is a valuable mathematical notion, but it is a concept which becomes only useful in follow-up courses which build on multivariable calculus like Riemannian geometry. The notion of "differentials" comes from a time when calculus was still foggy in some areas. Unfortunately it has survived and appears even in some calculus books.

Homework

1 If $2x+3y+2z = 9$ is the tangent plane to the graph of $z = f(x,y)$ at the point $(1,1,2)$. Extimate $f(1.01, 0.98)$.

2 Estimate $1000^{1/5}$ using linear approximation

3 Find $f(0.01, 0.999)$ for $f(x,y) = \cos(\pi xy)y + \sin(x+\pi y)$.

4 Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

5 Sketch a contour map of the function

$$f(x,y) = x^2 + 9y^2$$

find the **gradient vector** $\nabla f = \langle f_x, f_y\rangle$ of f at the point $(1,1)$. Draw it together with the tangent line $ax+by=d$ to the curve at $(1,1)$.

Lecture 11: Chain rule

If f and g are functions of one variable t, the **single variable chain rule** tells us that $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$.
It can be proven by linearizing the functions f and g and verifying the chain rule in the linear case. The **chain rule** is also useful:

For example, to find $\arccos'(x)$, we write $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1-\sin^2(\arccos(x))} \arccos'(x) = \sqrt{1-x^2} \arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

Define the **gradient** $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$.

If $\vec{r}(t)$ is curve and f is a function of several variables we can build a function $t \mapsto f(\vec{r}(t))$ of one variable. Similarly, If $\vec{r}(t)$ is a parametrization of a curve in the plane and f is a function of two variables, then $t \mapsto f(\vec{r}(t))$ is a function of one variable.

The **multivariable chain rule** is $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$.

Proof. When written out in two dimensions, it is

$$\frac{d}{dt} f(x(t),y(t)) = f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t) \ .$$

Now, the identity

$$\frac{f(x(t+h),y(t+h)) - f(x(t),y(t))}{h} = \frac{f(x(t+h),y(t+h)) - f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h)) - f(x(t),y(t))}{h}$$

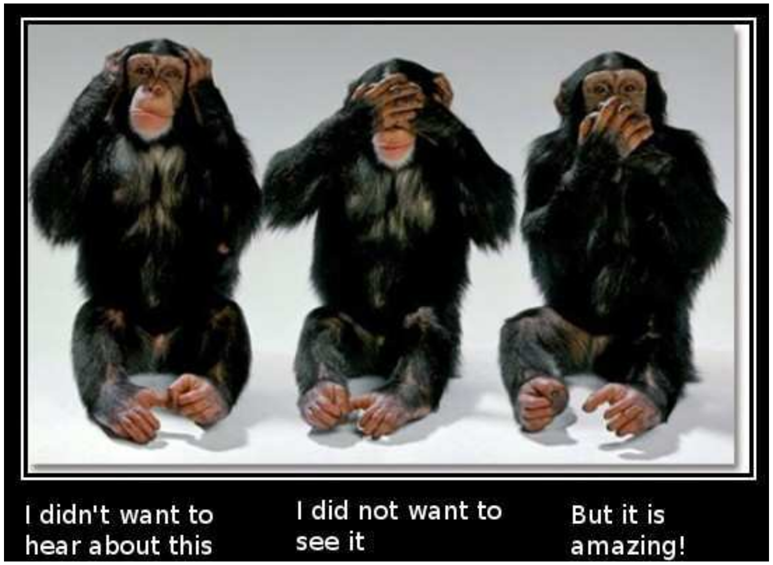
holds for every $h > 0$. The left hand side converges to $\frac{d}{dt} f(x(t),y(t))$ in the limit $h \to 0$ and the right hand side to $f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t)$ using the single variable chain rule twice. Here is the proof of the later, when we differentiate f with respect to t and y is treated as a constant:

$$\frac{f(\text{x(t+h)}) - f(x(t))}{h} = \frac{[f(\text{x(t) + (x(t+h)-x(t))}) - f(x(t))]}{\text{[x(t+h)-x(t)]}} \cdot \frac{\text{[x(t+h)-x(t)]}}{h} \ .$$

Write $H(t) =$ x(t+h)-x(t) in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t)+H) - f(x(t))]}{H} \cdot \frac{x(t+h) - x(t)}{h} \ .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(x(t))$ and the second factor to $x'(t)$.



1 We move on a circle $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ on a table with temperature distribution $f(x,y) = x^2 - y^3$. Find the rate of change of the temperature $\nabla f(x,y) = (2x, -3y^2)$, $\vec{r}'(t) = (-\sin(t), \cos(t))$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.

From $f(x,y) = 0$ one can express y as a function of x. From $d/df(x,y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain $y' = -f_x/f_y$. Even so, we do not know $y(x)$, we can compute its derivative! Implicit differentiation works also in three variables. The equation $f(x,y,z) = c$ defines a surface. Near a point where f_z is not zero, the surface can be described as a graph $z = z(x,y)$. We can compute the derivative z_x without actually knowing the function $z(x,y)$. To do so, we consider y a fixed parameter and compute using the chain rule

$$f_x(x,y,z(x,y))1 + f_z(x,y)z_x(x,y) = 0$$

so that $z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$.

2 The surface $f(x,y,z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x,y)$ at the point $(x,y,z) = (2,1,1)$.
Solution: $z_x(x,y) = -f_x(2,1,1)/f_z(2,1,1) = -4/(2/9) = -18$.

The chain rule is powerful because it implies other differentation rules like the addition, product and quotient rule in one dimensions: $f(x,y) = x+y, x = u(t), y = v(t), d/dt(x+y) = f_x u' + f_y v' = u' + v'$.
$f(x,y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x,y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.
As in one dimensions, the chain rule follows from linearization. If f is a linear function $f(x,y) = ax + by - c$ and if the curve $\vec{r}(t) = \langle x_0 + tu, y_0 + tv \rangle$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0+tu) + b(y_0+tv)) = au + bv$ and this is the dot product of $\nabla f = (a,b)$ with $\vec{r}\,'(t) = (u,v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

Homework

1 You know that $d/dtf(\vec{r}(t)) = 2$ if $\vec{r}(t) = \langle t, t\rangle$ and $d/dtf(\vec{r}(t)) = 3$ if $\vec{r}(t) = \langle t, -t\rangle$. Find the gradient of f at $(0, 0)$.

2 The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2+y^2-z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t\rangle$. Using the chain rule, compute the rate of change of the pressure the observer measures at time $t = 2$.

3 Mechanical systems are determined by the energy function $H(x, y)$, a function of two variables. The first, x is the position and the second y is the momentum. The equations of motion for the curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ are

$$\begin{aligned} x'(t) &= H_y(x, y) \\ y'(t) &= -H_x(x, y) \end{aligned}$$

They are called called **Hamilton equations**. a) Using the chain rule, verify that in full generality, the energy of a Hamiltonian system is preserved: for every path $\vec{r}(t) = \langle x(t), y(t)\rangle$ solving the system, we have $H(x(t), y(t)) = const$.
b) Check this in the particular case of the **pendulum**, where $H(x, y) = y^2/2 - \sin(x)$.

4 Derive using implicit differentiation the derivative $d/dx \operatorname{arctanh}(x)$, where

$$\tanh(x) = \sinh(x)/\cosh(x)\ .$$

The **hyperbolic sine** and **hyperbolic cosine** are defined as are $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$. We have $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\cosh^2(x) - \sinh^2(x) = 1$.

5 The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y. Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$. Estimate $g(1.01, 0.99)$ using linear approximation.

Lecture 12: Gradient

The **gradient** of a function $f(x,y)$ is defined as

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \, .$$

For functions of three dimensions, we define

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle \, .$$

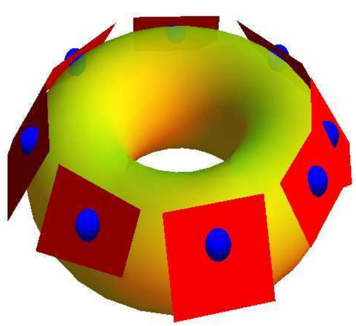
The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. Here is a very important fact:

Gradients are orthogonal to level curves and level surfaces.

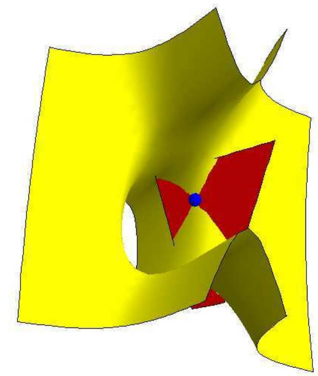
Proof. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$.
Because $\vec{n} = \nabla f(p,q) = \langle a,b \rangle$ is perpendicular to the level curve $f(x,y) = c$ through (p,q), the equation for the tangent line is $ax + by = d$, $a = f_x(p,q)$, $b = f_y(p,q)$, $d = ap + bq$. Compactly written, this is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

and means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane. It is one of the most important statements in multivariable calculus. since it provides a crucial link between calculus and geometry. The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines:



1 Compute the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$. **Solution:** $\nabla f(x,y,z) = \langle 6xy, 3x^2, 2z \rangle$. And $\nabla f(1,1,1) = \langle 6,3,2 \rangle$. The plane is $6x+3y+2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x+3y+2z = 11$.



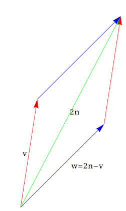
2 **Problem:** reflect the ray $\vec{r}(t) = \langle 1-t, -t, 1 \rangle$ at the surface

$$x^4 + y^2 + z^6 = 6 \, .$$

Solution: $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1,-2,1)$. The velocity vector in that ray is $\vec{v} = \langle -1,-1,0 \rangle$ The normal vector at this point is $\nabla f(-1,-2,1) = \langle -4,4,6 \rangle = \vec{n}$. The reflected vector is

$$R(\vec{v} = 2\mathrm{Proj}_{\vec{n}}(\vec{v}) - \vec{v} \, .$$

We have $\mathrm{Proj}_{\vec{n}}(\vec{v}) = 8/68 \langle -4,-4,6 \rangle$. Therefore, the reflected ray is $\vec{w} = (4/17)\langle -4,-4,6 \rangle - \langle -1,-1,0 \rangle$.



If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$.

The name directional derivative is related to the fact that every unit vector gives a direction. If $\vec{v}$ is a unit vector, then the chain rule tells us $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

The directional derivative tells us how the function changes when we move in a given direction. Assume for example that $T(x,y,z)$ is the temperature at position (x,y,z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}}T$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $h(x,y)$, then $D_{\vec{v}}h(x,y)$ gives us the slope we drive on.

3 If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)}f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

4 For $\vec{v} = (1,0,0)$, then $D_{\vec{v}}f = \nabla f \cdot v = f_x$, the directional derivative is a generalization of the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

The directional derivative satisfies $\boxed{|D_{\vec{v}}f| \leq |\nabla f||\vec{v}|}$ because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}|\cos(\phi)| \leq |\nabla f||\vec{v}|$.

The direction $\vec{v} = \nabla f/|\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

If $\vec{v} = \nabla f/|\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f/|\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

5 You are on a trip in a air-ship over Cambridge at $(1,2)$ and you want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x,y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest?
Solution: The gradient $\nabla p(x,y) = \langle 2x, 4y\rangle$ at the point $(1,2)$ is $\langle 2,8\rangle$. Normalize to get the direction $\langle 1,4\rangle/\sqrt{17}$.

The directional derivative has the same properties than any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f+g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + fD_v(g)$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x,y), where $\nabla f(x,y) = (0,0)$ are called **critical points** and help to understand the function f.

6 The Matterhorn is a 4'478 meter high mountain in Switzerland. It is quite easy to climb with a guide because there are ropes and ladders at difficult places. Evenso there are quite many climbing accidents at the Matterhorn, this does not stop you from trying an ascent. In suitable units on the ground, the height $f(x,y)$ of the Matterhorn is approximated by the function $f(x,y) = 4000 - x^2 - y^2$. At height $f(-10,10) = 3800$, at the point $(-10,10,3800)$, you rest. The climbing route continues into the south-east direction $v = \langle 1,-1\rangle/\sqrt{2}$. Calculate the rate of change in that direction. We have $\nabla f(x,y) = \langle -2x,-2y\rangle$, so that $\langle 20,-20\rangle \cdot \langle 1,-1\rangle/\sqrt{2} = 40/\sqrt{2}$. This is a place, with a ladder, where you climb $40/\sqrt{2}$ meters up when advancing 1m forward.
The rate of change in all directions is zero if and only if $\nabla f(x,y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f/|\nabla f|$ and get $D_{\nabla f} f = |\nabla f|$.

7 Assume we know $D_v f(1,1) = 3/\sqrt{5}$ and $D_w f(1,1) = 5/\sqrt{5}$, where $v = \langle 1,2\rangle/\sqrt{5}$ and $w = \langle 2,1\rangle/\sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f.
Solution: Let $\nabla f(1,1) = \langle a,b\rangle$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.

Homework

1 A surface $x^2 + y^2 - z = 1$ radiates light away. It can be parametrized as $\vec{r}(x,y) = \langle x, y, x^2 + y^2 - 1\rangle$. Find the parametrization of the wave front which is distance 1 from the surface.

2 Find the directional derivative $D_{\vec{v}}f(2,1) = \nabla f(2,1) \cdot \vec{v}$ into the direction $\vec{v} = \langle -3,4\rangle/5$ for the function $f(x,y) = x^5 y + y^3 + x + y$.

3 Assume $f(x,y) = 1 - x^2 + y^2$. Compute the directional derivative $D_{\vec{v}}(x,y)$ at $(0,0)$ where $\vec{v} = \langle \cos(t), \sin(t)\rangle$ is a unit vector. Now compute

$$D_v D_v f(x,y)$$

at $(0,0)$, for any unit vector. For which directions is this **second directional derivative** positive?

4 The **Kitchen-Rosenberg formula** gives the curvature of a level curve $f(x,y) = c$ as

$$\kappa = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x^2 + f_y^2)^{3/2}}$$

Use this formula to find the curvature of the ellipsoid $f(x,y) = x^2 + 2y^2 = 1$ at the point $(1,0)$.
P.S. This formula is known since a hundred years at least but got revived in computer vision. If you want to derive the formula, you can check that the angle

$$g(x,y) = \arctan(f_y/f_x)$$

of the gradient vector has κ as the directional derivative in the direction $\vec{v} = \langle -f_y, f_x\rangle/\sqrt{f_x^2 + f_y^2}$ tangent to the curve.

5 One numerical method to find the maximum of a function of two variables is to move in the direction of the gradient. This is called the **steepest ascent method**. You start at a point (x_0,y_0) then move in the direction of the gradient for some time c to be at $(x_1,y_1) = (x_0,y_0) + c\nabla f(x_0,y_0)$. Now you continue to get to $(x_2,y_2) = (x_1,y_1) + c\nabla f(x_1,y_1)$. This works well in many cases like the function $f(x,y) = 1 - x^2 - y^2$. It can have problems if the function has a flat ridge like in the **Rosenbrock function**

$$f(x,y) = 1 - (1-x)^2 - 100(y - x^2)^2 .$$

Plot the Contour map of this function on $-0.6 \leq x \leq 1, -0.1 \leq y \leq 1.1$ and find the directional derivative at $(1/5,0)$ in the direction $(1,1)/\sqrt{2}$. Why is it also called the **banana function**?

