Maths 21a First Midterm Review

Plan:

Lines and Planes

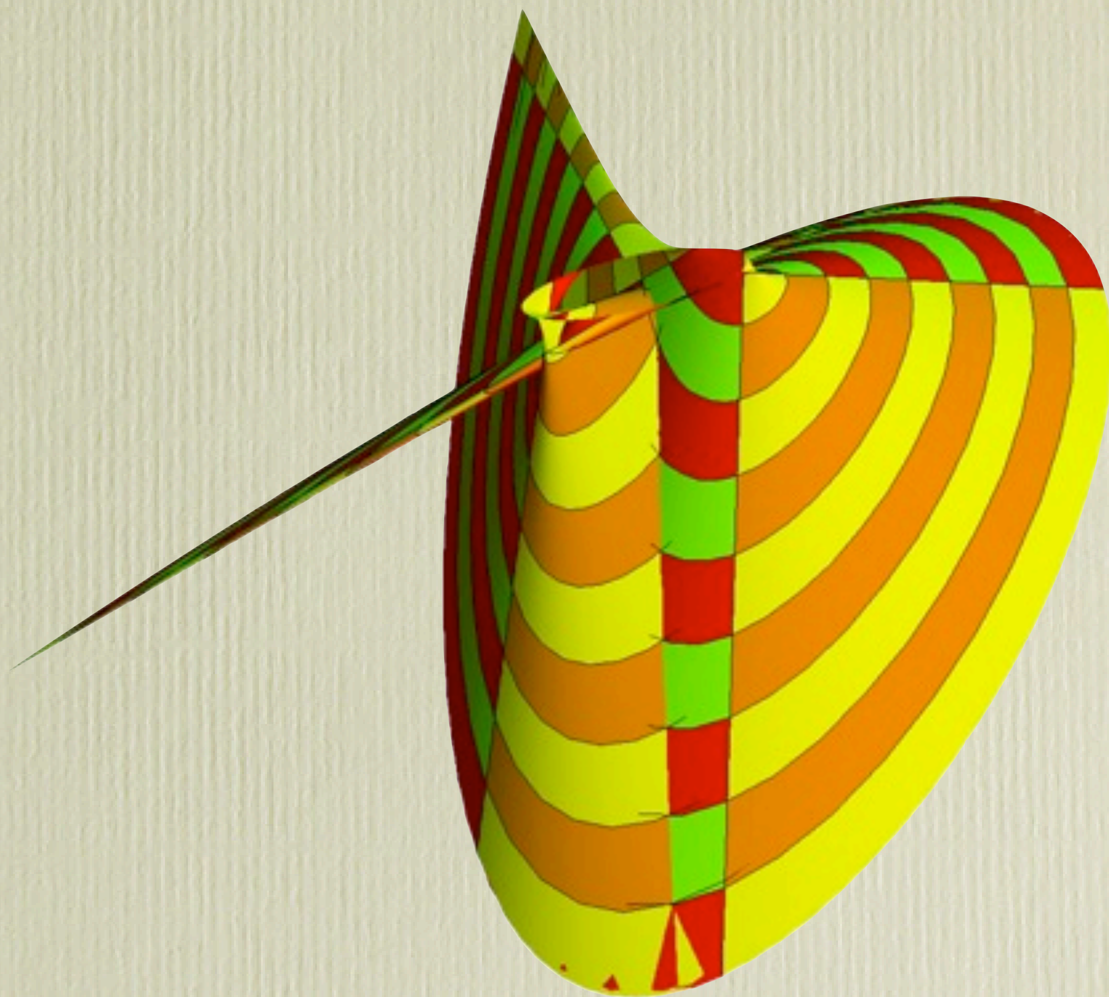
Distance formulas

Parametrized curves

Surfaces Quadrics

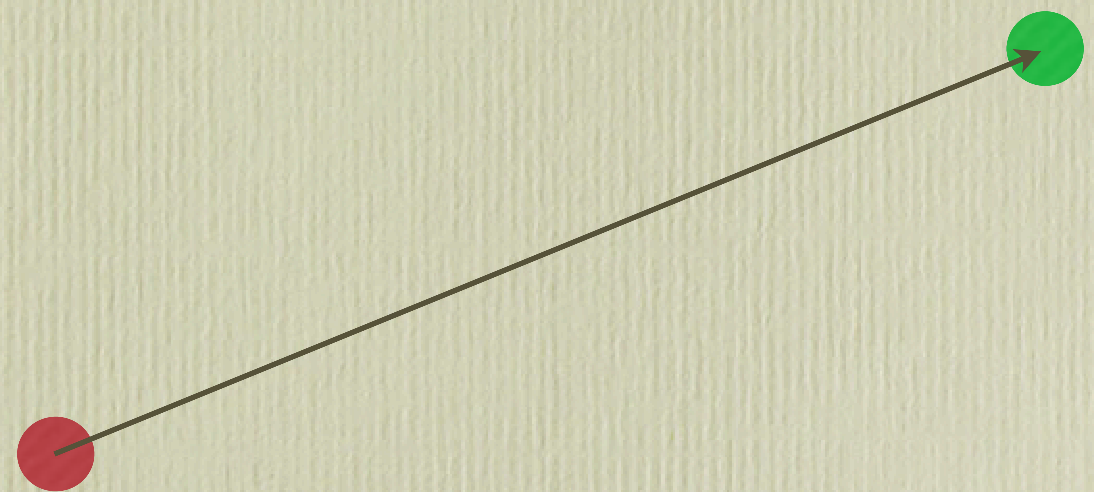
Other coordinates

Parametric Surfaces



Oliver Knill, July 12, 2011

Points and Vectors



Q = (2,5,7)