Lecture 13: Extrema

An important problem in multi-variable calculus is to **extremize** a function $f(x,y)$ of two variables. As in one dimensions, in order to look for maxima or minima, we consider points, where the "derivative" is zero.

> A point (a,b) in the plane is called a **critical point** of a function $f(x,y)$ if $\nabla f(a,b) = \langle 0,0 \rangle$.

> Critical points are candidates for extrema because at critical points, all directional derivatives $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ are zero. We can not increase the value of f by moving into a direction.

This definition does not include points, where f or its derivative is not defined We usually assume that a function is arbitrarily often differentiable. Points where the function has no derivatives do not belong to the domain and need to be studied separately. For the function $f(x,y) = |xy|$ for example, we would have to look at the points on the coordinate axes separately.

1 Find the critical points of $f(x,y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x,y) = \langle 4(x^3 - y), 4(y^3 - x) \rangle$ with critical points $(0,0), (1,1), (-1,-1)$.

2 $f(x,y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x,y) = \langle 2x\cos(x^2+y), \cos(x^2+y) + 1 \rangle$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $...(0,-\pi), (0,\pi), (0,3\pi), ...$

3 The graph of $f(x,y) = (x^2+y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = \langle 2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2) \rangle e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2 + y^2 = 1$. This function has infinitely many critical points.

4 The function $f(x,y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g\sin(x)) = \langle (0,0 \rangle$ for $(x,y) = \dots, (-\pi,0), (0,0), (\pi,0), (2\pi,0), \dots$. These points are equilibrium points, angles for which the pendulum is at rest.

5 The function $f(x,y) = a\log(y) - by + c\log(x) - dx$ is left invariant by the flow of the Volterra Lodka differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$. The point $(c/d, a/b)$ is a critical point. It is a place where the differential equation has stationary points.

6 The function $f(x,y) = |x| + |y|$ is smooth on the first quadrant. It does not have critical points there. The function has a minimum at $(0,0)$ but it is not in the domain, where f and ∇f are defined.

In one dimension, we needed $f'(x) = 0, f''(x) > 0$ to have a local minimum, $f'(x) = 0, f''(x) < 0$ for a local maximum. If $f'(x) = 0, f''(x) = 0$, then the critical point was undetermined and could be a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat inflection point like for $f(x) = x^3$.

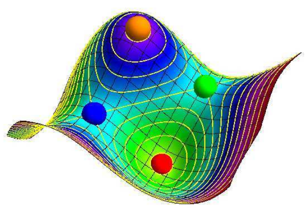
> Let now $f(x,y)$ be a function of two variables with a critical point (a,b). Define $D = f_{xx}f_{yy} - f_{xy}^2$. It is called the **discriminant** of the critical point.

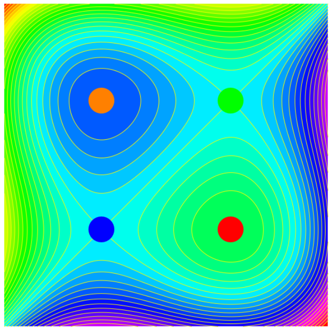
Remark: You might want to remember it better if you see that it is the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$.

> **Second derivative test**. Assume (a,b) is a critical point for $f(x,y)$.
> If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum.
> If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum.
> If $D < 0$ then (a,b) is a saddle point.

In the case $D = 0$, we need higher derivatives to determine the nature of the critical point.

7 The function $f(x,y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x,y) = \langle x^2 - 1, -y^2 + 1 \rangle$. There are 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1,1)$ we have $D = -4$ and so a saddle point,
For $(-1,1)$ we have $D = 4, f_{xx} = -2$ and so a local maximum,
For $(1,-1)$ we have $D = 4, f_{xx} = 2$ and so a local minimum.
For $(-1,-1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





To determine the maximum or minimum of $f(x,y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see next time how to get extrema on the boundary.

8 Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3,0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3,0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and 4/3 is a local maximum on the line $y = -1$. Comparing $f(4/3,0), f(4/3,-1)$ shows that $(4/3,0)$ is the global maximum.

As in one dimensions, knowing the critical points helps to understand the function. Critical points are also physically relevant. Examples are configurations with lowest energy. Many physical laws are based on the principle that the equations are critical points. Newton equations in Classical mechanics are an example: a particle of mass m moving in a field V along a path $\gamma : t \mapsto \vec{r}(t)$ extremizes the integral $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\, dt$ among all possible paths. Critical points γ satisfy the Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$.

Why is the second derivative test true? Assume $f(x, y)$ has the critical point $(0, 0)$ and is a quadratic function satisfying $f(0, 0) = 0$. Then

$$ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)$$

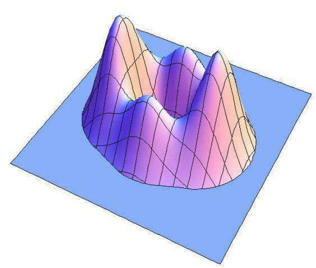
with $A = (x + \frac{b}{a}y), B = b^2/a^2$ and discriminant D. You see that if $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x, y) \neq (0, 0)$. The point $(0, 0)$ is a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x, y) \neq (0, 0)$ and the point (x, y) is a local maximum. If $D < 0$, then the function can take both negative and positive values. A general smooth function can be approximated by a quadratic function near $(0, 0)$.

Sometimes, we want to find the overall maximum and not only the local ones.

A point (a, b) in the plane is called a **global maximum** of $f(x, y)$ if $f(x, y) \leq f(a, b)$ for all (x, y). For example, the point $(0, 0)$ is a global maximum of the function $f(x, y) = 1 - x^2 - y^2$. Similarly, we call (a, b) a **global minimum**, if $f(x, y) \geq f(a, b)$ for all (x, y).

9 Does the function $f(x, y) = x^4 + y^4 - 2x^2 - 2y^2$ have a global maximum or a global minimum? If yes, find them. **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x, 0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x, y) = (x^2 - 1)^2 + (y - 1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

10 Here is a curious remark: let $f(x, y)$ be the height of an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a $+1$ if it is a maximum or minimum, and with -1 if it is a saddle point. Sum up all these number and you will get 1, independent of the function. This theorem of Poincare-Hopf is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



The following remarks can be skipped without problem:
1) for those of you who have taken linear algebra, you notice that the discriminant D is a determinant $\det(H)$ of the matrix $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. Besides the determinant, also the trace $f_{xx} + f_{yy}$ is independent of the coordinate system. The determinant is the product $\lambda_1\lambda_2$ of the eigenvalues of H and the trace is the sum of the eigenvalues. If the determinant D is positive, then λ_1, λ_2 have the same sign and this is also the sign of the trace. If the trace is positive then both eigenvalues are positive. This means that in in the eigendirections, the graph is concave up. We have a minimum. On the other hand, if the determinant D is negative, then λ_1, λ_2 have different signs and the function is concave up in one eigendirection and concave down in an other. In any case, if D is not zero, we have an orthonormal eigenbasis of the symmetric matrix A. In that basis, the matrix H is diagonal.
2) The discriminant D can be considered also at points where we have no critical point. The number $K = D/(1 + |\nabla f|^2)^2$ is called the **Gaussian curvature** of the surface. It is remarkable quantity since it only depends on the intrinsic geometry of the surface and not on the way how the surface is embedded in space. This is the famous Theorema Egregia (=great theorem) of Gauss. Note that at a critical point $\nabla f(x) = \vec{0}$, the discriminant agrees with the curvature $D = K$ at that point Since we mentioned one theorem for islands already: here is an other one, which follows from the famous Gauss-Bonnet theorem: assume you measure the curvature K at each point of an island and assume there is a nice beach all around so that the land disappears flat into the water. In that case the average curvature over the entire island is zero.
3) You might wonder what happens in higher dimensions. The second derivative test then does not work well any more without linear algebra. In three dimensions for example, one can form the second derivative matrix H again and look at all the eigenvalues of H. If all eigenvalues are negative, we have a local maximum, if all eigenvalues are positive, we have a local minimum. If the eigenvalues have different signs, we have a saddle point situation where in some directions the function increases and other directions the function decreases.

Homework

1 Find all the extrema of the function $f(x, y) = 2x^3 + 4y^2 - 2y^4 - 6x$ and determine whether they are maxima, minima or saddle points.

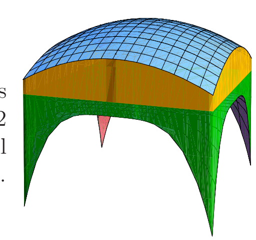
2 Where on the parametrized surface $\vec{r}(u, v) = \langle u^2, v^3, uv \rangle$ is the temperature $T(x, y, z) = 12x + y - 12z$ minimal? To find the minimum, look where the function $f(u, v) = T(\vec{r}(u, v))$ has an extremum. Find all local maxima, local minima or saddle points of f.

Remark. After you have found the function $f(u, v)$, you could replace the variables u, v again with x, y if you like and look at a function $f(x, y)$.

3 Find and classify all the extrema of the function $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$.

4 Find all extrema of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

5 The thickness of the region enclosed by the two graphs $f_1(x, y) = 10 - 2x^2 - 2y^2$ and $f_2(x, y) = -x^4 - y^4 - 2$ is denoted by $f(x, y) = f_1(x, y) - f_2(x, y)$. Classify all critical points of f and find the global minimal thickness.



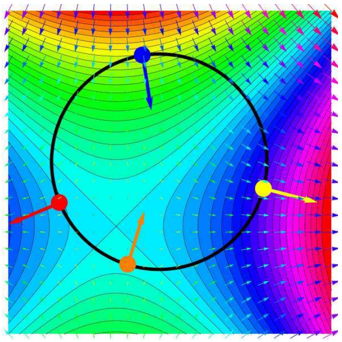
Lecture 14: Lagrange

We aim to find maxima and minima of a function $f(x,y)$ in the presence of a **constraint** $g(x,y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel because otherwise the we can move along the curve g and increase f. The directional derivative of f in the direction tangent to the level curve is zero if and only if the tangent vector to g is perpendicular to the gradient of f or if there is no tangent vector.

The system of equations $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = 0$. for the three unknowns x, y, λ are called **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

Lagrange theorem: Extrema of $f(x,y)$ on the curve $g(x,y) = c$ are either solutions of the Lagrange equations or critical points of g.

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla f = 0$ or $\nabla g = 0$. The case $\nabla f = 0$ can be included in the Lagrange equation case with $\lambda = 0$.



1 Minimize $f(x,y) = x^2 + 2y^2$ under the constraint $g(x,y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

2 Find the shortest distance from the origin to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize the function $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = \langle 2x, 2y \rangle, \nabla g = \langle 6x^5, 6y \rangle$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or 1 or -1. From the constraint equation $g = 1$, we obtain $y = \sqrt{(1-x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1,0), (-1,0)$. To see which is the minimum, just evaluate f on each of the points. We see that $(0, \pm\sqrt{1/3})$ are the minima.

3 Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r,h) = h\pi r^2 = 1$. The surface area is $A(r,h) = 2\pi r h + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x,y) = 2xy + 2\pi y^2$ under the constrained $g(x,y) = xy^2 = 1$. Calculate $\nabla f(x,y) = (2y, 2x + 4\pi y), \nabla g(x,y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$. Remark: Other factors can influence the shape. For example, the can has to withstand a pressure up to 100 psi. A typical can of "Coca-Cola classic" with 3.7 volumes of CO_2 dissolve has at 75F an internal pressure of 55 psi, where PSI stands for pounds per square inch.

4 On the curve $g(x,y) = x^2 - y^3$ the function $f(x,y) = x$ obviously has a minimum $(0,0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x,y) = 0$.

Remarks.
1) Either of the two properties equated in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.
2) With $g(x,y) = 0$, the Lagrange equations can also be written as $\nabla F(x,y,\lambda) = 0$ where $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$.
3) Either of the two properties equated in the Lagrange theorem are equivalent to "$\nabla g = \lambda \nabla f$ or f has a critical point".
4) Constrained optimization problems work also in higher dimensions. The proof is the same:

Extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$.

5 Find the extrema of $f(x,y,z) = z$ on the sphere $g(x,y,z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x,y,z) = (0,0,1), \nabla g(x,y,z) = (2x, 2y, 2z)$ and solve $(0,0,1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0,0,-1)$ the maximum the north pole $(0,0,1)$.

6 A dice shows k eyes with probability p_k with k in $\Omega = \{1,2,3,4,5,6\}$. A probability distribution is a nonnegative function p on Ω which sums up to 1. It can be written as a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The **entropy** of the probability vector $\vec{p}$ is defined as $f(\vec{p}) = -\sum_{i=1}^6 p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - ... - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + ... + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = 6\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_i = 1/6$. The distribution, where each event has the same probability is the distribution of maximal entropy. Maximal entropy means **least information content**. An unfair dice allows a cheating gambler or casino to gain profit. Cheating through asymmetric weight distributions can be avoided by making the dices transparent.

7 Assume that the probability that a physical or chemical system is in a state k is p_k and that the energy of the state k is E_k. Nature tries to minimize the **free energy** $f(p_1, \dots, p_n) =$

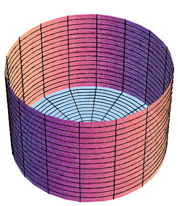
$-\sum_i [p_i \log(p_i) - E_i p_i]$ if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1-\lambda)$. The constraint $p_1 + \cdots + p_n = 1$ gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The Gibbs solution is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$.

Remarks:
1) Can we avoid Lagrange? This is often done in single variable calculus. To extremize $f(x,y)$ under the constraint $g(x,y) = 0$ we find $y = y(x)$ from the second equation and extremize the single variable problem $f(x, y(x))$. This needs to be done carefully and the boundaries must be considered. To extremize $f(x,y) = y$ on $x^2 + y^2 = 1$ for example we need to extremize $\sqrt{1-x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution. To extremize $f(x,y) = x^2 + y^2$ with constraint $g(x,y) = x^4 + 3y^2 - 1 = 0$ for example, we solve $y^2 = (1-x^4)/3$ and minimize $h(x) = f(x, y(x)) = x^2 + (1-x^4)/3$. $h'(x) = 0$ gives $x = 0$. The find the maximum $(\pm 1, 0)$, we had to maximize $h(x)$ on $[-1, 1]$, which occurs at ± 1.
To extremize $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = p(x) + p(y) = 1$, where p is a complicated function in x which satisfies $p(0) = 0, p'(1) = 2$,the Lagrange equations $2x = \lambda p'(x), 2y = \lambda p'(y), p(x) + p(y) = 1$ can be solved with $x = 0, y = 1, \lambda = 1$. We can not solve $g(x,y) = 1$ however for y in an explicit way.
2) How do we determine whether a solution of the Lagrange equations is a maximum or minimum? Instead of using a second derivative test, we make a list of critical points and pick the maximum and minimum. A second derivative test can be designed using second directional derivative in the direction of the tangent.
3) The Lagrange method also works with more constraints. With two constraints the constraint $g = c, h = d$ defines a curve in space. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f. Here is a formulation in three dimensions.

Extrema of $f(x,y,z)$ under the constraint $g(x,y,z) = c, h(x,y,z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x,y,z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

Homework

1 Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to extremize $f(x,y) = \pi x^2 y$ under the constraint $g(x,y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.



2 Find the extrema of the same function

$$f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$$

as in problem 4.1.3 but now on the entire disc $\{x^2 + y^2 \leq 4\ \}$ of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

3 Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \ .$$

Is there a global maximum or a global minimum for $f(x,y)$?

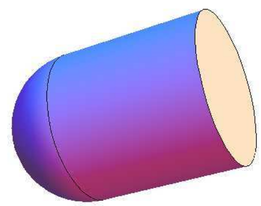
4 A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

$$f(h,r) = 3r^2 + 2rh$$

under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 \ .$$

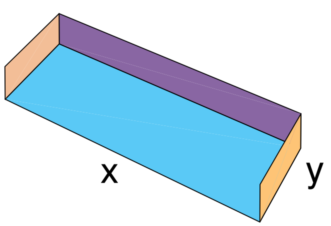
Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.



5 Minimize the material cost of an office tray

$$f(x,y) = xy + x + 2y$$

of length x, width y and height 1 under the constraint that the volume $g(x,y) = xy$ is



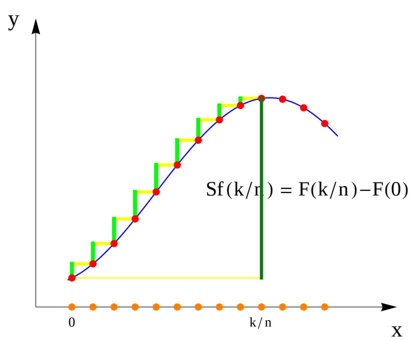
constant and equal to 4.

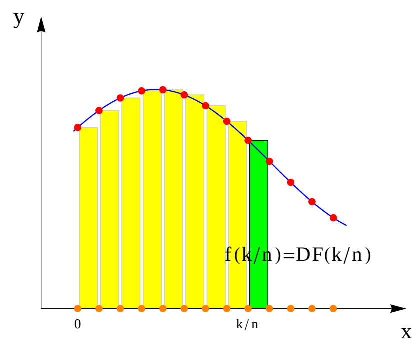
Lecture 15: Double integrals

Here is a one paragraph summary of single variable calculus: if $f(x)$ is a differentiable function, then the **Riemann integral** $\int_a^b f(x)\, dx$ is defined as the limit of the **Riemann sum** $S_n f(x) = \frac{1}{n}\sum_{k/n\in[a,b]} f(k/n)$ for $n \to \infty$. The derivative is the limit of difference quotients $D_n f(x) = n[f(x+1/n) - f(x)]$ as $n \to \infty$. The integral $\int_a^b f(x)\, dx$ is the **signed area** under the graph of f, where "signed" indicates that it can become negative too. The function $F(x) = \int_0^x f(y)\, dy$ is called an **anti-derivative** of f and determined up a constant. The **fundamental theorem of calculus** states

$$F'(x) = f(x), \int_0^x f(x) = F(x) - F(0)\ .$$

This theorem is obtained from the **quantum fundamental theorem** $DF(k/n) = f(k/n) - f(0), Sf(k/n) = F(k/n)$ (which holds for all functions!) in the limit $n \to \infty$ and allows to compute integrals by inverting differentiation. Differentiation rules become integration rules: the product rule becomes integration by parts, the chain rule becomes partial integration.





If $f(x,y)$ is differentiable on a region R, the integral $\int_R f(x,y)\, dxdy$ is defined as the limit of the Riemann sum

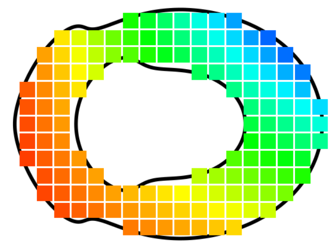
$$\frac{1}{n^2}\sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

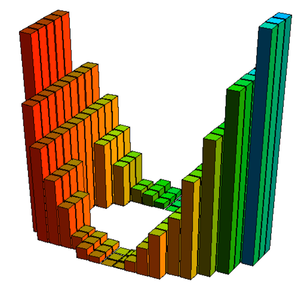
when $n \to \infty$. We write also $\int_R f(x,y)\, dA$ and think of dA as an area element.

1 If we integrate $f(x,y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over y to get $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2 If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside R.

3 The integral $\iint_R f(x,y)\, dA$ divided by the area of R is the **average** value of f on R.





4 One can interpret $\int\int_R f(x,y)\, dydx$ as the **signed volume** of the solid below the graph of f and above R in the $x-y$ plane. As in 1D integration, the volume of the solid below the xy-plane is counted negatively.

Fubini's theorem allows to switch the order of integration over a rectangle if the function f is continuous: $\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$.

Proof. We have for every n the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]}\sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]}\sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

which holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$.

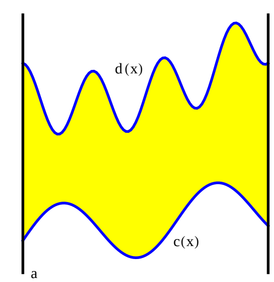
Fubini's theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

A **type I region** is of the form

$$R = \{(x,y) \mid a \leq x \leq b,\ c(x) \leq y \leq d(x)\ \}\ .$$

An integral over such a region is called a **type I integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx\ .$$

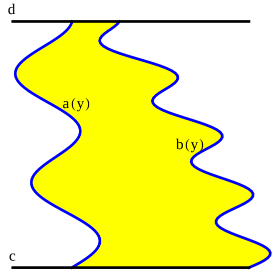


A **type II region** is of the form

$$R = \{(x,y) \mid c \le y \le d, a(y) \le x \le b(y)\ \} .$$

An integral over such a region is called a **type II integral**

$$\int\int_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$



5 Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \le x \le \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

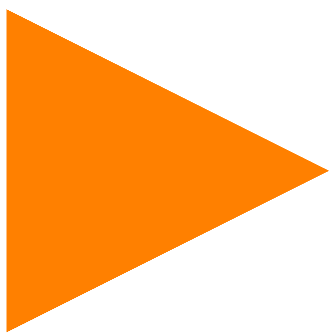
$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx$$

We have now an integral, which we can solve by substitution

$$= -\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3} .$$

6 Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of y: they are $x = y-1$ and $x = 1-y$

$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(\frac{1}{4}-\frac{1}{3}) = \frac{1}{10} .$$

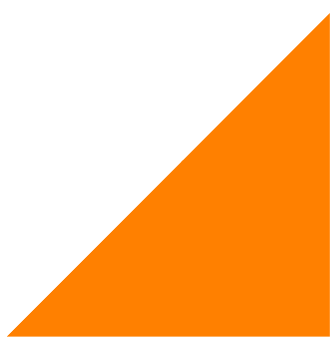


7 Let R be the triangle $1 \ge x \ge 0, 0 \le y \le x$. What is

$$\int\int_R e^{-x^2}\ dxdy\ ?$$

The type II integral $\int_0^1 [\int_y^1 e^{-x^2}\ dx] dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions.
The type I integral $\int_0^1 [\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... .$$



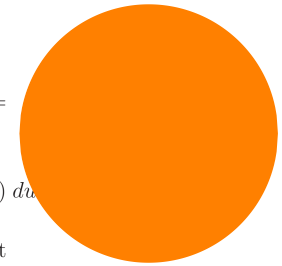
8 The area of a disc of radius R is

$$\int_{-R}^R \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\ dydx = \int_{-R}^R 2\sqrt{R^2-x^2}\ dx .$$

This integral can be solved with the substitution $x = R\sin(u), dx = R\cos(u)$

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)}R\cos(u)\ du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u)\ du$$

Using a double angle formula we get $R^2\int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u))}{2}\ du = R^2\pi$. We will now see how to do that better in polar coordinates.



Remark: The Riemann integral just defined works well for continuous functions. In other branches of mathematics like probability theory, a better integral is needed. The **Lebesgue integral** fits the bill. Its definition is close to the Riemann integral which we have given as the limit $n^{-2}\int_{(x_k,y_l)\in R} f(x_k,y_l)$ where $x_k = k/n, y_l = l/n$. The Lebesgue integral replaces the regularly spaced (x_k,y_l) grid with random points x_k, y_l and uses the same formula. The following Mathematica code computes the integral $\int_0^1\int_0^1 x^2y$ using this **Monte Carlo definition** of the Lebesgue integral.

```
M=10000;  R:=Random[];  f[x_,y_]:=x^2 y;  Sum[f[R,R],{M}]/M
```

It is as elegant than the numerical Riemann sum computation

```
M=100;  f[x_,y_]:=x^2 y;  Sum[f[k/M,l/M],{k,M},{l,M}]/M^2
```

but the Lebesgue integral is usually closer to the actual answer 1/6 than the Riemann integral. Note that for all continuous functions, the Lebesgue integral gives the same results than the Riemann integral. It does not change calculus. But it is useful for example to compute nasty integrals like the area of the Mandelbrot set.

Homework

1 Find the double integral $\int_1^4\int_0^2 (3x - \sqrt{y})\ dxdy$.

2 Find the area of the region

$$R = \{(x,y) \mid 0 \le x \le 2\pi, \sin(x) - 1 \le y \le \cos(x) + 2\}$$

and use it to compute the average value $\int\int_R f(x,y)\ dxdy/area(R)$ of $f(x,y) = y$ over that region.

3 Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2,2] \times [-3,3] = \{(x,y) \mid -2 \le x \le 2, -3 \le y \le 3\ \}$.

4 Calculate the iterated integral $\int_0^1\int_x^{2-x}(x^2-y)\ dydx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

5 Evaluate the double integral

$$\int_0^2 \int_{x^2}^4 \frac{x}{e^{y^2}}\ dydx .$$

Lecture 16: Surface area

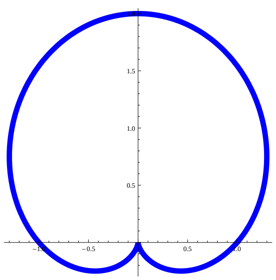
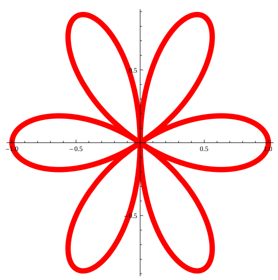
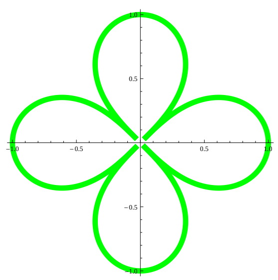
A **polar region** is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

In Cartesian coordinates the parametrization of the boundary curve is $\vec{r}(t) = \langle r(t)\cos(\theta(t), r(t)\sin(\theta(t)\rangle$. We are especially interested in regions which are bound by **polar graphs**, where $\theta(t) = t$.

1 The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**.

2 The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

3 The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**. It looks like an infinity sign.

To integrate in polar coordinates, we evaluate the integral

$$\iint_R f(x,y)\,dxdy = \iint_R f(r\cos(\theta), r\sin(\theta)r\,drd\theta$$

4 Integrate

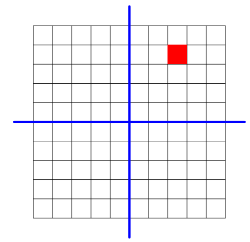
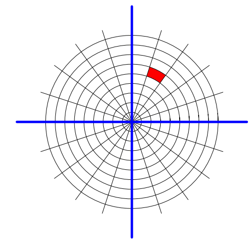
$$f(x,y) = x^2 + x^2 + xy ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\,dxdy = \int_0^1\int_0^{2\pi}(r^2 + r^2\cos(\theta)\sin(\theta))r\,d\theta dr = 2\pi/4$.

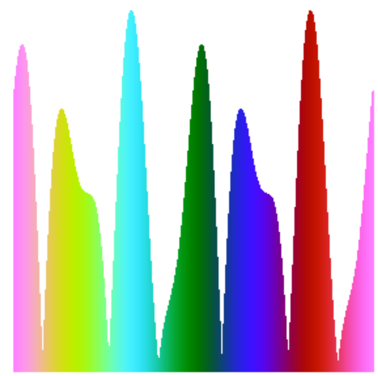
5 We have earlier computed area of the disc $\{x^2 + y^2 \leq 1\}$ using substitution. It is more elegant to do this integral in polar coordinates:

$$\int_0^{2\pi}\int_0^1 r\,drd\theta = 2\pi r^2/2|_0^1 = \pi .$$

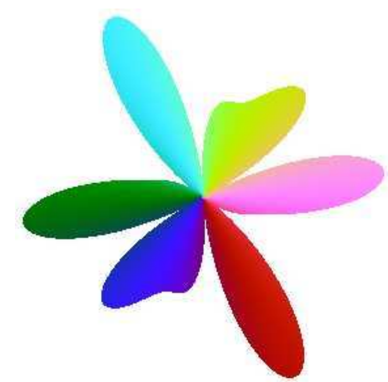
Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has the area $r\,d\theta dr$. If you have seen some linear algebra, note that the Jacobean matrix dT has the determinant r.

We can now integrate over type I or type II regions in the (θ, r) plane. like **flowers**: $\{(\theta, r)\ |0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



A polar region shown in polar coordinates. It is a type I region.



The same region in the xy coordinate system is not type I or II.

6 Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\,dxdy = \int_0^{2\pi}\int_0^{\cos(3\theta)} r\,dr\,d\theta = \int_0^{2\pi}\frac{\cos(3\theta)^2}{2}\,d\theta = \pi/2 .$$

7 Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y)\ |\ 1 < x^2 + y^2 < 4, y > 0\ \}$.

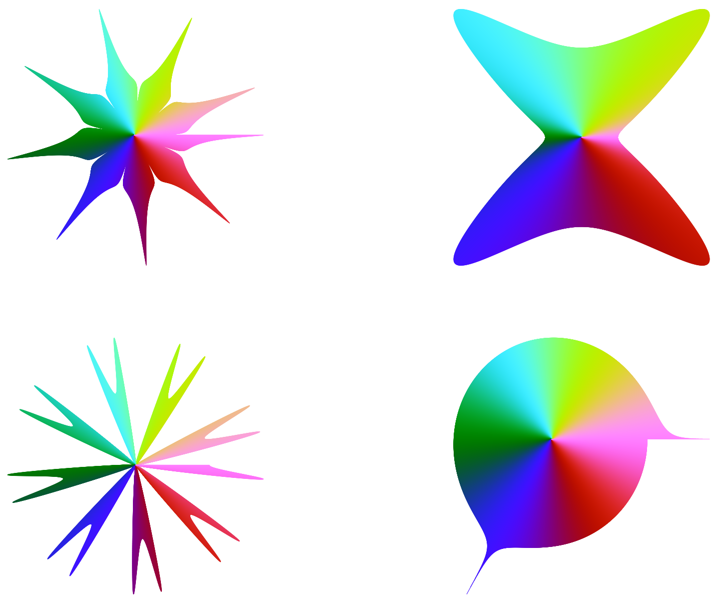
$$\int_1^2\int_0^{\pi} r\sin(\theta)r\,r\,d\theta dr = \int_1^2 r^3\int_0^{\pi}\sin(\theta)\,d\theta dr = \frac{(2^4-1^4)}{4}\int_0^{\pi}\sin(\theta)\,d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

8 The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b}\right)^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



A surface $\vec{r}(u,v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv \, .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$, the two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u+du] \times [v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r}+\vec{r}_u]$ and $[\vec{r}, \vec{r}+\vec{r}_v]$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \, dudv$.

9 The parametrized surface $\vec{r}(u,v) = \langle 2u, 3v, 0\rangle$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = \langle 0,0,6\rangle$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

[1] "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

10 The map $\vec{r}(u,v) = \langle L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)\rangle$ maps the rectangle $G = [0,2\pi] \times [0,\pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1 \, dS = \int_0^{2\pi}\int_0^{\pi} L^2\sin(v) \, dvdu = 4\pi L^2$

11 For graphs $(u,v) \mapsto \langle u, v, f(u,v)\rangle$, we have $\vec{r}_u = (1, 0, f_u(u,v))$ and $\vec{r}_v = (0,1,f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1+f_u^2+f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1+f_u^2+f_v^2} \, dudv$.

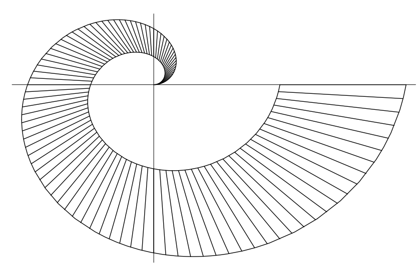
12 Lets take a surface of revolution $\vec{r}(u,v) = \langle v, f(v)\cos(u), f(v)\sin(u)\rangle$ on $R = [0,2\pi] \times [a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u))$, $\vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v| \, dudv = 2\pi\int_a^b |f(v)|\sqrt{1+f'(v)^2} \, dv$.

Homework

1 Integrate $f(x,y) = x^2$ over the unit disc $\{x^2+y^2 \leq 1\}$ in two ways, first using Cartesian coordinates, then using polar coordinates.

2 Find $\int\int_R (x^2+y^2)^{10} \, dA$, where R is the part of the unit disc $\{x^2+y^2 \leq 1\}$ for which $y > x$.

3 What is the area of the region which is bounded by the following three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0,2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0,2\pi]$ and third by the positive x-axis?



4 Find the average value of $f(x,y) = x^2+y^2$ on the annular region $R : 1 \leq |(x,y)| \leq 2$. The average is $\int_R f \, dxdy / \int_R 1 \, dxdy$.

5 Find the surface area of the part of the paraboloid $x = y^2+z^2$ which is inside the cylinder $y^2+z^2 \leq 9$.

Lecture 17: Triple integrals

If $f(x,y,z)$ is a function of three variables and E is a **solid region** in space, then $\int\int\int_E f(x,y,z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(i/n,j/n,k/n)\in E} f(\frac{i}{n},\frac{j}{n},\frac{k}{n}) .$$

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1 Assume E is the box $[0,1]\times[0,1]\times[0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

To compute the integral we start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the outer layer: $\int_0^1 3x^2 dx$ =1. When we calculate the most inner integral, we fix x and y. The integral is integrating up $f(x,y,z)$ along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these two dimensional sections.

There are two important methods to compute volume: the "washer method" and the "sandwich method". The washer method from single variable calculus reduces the problem directly to a one dimensional integral. The new sandwich method reduces the problem to a two dimensional integration problem.

The **washer method** slices the solid along a line. If $g(z)$ is the double integral along the two dimensional slice, then $\int_a^b g(z)\ dz$. The **sandwich method** sees the solid sandwiched between the graphs of two functions $g(x,y)$ and $h(x,y)$ over a common two dimensional region R. The integral becomes $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz] dA$.

2 An important special case of the sandwich method is the volume

$$\int\int_R \int_0^{f(x,y)} 1\ dzdxdy .$$

under the graph of a function $f(x,y)$ and above a region R. It is the integral $\int\int_R f(x,y)\ dA$. What we actually have computed is a triple integral

3 Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

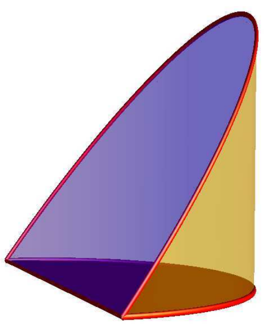
$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz] dA .$$

which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2}\ dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}r\ drd\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2)\ dz = 4\pi/3$.

4 The mass of a body with density $\rho(x,y,z)$ is defined as $\int\int\int_R \rho(x,y,z)\ dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1. **Solution:**

$$\begin{aligned} \int_0^2 \int_0^6 \int_0^{4-x^2} dz\ dy\ dx &= \int_0^2 \int_0^6 (4-x^2)\ dydx \\ = 6\int_0^2 (4-x^2)\ dx &= 6(4x - x^3/3)|_0^2 = 32 \end{aligned}$$

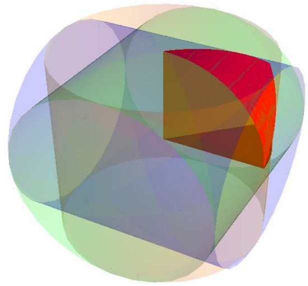
5 The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed his Riemann sum integration technique. It appears in every calculus text book. Find the volume. **Solution**. Look from the situation from above and picture it in the $x - y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\ dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos(\theta)\ drd\theta = 2/3$.



6 Finding the volume of the solid region bound by the three cylinders $x^2+y^2 = 1$, $x^2+z^2 = 1$ and $y^2+z^2 = 1$ is one of the most famous volume integration problems. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \le \theta \le \pi/4, r \le 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration problem

$$16\int_0^{\pi/4}\int_0^1 \sqrt{1 - r^2\cos^2(\theta)}r\ drd\theta$$

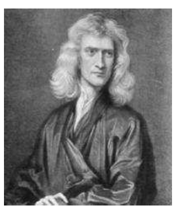
has an inner r-integral of $(16/3)(1-\sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x)+\sec(x)+\tan(x)$. The result is $16 - 8\sqrt{2}$.



The problem of computing volumes has been tackled early in mathematics:

Archimedes (287-212 BC) designed an integration method which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of **exhaustion** is close to the numerical method of integration by Riemann sum. In our terminology, Archimedes used the **washer method** to reduce the problem to a single variable problem. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. His **displacement method** later would morph into Cavalieri principle and modern rearrangement techniques in modern analysis. Heureka!

Cavalieri (1598-1647) would build on Archimedes ideas and determine area and volume using tricks like the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibniz(1646-1716): Newton and Leibniz, developed calculus independently. The new tool made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools as we do here.

Remarks which can as usual be skipped. 1) Here is an other way to compute integrals: the Lebesgue integral is more powerful than the Riemann integral: suppose we want to calculate the volume of some solid body R which we assumed to be contained inside the unit cube $[0,1] \times [0,1] \times [0,1]$. The **Monte Carlo method** shoots randomly n times onto the unit cube and count the number k of times, we hit the solid. The result k/n approximates the volume. Here is a Mathematica example with one eights of the unit ball:

$$\mathrm{R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of n=10000 times. The volume so measured is 0.5277. The actual volume of 1/8'th of the sphere is $\pi/6 = 0.524$. For $n \to \infty$ the Monte Carlo computation gives the actual volume. The Monte-Carlo integral is stronger than the Riemann integral. It is equivalent to the **Lebesgue integral** and allows to measure much more sets than solids with piecewise smooth boundaries.
2) Is there an "integral' which is able to measure **every solid** in space and which has the property that the volume of a rotated or translated body remains the same? No! Most sets one can define are "crazy" in the sense that one can not measure their volume. An example is the **paradox of Banach and Tarski** which tells that one can slice up the unit ball $x^2 + y^2 + z^2 \leq 1$ into 5 pieces A, B, C, D, E, rotate and translate them in space so that the pieces A, B, C fit together to be a unit ball and D, E again form an other unit ball. Since the volume has obviously doubled and volume should be additive in the sense that the volume of two disjoint sets should be the sum of the volumes, some of the sets A, B, C, D, E have no defined volume.

Homework

1 Evaluate the triple integral

$$\int_0^1 \int_0^z \int_0^{2y} z e^{-y^2} \, dxdydz \; .$$

2 Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 9 - (x^2 + y^2)$ and satisfying $x \geq 0$.

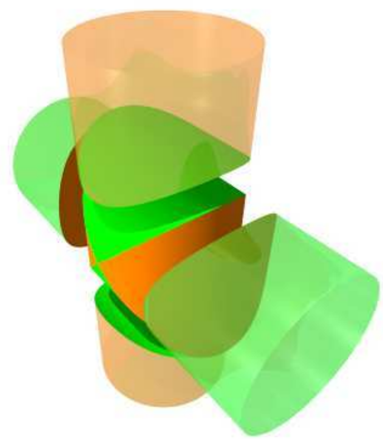
3 Find the **moment of inertia** $\int\int\int_E (x^2 + y^2) \, dV$ of a cone

$$E = \{ x^2 + y^2 \leq z^2 \; 0 \leq z \leq 1 \} ,$$

which has the z-axis as its center of symmetry.

4 Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices $(0, 0, 0), (1, 1, 0), (0, 1, 0), (0, 0, 3)$.

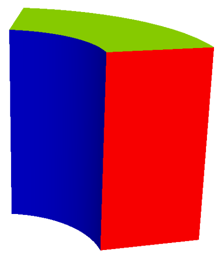
5 What is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$?



Lecture 18: Spherical Coordinates

Remember that **cylindrical coordinates** are coordinates in space in which polar coordinates are chosen in the xy plane and the z-coordinate is left untouched. A surface of revolution can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$, produces the same integration factor r as in polar coordinates.

$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(r,\theta,z)\ \boxed{r}\ drd\theta dz$$

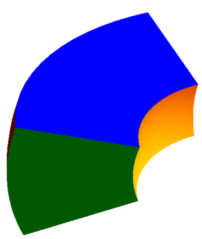


Remember also that spherical coordinates use ρ, the distance to the origin as well as two angles: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x,y,z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\ .$$

The integration factor can be seen by measuring the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

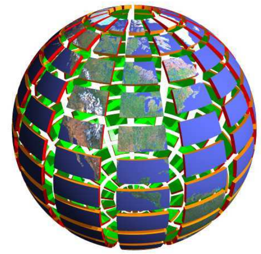
$$\iint_{T(R)} f(x,y,z)\ dxdydz = \iint_R g(\rho,\theta,z)\ \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d$$



1 A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\ d\phi d\theta d\rho\ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.



The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x,y,z)^2\ dzdydx$, where $r(x,y,z) = R\sin(\phi)$ is the distance from the axis L.

2 For a sphere of radius R we obtain with respect to the z-axis:

$$\begin{aligned} I &= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin^2(\phi)\rho^2\sin(\phi)\ d\phi d\theta d\rho \\ &= (\int_0^\pi \sin^3(\phi)\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2\pi}\ d\theta) \\ &= (\int_0^\pi \sin(\phi)(1-\cos^2(\phi))\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2pi}\ d\theta) \\ &= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15}\ . \end{aligned}$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8\cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8\cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned} \overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\cos(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\ \overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\sin^2(\phi)\sin(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\ \overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3\cos(\phi)\sin(\phi)\ d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V} \end{aligned}$$

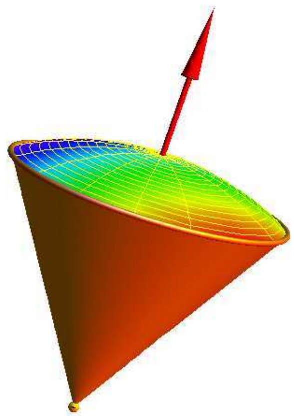
4 Find $\int\int\int_R z^2\ dV$ for the solid obtained by intersecting $\{1 \le x^2+y^2+z^2 \le 4\ \}$ with the double cone $\{z^2 \ge x^2+y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with

2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi)\, d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5})2\pi(\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi\frac{31}{5}(1 - 2^{-3/2}) .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.



Remarks: There are other coordinate systems besides Euclidean, cylindrical and spherical. One of them are **toral coordinates**, where $T(r, \phi, \theta) = (1+r\cos(\phi))\cos(\theta), (1+r\cos(\phi)\sin(\theta), r\sin(\phi))$, a coordinate system which works inside the solid torus $r \leq 1$.

Are there spherical coordinates in higher dimensions? Yes, of course. They are called **hyper-spherical coordinates**. In four dimensions for example we would have a third angle ψ and get

$$(x, y, z, w) = (\rho\sin(\psi)\sin(\phi)\sin(\theta), \rho\sin(\psi)\sin(\phi)\cos(\theta), \rho\sin(\psi)\cos(\phi), \rho\cos(\psi)) .$$

The four dimensional case is especially interesting because one can write the sphere S^3 in four dimensions as the set of pairs of complex numbers z, w with $|z|^2+|w|^2 = 1$. The 3 sphere is special because it is equal to the group $SU(2)$ of all unitary 2×2 matrices of determinant 1. It is also the set of all quaternions of length 1. The quaternions are historically interesting for multivariable calculus because they predated vector calculus we teach today and incorporate both the dot and cross product.

Homework

1 The density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1.$ is given by the forth power of the distance to the z-axes: $\sigma(x, y, z) = (x^2 + y^2)^2$. Find its mass

$$M = \int\int\int_E (x^2 + y^2)^2\, dxdydz .$$

2 Find the moment of inertia $\int\int\int_E (x^2 + y^2)\, dV$ of the body E whose volume is given by the integral

$$\mathrm{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi)\, d\rho d\theta d\phi .$$

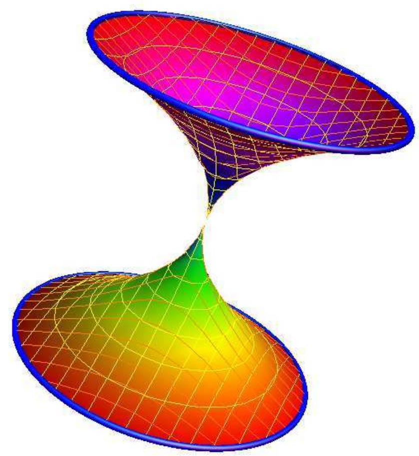
3 A solid is described in spherical coordinates by the inequality $\rho \leq \sin(\phi)$. Find its volume.

4 Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

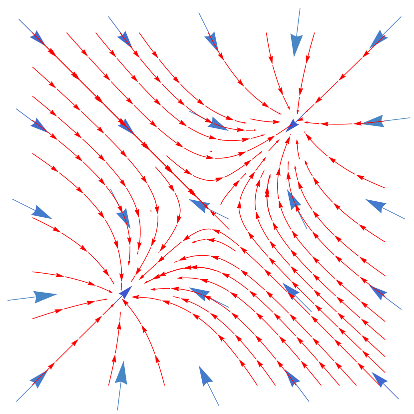
5 Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 1$.



Lecture 19: Vectorfields

A **vector field** in the plane is a map, which assigns to each point (x, y) in the plane a vector $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$. A vector field in space is a map, which assigns to each point (x, y, z) in space a vector $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

For example $\vec{F}(x, y) = \langle x-1, y\rangle/((x-1)^2+y^2)^{3/2} - \langle x+1, y\rangle/((x+1)^2+y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called **dipole field**. It is shown in the picture below



If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.

When is a vector field a gradient field? $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \nabla f(x, y)$ implies $Q_x(x, y) = P_y(x, y)$. If this does not hold at some point, F is no gradient field.

Clairot test: If $Q_x(x, y) - P_y(x, y)$ is not zero at some point, then $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is not a gradient field.

We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for F to be a gradient field. In class, we see more examples on how to construct the potential f from the gradient field F.

1 Is the vector field $\vec{F}(x, y) = \langle P, Q\rangle = \langle 3x^2y + y + 2, x^3 + x - 1\rangle$ a gradient field? **Solution**: the Clairot test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x, y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^2 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

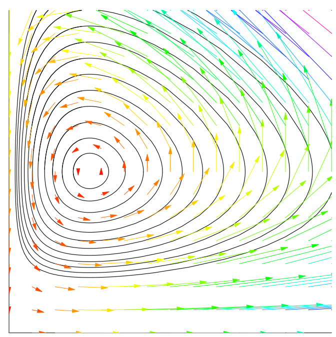
2 Is the vector field $\vec{F}(x, y) = \langle xy, 2xy^2\rangle$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

Vector fields are important in differential equations. We look at some examples in population dynamics and mechanics. You can skip this part. This is more motivational.

3 Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $-cy(t) + dxy$, where c, d are positive. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \ . \end{aligned}$$

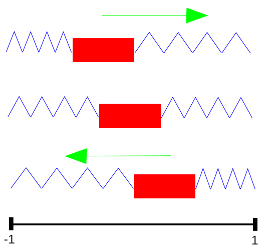
Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x, y) = \langle x(t), y(t)\rangle$, there is a curve $= \vec{r}(t) = \langle x(t), y(t)\rangle$ through that point for which the tangent $\vec{r}\,'(t) = (x'(t), y'(t)$ is the vector $\langle 0.4x - 0.4xy, -0.1y + 0.2xy\rangle$.

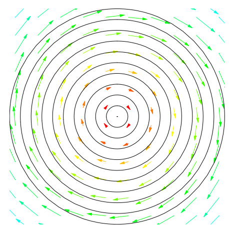


4 A class vector fields important in mechanics are **Hamiltonian fields**: If $H(x, y)$ is a function of two variables, then $\langle H_y(x, y), -H_x(x, y)\rangle$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x, y) = x^2+y^2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of H (as you have shown in th homework with the chain rule).

5 Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x, y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

6 We don't yet know yet the curve $t \mapsto \vec{r}(t) = \langle x(t), y(t)\rangle$, but we know the tangents $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle = \langle y(t), -x(t)\rangle$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE's) More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x, y), the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r\sin(t), r\cos(t))$ are solutions. They form a family of curves.





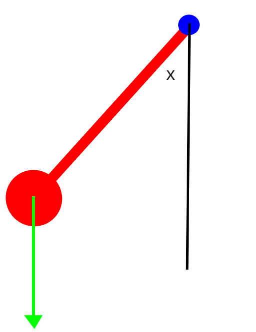
7 If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $G(x) = -gm\sin(x)$, where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero.
The Newton equation "mass times acceleration = force" gives

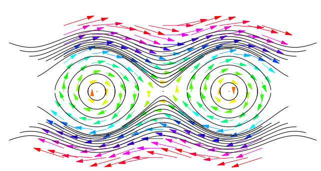
$$\ddot{x}(t) = -g\sin(x(t)) \, .$$

8 The equation of motion for the pendulum $\ddot{x}(t) = -g\sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g\sin(x(t)) \, .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $\vec{r}(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like $\sin, \cos, \exp, \log$ etc. However, one still can understand the curves.





Curves on the top of the picture represent situations where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum.

Vector fields in weather forecast On weather maps, one can see **isoterms**, curves of constant temperature or **isobars**, curves $p(x, y) = c$ of constant pressure. These are level curves. The wind velocity $\vec{F}(x, y)$ is close but not always exactly perpendicular to the **isobars**, the lines of equal pressure p. In reality, the scalar pressure field p and the velocity field $\vec{F}$ also depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

$$\frac{d}{dt}\vec{F} + \vec{F} \cdot \nabla\vec{F} = \nu\Delta\vec{F} - \nabla p + f, \mathrm{div}\vec{F} = 0$$

(where Δ and div are defined later. This is an other example of a **partial differential equation**. It is one of the millenium problems to prove that these equations have smooth solutions in space.

Homework

1 The vector field $\vec{F}(x, y) = \langle x/r^3, y/r^3\rangle$ appears in electrostatics, where $r = \sqrt{x^2 + y^2}$ is the distance to the charge. Find a function $f(x, y)$ such that $\vec{F} = \nabla f$. Hint. Write out the vector field $F = \langle P, Q\rangle$ where P, Q are functions of x, y. Then integrate P with respect to x.

2 a) Draw the gradient vector field of the function $f(x, y) = \sin(x + y)$.
b) Draw the gradient vector field of the function $f(x, y) = (x - 1)^2 + (y - 2)^2$.
Hint: In both cases, draw first a contour map of f and use a property of gradients to draw the vector field $F(x, y) = \nabla f$.

3 a) Is the vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle xy, x^2\rangle$ a gradient field?
b) Is the vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \langle \sin(x) + y, \cos(y) + x\rangle$ a gradient field?
In both cases, give the potential $f(x, y)$ satisfying $\nabla f(x, y) = \vec{F}(x, y)$ if it exists and if there is no gradient, give a reason, why it is not a gradient field.

4 Which of the following vector fields $\vec{F} = \langle P, Q\rangle$ can be written as $\vec{F} = \langle P, Q\rangle = \langle f_x, f_y\rangle$? Make use of Clairot's identity which implies that $Q_x = P_y$, if a function f exists. If f exists, find the potential f.
a) $\vec{F}(x, y) = \langle x^5, y^7\rangle$.
b) $\vec{F}(x, y) = \langle y^5, x^7\rangle$.
c) $\vec{F}(x, y) = \langle y, x\rangle$.
d) $\vec{F}(x, y) = \langle y^2 + x^2, y^2 + x^2\rangle$.
e) $\vec{F}(x, y) = \langle 5 - y^2 + 4x^3y^3, -2xy + 3x^4y^2\rangle$.

5 a) The vector field

$$\vec{F}(x, y, z) = \langle 5x^4y + z^4 + y\cos(x * y), x^5 + x\cos(xy), 4xz^3\rangle$$

is a gradient field. Find the potential function f.
b) Can you find conditions for a vector field $\vec{F} = \langle P, Q, R\rangle$ so that $\vec{F} = \nabla f$?

Lecture 20: Theorem of lineintegrals

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

The short-hand notation $\int_C \vec{F} \cdot d\vec{r}$ is also used. In physics, if $\vec{F}(x,y,z)$ is a force field, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ is called **work**. In electrodynamics, if $\vec{F}(x,y,z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ gives the **electric potential**.

1 Let $C : t \mapsto \vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ be a circle parametrized by $t \in [0, 2\pi]$ and let $\vec{F}(x,y) = \langle -y, x \rangle$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.
Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = 2\pi$

2 Let $\vec{r}(t)$ be a curve given in polar coordinates as $\vec{r}(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x,y) = (-xy, 0)$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$. **Solution:** In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $r'(t) = (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t))$. The line integral is

$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^\pi (\cos^3(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\, dt$$
$$= -2\int_0^\pi \sin^2(t)\cos^4(t)\, dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8 .$$

Here is the first generalization of the fundamental theorem of calculus to higher dimensions. It is called the **fundamental theorem of line integrals**.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a)) .$$

In other words, the line integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if $\vec{F}$ is a gradient field.

3 Let $f(x,y,z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

4 If $\vec{r}(t)$ is parallel to the level curve of f, then $d/dt f(\vec{r}(t)) = 0$ and $\vec{r}\,'(t)$ orthogonal to $\nabla f(\vec{r}(t))$.

5 If $\vec{r}(t)$ is orthogonal to the level curve, then $|d/dt f(\vec{r}(t))| = |\nabla f||\vec{r}\,'(t)|$ and $\vec{r}\,'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a)) .$$

For a gradient field, the line-integral along any closed curve is zero.

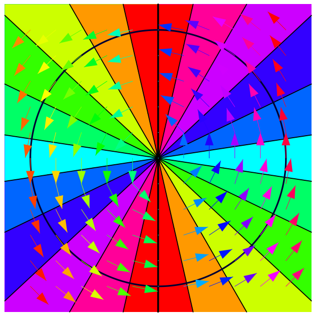
When is a vector field a gradient field? $\vec{F}(x,y) = \nabla f(x,y)$ implies $P_y(x,y) = Q_x(x,y)$. If this does not hold at some point, $\vec{F} = \langle P, Q \rangle$ is no gradient field. This is called the **component test** or Clairot test. We will see later that the condition $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$ implies that the field is conservative, if the region satisfies a certain property.

6 Let $\vec{F}(x,y) = \langle 2xy^2 + 3x^2, 2yx^2 \rangle$. Find a potential f of $\vec{F} = \langle P, Q \rangle$.
Solution: The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$. Find g, h from $f(x,y) = \int_0^x P(x,y)\, dx + h(y)$ and $f_y(x,y) = g(x,y)$.

7 Let $\vec{F}(x,y) = \langle P, Q \rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1 + y^2/x^2) = P, f_y = (1/x)/(1 + y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\, d\vec{r}$, where γ is the unit circle is

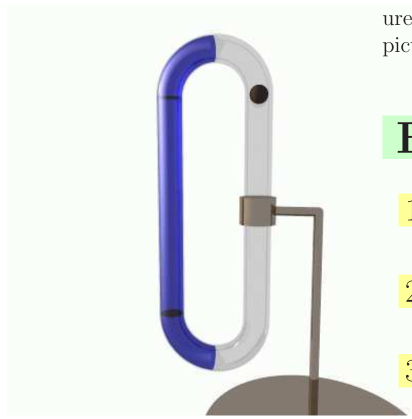
$$\int_0^{2\pi} \langle \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2} \rangle \cdot \langle -\sin(t), \cos(t) \rangle\, dt$$

which is $\int_0^{2\pi} 1\, dt = 2\pi$. What is wrong?
Solution: note that the potential f as well as the vector-field F are not differentiable everywhere. The curl of F is zero except at $(0,0)$, where it is not defined.

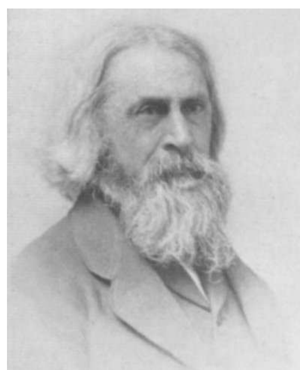


Remarks: The fundamental theorem of line integrals works in any dimension. You can formulate and check it yourself. The reason is that curves, vector fields, chain rule and integration along curves are easy to generalize in any dimensions. We will see later that if R is a simply connected region then $\vec{F}$ is a gradient field if and only if $\mathrm{curl}(F) = 0$ everywhere in R. A region R is called **simply connected**, if every curve in R can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annular region is an example which is not. Any region with a hole is not simply connected. For simply connected regions, the existence of a gradient field is equivalent to the field having curl zero everywhere.

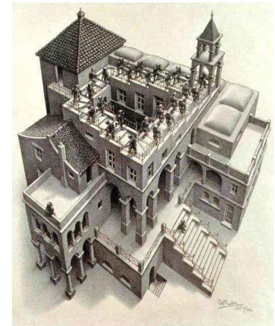
A device which implements a non gradient force field is called a **perpetual motion machine**. Mathematically, it realizes a force field for which along some closed loops the energy gain is nonnegative. By possibly changing the direction, the energy change is positive. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up with and to see why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.

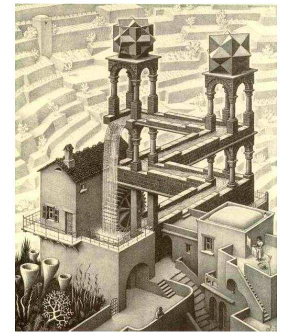


Why does this "perpetual motion machine" not work? The former Harvard professor Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the "antropic principle". "Such a series of motions would receive the technical name of a "perpetual motion" by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".



Nonconservative fields can also be generated by **optical illusion** as **M.C. Escher** did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"?





Homework

1 Let C be the space curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $t \in [0, 1]$ and let $\vec{F}(x, y, z) = \langle y, x, 5 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

2 Find the work done by the force field $F(x, y) = (x \sin(y), y)$ on a particle that moves along the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$.

3 Let $\vec{F}$ be the vector field $\vec{F}(x, y) = \langle -y, x \rangle/2$. Compute the line integral of F along an ellipse $\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$ with width $2a$ and height $2b$. The result should depend on a and b.

4 After this summer school, you relax in a Jacuzzi and move along curve C which is given by part of the curve $x^{10} + y^{10} = 1$ in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity $\vec{F}(x, y) = \langle x, y^4 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$, the energy you gain from the fluid force.

5 Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x^2 \rangle$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \neq 0$.

Lecture 21: Greens theorem

Green's theorem is the second integral theorem in the plane. This entire section deals with multivariable calculus in the plane, where we have two integral theorems, the fundamental theorem of line integrals and Greens theorem. First two reminders:

If $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a vector field and $C : \vec{r}(t) = \langle x(t), y(t)\rangle, t \in [a,b]$ is a curve, the **line integral**

$$\int_C \vec{F}\ d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\ dt$$

measures the work done by the field $\vec{F}$ along the path.

The **curl** of a two dimensional vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is defined as the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y)\ .$$

The curl(F) measures the **vorticity** of the vector field.

One can write $\nabla \times \vec{F} = \text{curl}(\vec{F})$ because the two dimensional cross product of (∂_x, ∂_y) with $\vec{F} = \langle P, Q\rangle$ is the scalar $Q_x - P_y$.

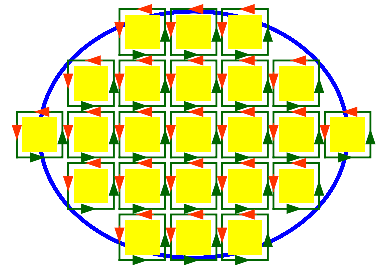
1 For $\vec{F}(x,y) = \langle -y, x\rangle$ we have curl$(F)(x,y) = 2$.

2 If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and curl$(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairot's theorem.

Green's theorem: If $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a smooth vector field and R is a region for which the boundary C is a curve parametrized so that R is "to the left". Then

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_G \text{curl}(F)\ dxdy\ .$$

Proof. Look first at a small square $G = [x, x+\epsilon] \times [y, y+\epsilon]$. The line integral of $\vec{F} = \langle P, Q\rangle$ along the boundary is $\int_0^\epsilon P(x+t, y)dt + \int_0^\epsilon Q(x+\epsilon, y+t)\ dt - \int_0^\epsilon P(x+t, y+\epsilon)\ dt - \int_0^\epsilon Q(x, y+t)\ dt$. This line integral measures the "circulation" at the place (x,y). Because $Q(x+\epsilon, y) - Q(x,y) \sim Q_x(x,y)\epsilon$ and $P(x, y+\epsilon) - P(x,y) \sim P_y(x,y)\epsilon$, the line integral is $(Q_x - P_y)\epsilon^2 \sim \int_0^\epsilon\int_0^\epsilon \text{curl}(F)\ dxdy$. All identities hold in the limit $\epsilon \to 0$. To prove the statement for a general region G, chop it into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vorticities on the squares is a Riemann sum approximation of the double integral.



George Green lived from 1793 to 1841. He was a physicist a self-taught mathematician and miller. His work greatly contributed to modern physics.

3 If $\vec{F}$ is a gradient field then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot d\vec{r}$ is zero by the fundamental theorem for line integrals. and $\int\int_G \text{curl}(F) \cdot dA$ is zero because curl$(F) =$ curl(grad$(f)) = 0$.

The already established Clairot identity

curl(grad$(f)) = 0$

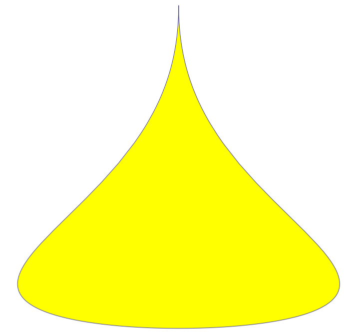
can also checked by writing it as $\nabla \times \nabla f$ and using that the cross product of two identical vectors is 0. Treating ∇ as a vector is called **nabla calculus**.

4 Find the line integral of $\vec{F}(x,y) = \langle x^2 - y^2, 2xy\rangle = \langle P, Q\rangle$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: curl$(\vec{F}) = Q_x - P_y = 2y - 2y = -4y$ so that $\int_C \vec{F}\ d\vec{r} = \int_0^2\int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

5 Find the area of the region enclosed by

$$\vec{r}(t) = \langle \frac{\sin(\pi t)^2}{t}, t^2 - 1\rangle$$

for $-1 \le t \le 1$. To do so, use Greens theorem with the vector field $\vec{F} = \langle 0, x\rangle$.



6 Green's theorem allows to express the coordinates of the **centroid** = center of mass

$$\left(\int\int_G x\ dA/A, \int\int_G y\ dA/A\right)$$

using line integrals. With the vector field $\vec{F} = \langle 0, x^2 \rangle$ we have

$$\int\int_G x\ dA = \int_C \vec{F}\ d\vec{r}\ .$$

7 An important application of Green is to **compute area**. With the vector fields $\vec{F}(x,y) = \langle P, Q \rangle = \langle -y, 0 \rangle$ or $\vec{F}(x,y) = \langle 0, x \rangle$ have vorticity $\text{curl}(\vec{F})(x,y) = 1$. For $\vec{F}(x,y) = \langle 0, x \rangle$, the right hand side in Green's theorem is the **area** of G:

$$\text{Area}(G) = \int_C x(t)\dot{y}(t)\ dt\ .$$

8 Let G be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of G is 0 from $(a,0)$ to $(b,0)$ because $\vec{F}(x,y) = \langle 0,0 \rangle$ there. The line integral is also zero from $(b,0)$ to $(b,f(b))$ and $(a,f(a))$ to $(a,0)$ because $N = 0$. The line integral along the curve $(t,f(t))$ is $-\int_a^b \langle -y(t), 0 \rangle \cdot \langle 1, f'(t) \rangle\ dt = \int_a^b f(t)\ dt$. Green's theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If $\vec{F}$ is a gradient field then $\text{curl}(F) = 0$ everywhere.

Is the converse true? Here is the answer:

A region R is called **simply connected** if every closed loop in R can be pulled together to a point in R.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\int_C \vec{F}\ d\vec{r} = 0$. So $\vec{F}$ has the closed loop property in G and is therefore a gradient field there.

In the homework, you look at an example of a not simply connected region where the $\text{curl}(\vec{F}) = 0$ does not imply that $\vec{F}$ is a gradient field.

An engineering application of Greens theorem is the **planimeter**, a mechanical device for measuring areas. We will demonstrate it in class. Historically it had been used in medicine to measure the size of the cross-sections of tumors, in biology to measure the area of leaves or wing sizes of insects, in agriculture to measure the area of forests, in engineering to measure the size of profiles. There is a vector field $\vec{F}$ associated to a planimeter which is obtained by placing a unit vector perpendicular to the arm).
One can prove that $\vec{F}$ has vorticity 1. The planimeter calculates the line integral of $\vec{F}$ along a given curve. Green's theorem assures it is the area.

Homework

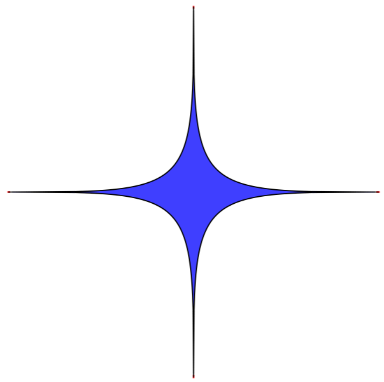
1 Calculate the line integral $\int_C \vec{F}\ d\vec{r}$ with $\vec{F} = \langle 2y + x\sin(y), x^2\cos(y) - 3y^{200\sin(y)} \rangle$ along a triangle C with edges $(0,0)$, $(\pi/2, 0)$ and $(\pi/2, \pi/2)$.

2 Evaluate the line integral of the vector field $\vec{F}(x,y) = \langle xy^2, x^2 \rangle$ along the rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$.

3 Find the area of the region bounded by the **hypocycloid**

$$\vec{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle$$

using Green's theorem. The curve is parameterized by $t \in [0, 2\pi]$.



4 Let G be the region $x^6 + y^6 \leq 1$. Compute the line integral of the vector field $\vec{F}(x,y) = \langle x^6, y^6 \rangle$ along the boundary.

5 Let $\vec{F}(x,y) = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$. Let $C : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle, t \in [0, 2\pi]$.
a) Compute $\int_C \vec{F} \cdot d\vec{r}$.
b) Show that $\text{curl}(\vec{F}) = 0$ everywhere for $(x,y) \neq (0,0)$.
c) Let $f(x,y) = \arctan(y/x)$. Verify that $\nabla f = \vec{F}$.
d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does a) and b) not contradict Green's theorem?

Lecture 22: Curl and Divergence

We have seen the curl in two dimensions: $\text{curl}(F) = Q_x - P_y$. By Greens theorem, it had been the average work of the field done along a small circle of radius r around the point in the limit when the radius of the circle goes to zero. Greens theorem so has explained what the curl is. In three dimensions, the curl is a vector:

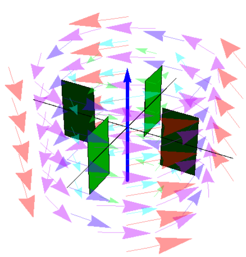
The **curl** of a vector field $\vec{F} = \langle P, Q, R\rangle$ is defined as the vector field

$$\text{curl}(P, Q, R) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \ .$$

Invoking nabla calculus, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$. Note that the third component of the curl is for fixed z just the two dimensional vector field $\vec{F} = \langle P, Q\rangle$ is $Q_x - P_y$. While the curl in 2 dimensions is a scalar field, it is a vector in 3 dimensions. In n dimensions, it would have dimension $n(n-1)/2$. This is the number of two dimensional coordinate planes in n dimensions. The curl measures the "vorticity" of the field.

If a field has zero curl everywhere, the field is called **irrotational**.

The curl is often visualized using a "paddle wheel". If you place such a wheel into the field into the direction v, its rotation speed of the wheel measures the quantity $\vec{F} \cdot \vec{v}$. Consequently, the direction in which the wheel turns fastest, is the direction of $\text{curl}(\vec{F})$. Its angular velocity is the length of the curl. The wheel could actually be used to measure the curl of the vector field at any point. In situations with large vorticity like in a tornado, one can "see" the direction of the curl near the vortex center.



In two dimensions, we had two derivatives, the gradient and curl. In three dimensions, there are three fundamental derivatives, the **gradient**, the **curl** and the **divergence**.

The **divergence** of $\vec{F} = \langle P, Q, R\rangle$ is the scalar field $\text{div}(\langle P, Q, R\rangle) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$.

The divergence can also be defined in two dimensions, but it is not fundamental.

The **divergence** of $\vec{F} = \langle P, Q\rangle$ is $\text{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

In two dimensions, the divergence is just the curl of a -90 degrees rotated field $\vec{G} = \langle Q, -P\rangle$ because $\text{div}(\vec{G}) = Q_x - P_y = \text{curl}(\vec{F})$. The divergence measures the "expansion" of a field. If a field has zero divergence everywhere, the field is called **incompressible**.

With the "vector" $\nabla = \langle \partial_x, \partial_y, \partial_z\rangle$, we can write $\text{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$. Formulating formulas using the "Nabla vector" and using rules from geometry is called **Nabla calculus**. This works both in 2 and 3 dimensions even so the ∇ vector is not an actual vector but an operator. The following combination of divergence and gradient often appears in physics:

$$\Delta f = \text{div}(\text{grad}(f)) = f_{xx} + f_{yy} + f_{zz} \ .$$

It is called the Laplacian of f. We can write $\Delta f = \nabla^2 f$ because $\nabla \cdot (\nabla f) = \text{div}(\text{grad}(f))$.

We can extend the Laplacian also to vector fields with

$\Delta \vec{F} = (\Delta P, \Delta Q, \Delta R)$ and write $\nabla^2 \vec{F}$.

Here are some identities:

$\text{div}(\text{curl}(\vec{F})) = 0$.
$\text{curlgrad}(\vec{F}) = \vec{0}$
$\text{curl}(\text{curl}(\vec{F})) = \text{grad}(\text{div}(\vec{F}) - \Delta(\vec{F}))$.

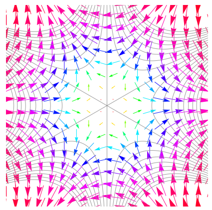
Proof. $\nabla \cdot \nabla \times \vec{F} = 0$.
$\nabla \times \nabla \vec{F} = \vec{0}$.
$\nabla \times \nabla \times \vec{F} = \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$.

1 **Question:** Is there a vector field $\vec{G}$ such that $\vec{F} = \langle x + y, z, y^2\rangle = \text{curl}(\vec{G})$?
Answer: No, because $\text{div}(\vec{F}) = 1$ is incompatible with $\text{div}(\text{curl}(\vec{G})) = 0$.

2 Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \text{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \text{grad}(f)$. Now, $\text{div}(F) = 0$ implies $\text{divgrad}(f) = \Delta f = 0$.

3 Find an example of a field which is both incompressible and irrotational. Solution. Find f which satisfies the Laplace equation $\Delta f = 0$, like $f(x, y) = x^3 - 3xy^2$, then look at its gradient field $\vec{F} = \nabla f$. In that case, this gives

$$\vec{F}(x, y) = \langle 3x^2 - 3y^2, -6xy\rangle \ .$$



4 If we rotate the vector field $\vec{F} = \langle P, Q \rangle$ by 90 degrees $= \pi/2$, we get a new vector field $\vec{G} = \langle -Q, P \rangle$. The integral $\int_C F \cdot ds$ becomes a **flux** $\int_\gamma G \cdot dn$ of G through the boundary of R, where dn is a normal vector with length $|r'|dt$. With $\text{div}(\vec{F}) = (P_x + Q_y)$, we see that

$$\text{curl}(\vec{F}) = \text{div}(\vec{G}) \ .$$

Green's theorem now becomes

$$\int\int_R \text{div}(\vec{G}) \ dxdy = \int_C \vec{G} \cdot \vec{dn} \ ,$$

where $dn(x, y)$ is a normal vector at (x, y) orthogonal to the velocity vector $\vec{r}\,'(x, y)$ at (x, y). This new theorem has a generalization to three dimensions, where it is called Gauss theorem or divergence theorem. Don't treat this however as a different theorem in two dimensions. It is just Green's theorem in disguise.

This result shows:

The divergence at a point (x, y) is the average flux of the field through a small circle of radius r around the point in the limit when the radius of the circle goes to zero

We have now all the derivatives together. In dimension d, there are d fundamental derivatives.

$$1$$

$$1 \xrightarrow{\text{grad}} 1$$

$$1 \xrightarrow{\text{grad}} 2 \xrightarrow{\text{curl}} 1$$

$$1 \xrightarrow{\text{grad}} 3 \xrightarrow{\text{curl}} 3 \xrightarrow{\text{div}} 1$$

They are incarnations of the same derivative, the so called **exterior derivative**.
To the end, let me stress that it is important you keep the dimensions. Many books treat two dimensional situations using terminology from three dimensions which leads to confusion. Geometry in two dimensions should be treated as a "flatlander"[1] in two dimensions only. Integral theorems become more transparent if you look at them in the right dimension. In one dimension, we had one theorem, the fundamental theorem of calculus. In two dimensions, there is the fundamental theorem of line integrals and Greens theorem. In three dimensions there are three theorems: the fundamental theorem of line integrals, Stokes theorem and the divergence theorem. We will look at the remaining two theorems next time.

[1] A. Abbott, Flatland, A romance in many dimensions,1884

Homework

1 Find a nonzero vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ in each of the following cases:
a) $\vec{F}$ is irrotational but not incompressible.
b) $\vec{F}$ is incompressible but not irrotational.
c) $\vec{F}$ is irrotational and incompressible.
d) $\vec{F}$ is not irrotational and not incompressible.

The terminology in this problem comes from fluid dynamics where fluids can be incompressible, irrotational.

2 The vector field $\vec{F}(x, y, z) = \langle x, y, -2z \rangle$ satisfies $\text{div}(\vec{F}) = 0$. Can you find a vector field $\vec{G}(x, y, z)$ such that $\text{curl}(\vec{G}) = \vec{F}$? Such a field $\vec{G}$ is called a **vector potential**.
Hint. Write $\vec{F}$ as a sum $\langle x, 0, -z \rangle + \langle 0, y, -z \rangle$ and find vector potentials for each of the summand using a vector field you have seen in class.

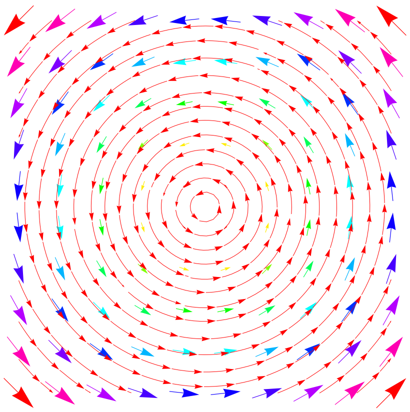
3 Evaluate the flux integral $\int\int_S \langle 0, 0, yz \rangle \cdot \vec{dS}$, where S is the surface with parametric equation $x = uv, y = u + v, z = u - v$ on $R : u^2 + v^2 \leq 1$.

4 Evaluate the flux integral $\iint_S \text{curl}(F) \cdot \vec{dS}$ for $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0, 1] \times [0, 1]$ and has an upward orientation.

5 a) What is the relation between the flux of the vector field $\vec{F} = \nabla g / |\nabla g|$ through the surface $S : \{g = 1\}$ with $g(x, y, z) = x^6 + y^4 + 2z^8$ and the surface area of S?
b) Find the flux of the vector field $\vec{G} = \nabla g \times \langle 0, 0, 1 \rangle$ through the surface S.

Remark This problem, both part a) and part do not need any computation. You can answer each question with one sentence. In part a) compare $\vec{F} \cdot \vec{dS}$ with dS in that case.

Lecture 22: Curl and Divergence



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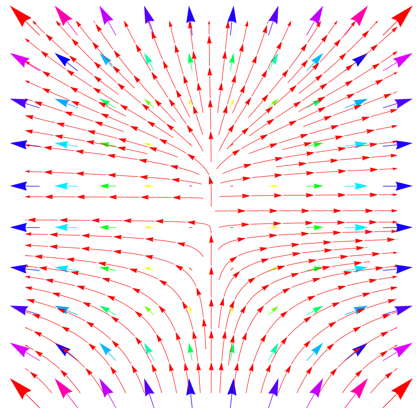

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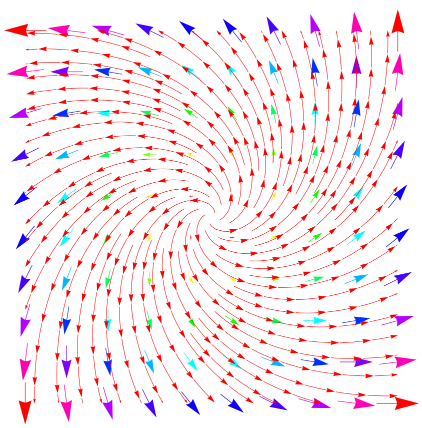

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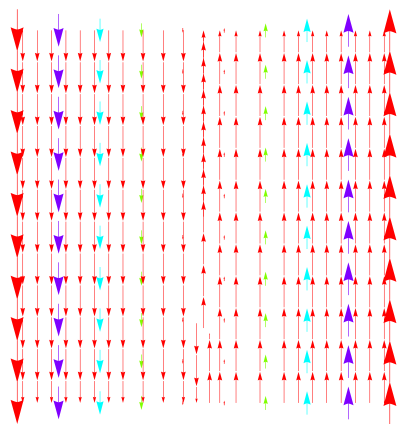

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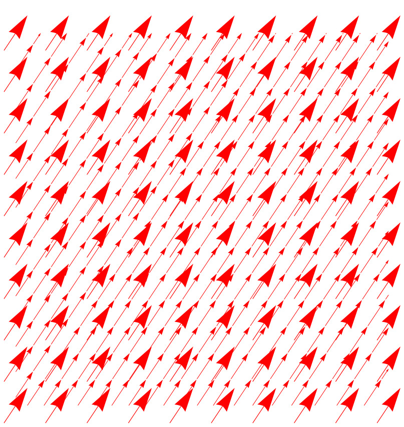

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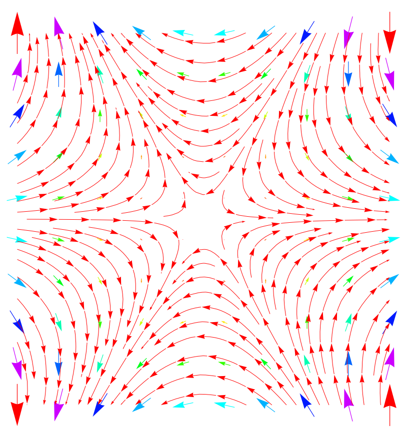


VI

1 Which vector fields can be incompressible?

2 Which vector fields can be irrotational?

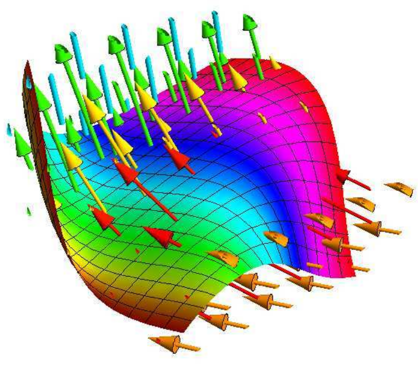
Lecture 23: Stokes Theorem

Assume a surface S is parametrized as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a domain G in the uv-plane.

The **flux integral** of $\vec{F}$ through S is defined as the double integral

$$\int\int_G \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dudv\ .$$

With the short hand notation $\vec{dS} = (\vec{r}_u \times \vec{r}_v)\ dudv$ representing an infinitesimal normal vector to the surface, this can be written as $\int\int_S \vec{F} \cdot \vec{dS}$. The interpretation is that if $\vec{F}$ = fluid velocity field, then $\int\int_S \vec{F} \cdot \vec{dS}$ is the amount of fluid passing through S in unit time.
Because $\vec{n} = \vec{r}_u \times \vec{r}_v / |\vec{r}_u \times \vec{r}_v|$ is a unit vector normal to the surface and on the surface, $\vec{F} \cdot \vec{n}$ is the normal component of the vector field with respect to the surface. One could write therefore also $\int\int_S \vec{F} \cdot \vec{dS} = \int\int \vec{F} \cdot \vec{n}\ dS$ where dS is the surface element we know from when we computed surface area. The function $\vec{F} \cdot \vec{n}$ is the scalar projection of $\vec{F}$ in the normal direction. Whereas the formula $\int\int 1\ dS$ gave the area of the surface with $dS = |\vec{r}_u \times \vec{r}_v| dudv$, the flux integral weights each area element dS with the normal component of the vector field with $\vec{F}(\vec{r}(u,v) \cdot \vec{n}(\vec{r}(u,v))$. We do not use this formula for computations because computing $\vec{n}$ gives additional work. We just determine the vectors $\vec{F}(\vec{r}(u,v))$ and $\vec{r}_u \times \vec{r}_v$ and integrate its dot product over the domain.



1 Compute the flux of $\vec{F}(x,y,z) = \langle 0, 1, z^2\rangle$ through the upper half sphere S parametrized by

$$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle\ .$$

Solution. We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u,v)) = \langle 0, 1, \cos^2(v)\rangle$ so that

$$\int_0^{2\pi}\int_0^{\pi} -\langle 0, 1, \cos^2(v)\rangle \cdot \langle \cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)\rangle\ dudv\ .$$

The flux integral is $\int_0^{2\pi}\int_{\pi/2}^{\pi} -\sin^2(v)\sin(u) - \cos^3(v)\sin(v)\ dudv$ which is $-\int_{\pi/2}^{\pi}\cos^3 v\sin(v)\ dv = \cos^4(v)/4|_0^{\pi/2} = -1/4$.

2 Calculate the flux of the vector field $\vec{F}(x,y,z) = \langle 1, 2, 4z\rangle$ through the paraboloid $z = x^2+y^2$ lying above the region $x^2+y^2 \leq 1$. **Solution:** We can parametrize the surface as $\vec{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r^2\rangle$ where $\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos(\theta), -2r^2\sin(\theta), r\rangle$ and $\vec{F}(\vec{r}(u,v)) = \langle 1, 2, 4r^2\rangle$. We get $\int_S \vec{F} \cdot \vec{dS} = \int_0^{2\pi}\int_0^1 (-2r^2\cos(v) - 4r^2\sin(v) + 4r^3)\ drd\theta = 2\pi$.

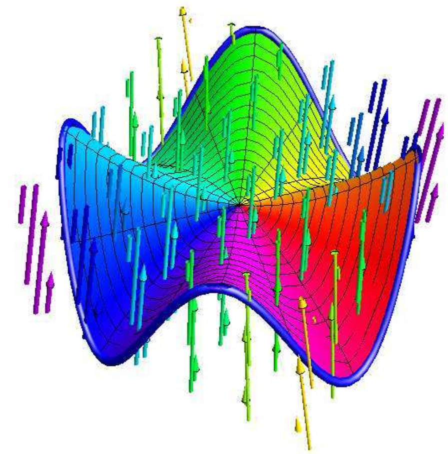
3 Compute the flux of $\vec{F}(x,y,z) = \langle 2, 3, 1\rangle$ through the torus parameterized as $\vec{r}(u,v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v)\rangle$, where both u and v range from 0 to $2pi$. **Solution.** There is no computation is needed. Think about what the flux means.

The following theorem is the second fundamental theorem of calculus in three dimensions. Since we integrate over two and one dimensional objects many concepts of multivariable calculus come together.

The **boundary** of a surface is a curve oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector v, a vector w pointing towards the surface and the normal vector n to the surface form a right handed coordinate system.

Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \text{curl}(\vec{F})\ \cdot \vec{dS} = \int_C \vec{F} \cdot \vec{dr}\ .$$



The boundary curve C is oriented such that if you walk along the surface. If your head points into the direction of the normal $\vec{r}_u \times \vec{r}_v$, then the surface to your left.

One can prove this similarly as Greens theorem. Chop up a as a union of small triangles. As before, the sum of the fluxes through all these triangles adds up to the flux through the surface and the sum of the line integrals along the boundaries adds up to the line integral of the boundary of S. Stokes theorem for a small triangle can be reduced to Greens theorem because with a coordinate system such that the triangle is in the $x-y$ plane, the flux of the field is the double integral $Q_x - P_y$.

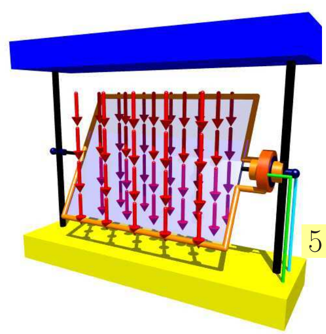
4 Let $\vec{F}(x,y,z) = \langle -y, x, 0\rangle$ and let S be the upper semi hemisphere, then $\mathrm{curl}(\vec{F})(x,y,z) = \langle 0,0,2\rangle$. The surface is parameterized by $\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v)\rangle$ on $G = [0,2\pi]\times[0,\pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}(u,v)$ so that $\mathrm{curl}(\vec{F})(x,y,z)\cdot \vec{r}_u\times\vec{r}_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi}\int_0^{\pi/2}\sin(2v)\; dvdu = 2\pi$.
The boundary C of S is parameterized by $\vec{r}(t) = \langle \cos(t), \sin(t), 0\rangle$ so that $\vec{dr} = \vec{r}\,'(t)dt = \langle -\sin(t), \cos(t), 0\rangle\; dt$ and $\vec{F}(\vec{r}(t))\; \vec{r}\,'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F}\cdot\vec{dr}$ along the boundary is 2π.

5 If S is a surface in the xy-plane and $\vec{F} = \langle P, Q, 0\rangle$ has zero z component, then $\mathrm{curl}(\vec{F}) = \langle 0,0,Q_x - P_y\rangle$ and $\mathrm{curl}(\vec{F})\cdot\vec{dS} = Q_x - P_y\; dxdy$. We see that for a surface which is flat, Stokes theorem is a consequence of Green's theorem. If we put the coordinate axis so that the surface is in the xy-plane, then the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\mathrm{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\mathrm{curl}(\vec{F})\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$ is the two dimensional curl: $\vec{F}(\vec{r}(u,v))\cdot\langle 0,0,1\rangle = Q_x - P_y$. If C is the boundary of the surface, then $\int\int_S \vec{F}(\vec{r}(u,v))\cdot\langle 0,0,1\rangle\; dudv = \int_C \vec{F}(\vec{r}(t))\;\vec{r}\,'(t)dt$.

6 Calculate the flux of the curl of $\vec{F}(x,y,z) = \langle -y, x, 0\rangle$ through the surface parameterized by $\vec{r}(u,v) = \langle \cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u+\pi/2)\rangle$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

7 For every surface bounded by a curve C, the flux of $\mathrm{curl}(\vec{F})$ through the surface is the same. Proof. The flux of the curl of a vector field through a surface S depends only on the boundary of S. Compare this with the earlier statement that for every curve between two points A, B the line integral of $\mathrm{grad}(f)$ along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C.

8 The electric field E and the magnetic field B are linked by the **Maxwell equation** $\mathrm{curl}(\vec{E}) = -\frac{1}{c}\dot{B}$. Take a closed wire C which bounds a surface S and consider $\int\int_S B\cdot dS$, the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt\int\int_S B\cdot dS = \int\int_S \dot{B}\cdot dS = \int\int_S -c\;\mathrm{curl}(\vec{E})\cdot\vec{dS} = -c\int_C \vec{E}\;\vec{dr} = U$, where U is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. This happens in a power-generator like an alternator in a car. Stokes theorem explains why we can generate electricity from motion.



Stokes theorem was found by Ampère in 1825. George Gabriel Stokes (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840.

George Gabriel Stokes

André Marie Ampere

Homework

1 Find $\int_C \vec{F}\cdot\vec{dr}$, where $\vec{F}(x,y,z) = \langle x^2y, x^3/3, xy\rangle$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

2 If S is the surface $x^6 + y^6 + z^6 = 1$ and assume $\vec{F}$ is a smooth vector field. Explain why $\iint_S \mathrm{curl}(\vec{F})\cdot\vec{dS} = 0$.

3 Evaluate the flux integral

$$\int\int_S \mathrm{curl}(\vec{F})\cdot\vec{dS}\;,$$

where $\vec{F}(x,y,z) = \langle xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x\rangle$ and where S is the part of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards.

4 Find the line integral $\int_C \vec{F}\;\vec{dr}$, where C is the circle of radius 3 in the xz-plane oriented counter clockwise when looking from the point $(0,1,0)$ onto the plane and where $\vec{F}$ is the vector field

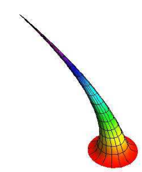
$$\vec{F}(x,y,z) = \langle 2x^2z + x^5, \cos(e^y), -2xz^2 + \sin(\sin(z))\rangle\;.$$

Use a convenient surface S which has C as a boundary.

5 Find the flux integral $\iint_S \mathrm{curl}(\vec{F})\cdot\vec{dS}$, where $\vec{F}(x,y,z) = \langle 2\cos(\pi y)e^{2x} + z^2, x^2\cos(z\pi/2) - \pi\sin(\pi y)e^{2x}, 2xz\rangle$ and S is the **thorn** surface parametrized by

$$\vec{r}(s,t) = \langle (1-s^{1/3})\cos(t) - 4s^2, (1-s^{1/3})\sin(t), 5s\rangle$$

with $0 \le t \le 2\pi, 0 \le s \le 1$ and oriented so that the normal vectors point to the outside of the thorn.



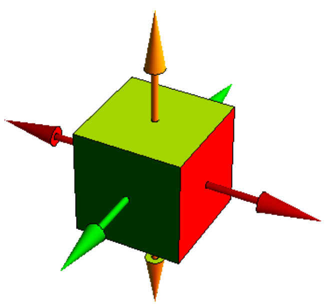
Lecture 24: Divergence theorem

There are three integral theorems in three dimensions. We have seen already the fundamental theorem of line integrals and Stokes theorem. Here is the divergence theorem, which completes the list of integral theorems in three dimensions:

Divergence Theorem. Let E be a solid with boundary surface S oriented so that the normal vector points outside. Let $\vec{F}$ be a vector field. Then

$$\int\int\int_E \operatorname{div}(\vec{F})\, dV = \int\int_S \vec{F} \cdot dS .$$

To prove this, one can look at a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. The flux of $\vec{F} = \langle P, Q, R \rangle$ through the faces perpendicular to the x-axes is $[\vec{F}(x+dx, y, z) \cdot \langle 1, 0, 0 \rangle + \vec{F}(x, y, z) \cdot \langle -1, 0, 0 \rangle] dydz = P(x+dx, y, z) - P(x, y, z) = P_x\, dxdydz$. Similarly, the flux through the y-boundaries is $P_y\ dydxdz$ and the flux through the two z-boundaries is $P_z\ dzdxdy$. The total flux through the faces of the cube is $(P_x + P_y + P_z)\ dxdydz = \operatorname{div}(\vec{F})\ dxdydz$. A general solid can be approximated as a union of small cubes. The sum of the fluxes through all the cubes consists now of the flux through all faces without neighboring faces. and fluxes through adjacent sides cancel. The sum of all the fluxes of the cubes is the flux through the boundary of the union. The sum of all the $\operatorname{div}(\vec{F})\ dxdydz$ is a Riemann sum approximation for the integral $\int\int\int_G \operatorname{div}(\vec{F})\ dxdydz$. In the limit, where dx, dy, dz goes to zero, we obtain the divergence theorem.



The theorem explains what divergence means. If we average the divergence over a small cube is equal the flux of the field through the boundary of the cube. If this is positive, then more field exists the cube than entering the cube. There is field "generated" inside. The divergence measures the expansion of the field.

1 Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and let S be sphere. The divergence of $\vec{F}$ is the constant function $\operatorname{div}(\vec{F}) = 3$ and $\int\int\int_G \operatorname{div}(\vec{F})\ dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v\ dudv = \int\int_S |\vec{r}(u, v)|^2 \sin(v)\ dudv = \int_0^\pi \int_0^{2\pi} \sin(v)\ dudv = 4\pi$ also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

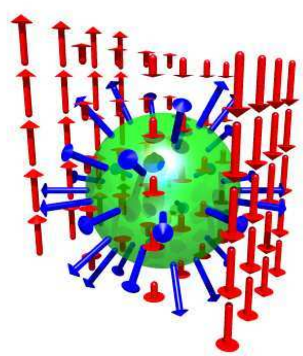
2 What is the flux of the vector field $\vec{F}(x, y, z) = \langle 2x, 3z^2 + y, \sin(x) \rangle$ through the solid $G = [0,3] \times [0,3] \times [0,3] \setminus ([0,3] \times [1,2] \times [1,2] \cup [1,2] \times [0,3] \times [1,2] \cup [0,3] \times [0,3] \times [1,2])$ which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the divergence theorem: $\operatorname{div}(\vec{F}) = 2$ and so $\int\int\int_G \operatorname{div}(\vec{F})\ dV = 2\int\int\int_G\ dV = 2\mathrm{Vol}(G) = 2(27 - 7) = 40$. Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

3 Find the flux of $\operatorname{curl}(F)$ through a torus if $\vec{F} = \langle yz^2, z + \sin(x) + y, \cos(x) \rangle$ and the torus has the parametrization

$$\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi)) \cos(\theta), (2 + \cos(\phi)) \sin(\theta), \sin(\phi) \rangle .$$

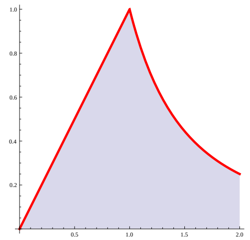
Solution: The answer is 0 because the divergence of $\operatorname{curl}(F)$ is zero. By the divergence theorem, the flux is zero.

4 Similarly as Green's theorem allowed to calculate the area of a region by passing along the boundary, the volume of a region can be computed as a flux integral: Take for example the vector field $\vec{F}(x, y, z) = \langle x, 0, 0 \rangle$ which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: $\int\int_{\delta G} \langle x, 0, 0 \rangle \cdot \vec{dS} = \mathrm{Vol}(G)$.



5 How heavy are we, at distance r from the center of the earth?
Solution: The law of gravity can be formulated as $\operatorname{div}(\vec{F}) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the

gravitational force is normal to the surface: $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/||\vec{x}||$. The flux of $\vec{F}$ through a ball of radius r is $\int\int_{S_r} \vec{F}(x) \cdot \vec{dS} = 4\pi r^2 \vec{F}(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi \int\int\int_{B_r} \rho(x)\ dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2\rho r^3/3 = 4\pi r^2 \vec{F}(r)$ for $r < R$ and $(4\pi)^2\rho R^3/3 = 4\pi r^2 \vec{F}(r)$ for $r \geq R$. Inside the earth, the gravitational force $\vec{F}(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $\vec{F}(r) = M/r^2$ with $M = 4\pi R^3\rho/3$.



We are now at the end of the course. Lets have an overview over the integral theorems and give an other typical example in each case.

The fundamental theorem for line integrals, Green's theorem, Stokes theorem and divergence theorem are all incarnation of **one single theorem** $\int_A dF = \int_{\delta A} F$, where dF is a **exterior derivative** of F and where δA is the **boundary** of A. They all generalize the fundamental theorem of calculus.

Fundamental theorem of line integrals: If C is a curve with boundary $\{A, B\}$ and f is a function, then

$$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A)$$

Remarks.

1) For closed curves, the line integral $\int_C \nabla f \cdot \vec{dr}$ is zero.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then the line integral between two points P and Q does not depend on the path connecting the two points.
3) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus** $\int_a^b f'(x)\ dx = f(b) - f(a)$
4) The theorem justifies the name **conservative** for gradient vector fields.
5) The term "potential" was coined by George Green who lived from 1783-1841.

Example. Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of the vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = \langle\cos(5t), \sin(2t), t^2\rangle$ from $t = 0$ to $t = 2\pi$.

Solution. $\vec{r}(0) = \langle 1, 0, 0\rangle$ and $\vec{r}(2\pi) = \langle 1, 0, 4\pi^2\rangle$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f\ \vec{dr} = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

Green's theorem. If R is a region with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_R \text{curl}(\vec{F})\ dxdy = \int_C \vec{F} \cdot \vec{dr}\ .$$

Remarks.

1) Greens theorem allows to switch from double integrals to one dimensional integrals.
2) The curve is oriented in such a way that the region is to the left.
3) The boundary of the curve can consist of piecewise smooth pieces.
4) If $C : t \mapsto \vec{r}(t) = \langle x(t), y(t)\rangle$, the line integral is $\int_a^b \langle P(x(t), y(t)), Q(x(t), y(t))\rangle \cdot \langle x'(t), y'(t)\rangle\ dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If $\text{curl}(\vec{F}) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
7) Taking $\vec{F}(x, y) = \langle -y, 0\rangle$ or $\vec{F}(x, y) = \langle 0, x\rangle$ gives **area formulas**.

Example. Find the line integral of the vector field $\vec{F}(x, y) = \langle x^4 + \sin(x) + y, x + y^3\rangle$ along the path $\vec{r}(t) = \langle\cos(t), 5\sin(t) + \log(1 + \sin(t))\rangle$, where t runs from $t = 0$ to $t = \pi$.

Solution. $\text{curl}(\vec{F}) = 0$ implies that the line integral depends only on the end points $(0, 1), (0, -1)$ of the path. Take the simpler path $\vec{r}(t) = \langle -t, 0\rangle, -1 \leq t \leq 1$, which has velocity $\vec{r}\,'(t) = \langle -1, 0\rangle$. The line integral is $\int_{-1}^1 \langle t^4 - \sin(t), -t\rangle \cdot \langle -1, 0\rangle\ dt = -t^5/5|_{-1}^1 = -2/5$.

Remark We could also find a potential $f(x, y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\text{grad}(f) = F$. Again, we get $f(0, -1) - f(0, 1) = -1/5 - 1/5 = -2/5$.

Stokes theorem. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \text{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot \vec{dr}\ .$$

Remarks.
1) Stokes theorem allows to derive Greens theorem: if $\vec{F}$ is z-independent and the surface S is contained in the xy-plane, one obtains the result of Green.
2) The orientation of C is such that if you walk along C and have your head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903).
4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

Example. Compute the line integral of $\vec{F}(x, y, z) = \langle x^3 + xy, y, z\rangle$ along the polygonal path C connecting the points $(0, 0, 0), (2, 0, 0), (2, 1, 0), (0, 1, 0)$.
Solution. The path C bounds a surface $S : \vec{r}(u, v) = \langle u, v, 0\rangle$ parameterized by $R = [0, 2] \times [0, 1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(\vec{F})(x, y, z) = \langle 0, 0, -x\rangle$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle = \langle 0, 0, 1\rangle$ so that $\int\int_S \text{curl}(\vec{F})\ \vec{dS} = \int_0^2\int_0^1 \langle 0, 0, -u\rangle \cdot \langle 0, 0, 1\rangle\ dudv = \int_0^2\int_0^1 -u\ dudv = -2$.

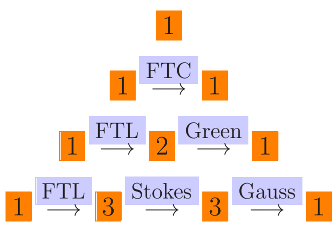
Divergence theorem: If S is the boundary of a region E in space and $\vec{F}$ is a vector field, then

$$\int\int\int_B \mathrm{div}(\vec{F})\, dV = \int\int_S \vec{F} \cdot d\vec{S} \; .$$

Remarks.
1) The divergence theorem is also called **Gauss theorem**.
2) It can be helpful to determine the flux of vector fields through surfaces.
3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

Example. Compute the flux of the vector field $\vec{F}(x,y,z) = \langle -x, y, z^2 \rangle$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
Solution. By Gauss theorem, the flux is equal to the triple integral of $\mathrm{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \; dxdydz = (3-0)(2-(-1))(4-1) = 27$.

How do these theorems fit together? In n-dimensions, there are n theorems. We have here seen the situation in dimension n=2 and n=3, but one could continue. The fundamental theorem of line integrals generalizes directly to higher dimensions. Also the divergence theorem generalizes directly since an n-dimensional integral in n dimensions. The generalization of curl and flux is more subtle, since in 4 dimensions already, the curl of a vector field is a 6 dimensional object. It is a $n(n-1)/2$ dimensional object in general.



In one dimensions, there is one derivative $f(x) \to f'(x)$ from scalar to scalar functions. It corresponds to the entry $1-1$ in the Pascal triangle. The next entry $1-2-1$ corresponds to differentiation in two dimensions, where we have the gradient $f \to \nabla f$ mapping a scalar function to a vector field with 2 components as well as the curl, $F \to \mathrm{curl}(F)$ which corresponds to the transition $2-1$. The situation in three dimensions is captured by the entry $1-3-3-1$ in the Pascal triangle. The first derivative $1-3$ is the gradient. The second derivative $3-3$ is the curl and the third derivative $3-1$ is the divergence. In $n=4$ dimensions, we would have to look at $1-4-6-4-1$. The first derivative $1-4$ is still the gradient. Then we have a first curl, which maps a vector field with 4 components into an object with 6 components. Then there is a second curl, which maps an object with 6 components back to a vector field, we would have to look at $1-4-6-4-1$. When setting up calculus in dimension n, one talks about **differential forms** instead of scalar fields or vector fields. Functions are 0 forms or n-forms. Vector fields can be described by 1 or $n-1$ forms. The general formalism defines a derivative d called **exterior derivative** on differential forms as well as integration of such k forms on k dimensional objects. There is a **boundary operation** δ which maps a k-dimensional object into a $k-1$ dimensional object. This boundary operation is dual to differentiation. They both satisfy the same relation $dd(F) = 0$ and $\delta\delta G = 0$. Differentiation and integration are linked by the general Stokes theorem:

$$\int_{\delta G} F = \int_G dF$$

which becomes a single theorem called **fundamental theorem of multivariable calculus**. The theorem becomes much simpler in quantum calculus, where geometric objects and differential forms are on the same footing. It tuns out that the theorem becomes then

$$< \delta G, F > = < G, dF >$$

which you might see in linear algebra in the form $< A^T v, w > = < v, Aw >$, where A is a matrix and $< v, w >$ is the dot product. If we deal with "smooth" functions and fields that we have to pay a prize and consider in turn "singular" objects like points or curves and surfaces. These are idealized objects which have zero diameter, radius or thickness. Nature likes simplicity and elegance [1]: and has chosen quantum mathematics to be more fundamental but it manifests only in the very small. While it is well understood mathematically, it will take a while until this formalism will enter calculus courses.

Homework

1 Compute using the divergence theorem the flux of the vector field $\vec{F}(x,y,z) = \langle 3y, xy, 2yz \rangle$ through the unit cube $[0,1] \times [0,1] \times [0,1]$.

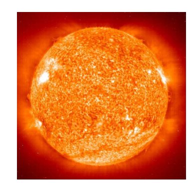
2 Find the flux of the vector field $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$ through the solid cylinder $x^2+y^2 \leq 1$, $0 \leq z \leq 1$.

3 Use the divergence theorem to calculate the flux of $\vec{F}(x,y,z) = \langle x^3, y^3, z^3 \rangle$ through the sphere $S : x^2+y^2+z^2 = 1$ where the sphere is oriented so that the normal vector points outwards.

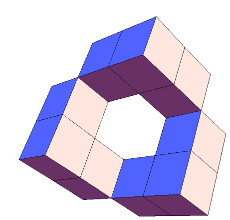
4 Assume the vector field

$$\vec{F}(x,y,z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\iint_S \vec{F} \cdot d\vec{S}$.



5 Find $\int\int_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = \langle x, y, z \rangle$ and S is the boundary of the solid built with 9 unit cubes shown in the picture.

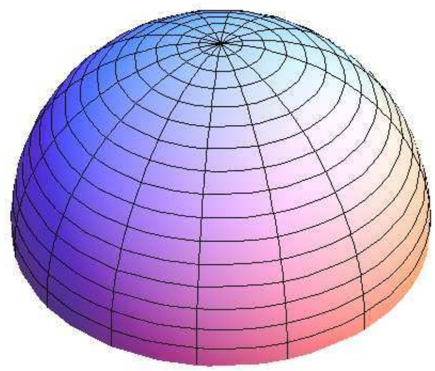


[1] Leibniz: 1646-1716

Divergence theorem

We consider the following objects:

- Assume $\vec{F} = \langle M, N, R \rangle$ is a vector field in space.
- Let E be the solid upper half ball $z \geq 0, x^2 + y^2 + z^2 \leq 1$.
- Let S be the surface which is the upper half sphere $z > 0, x^2 + y^2 + z^2 = 1$ oriented so that the normal vector points upwards.
- Let D be the surface formed by the disc $z = 0, x^2 + y^2 \leq 1$ oriented so that the normal vector points upwards.
- Let C be the curve $\vec{r}(t) = \rangle \cos(t), \sin(t), 0 \rangle$ which is a circle in the xy-plane.



Define the following integrals:

I)	$\int\int\int_E \mathrm{div}(\vec{F})\ dV$
II)	$\int\int_S \vec{F} \cdot dS$
III)	$\int_C \vec{F} \cdot ds$
IV)	$\int\int_D \vec{F} \cdot dS$
V)	$\int\int_S \mathrm{curl}(\vec{F}) \cdot dS$
VI)	$\int\int_D \mathrm{curl}(\vec{F}) \cdot dS$

1 Complete the following identities. You just have to fill in the signs + or − and give the name of the integral theorem, which justifies the identity:

a) I = $\boxed{}$ II + $\boxed{}$ IV by the $\boxed{}$ theorem

b) III = $\boxed{}$ V by the $\boxed{}$ theorem

c) III = $\boxed{}$ VI by the $\boxed{}$ theorem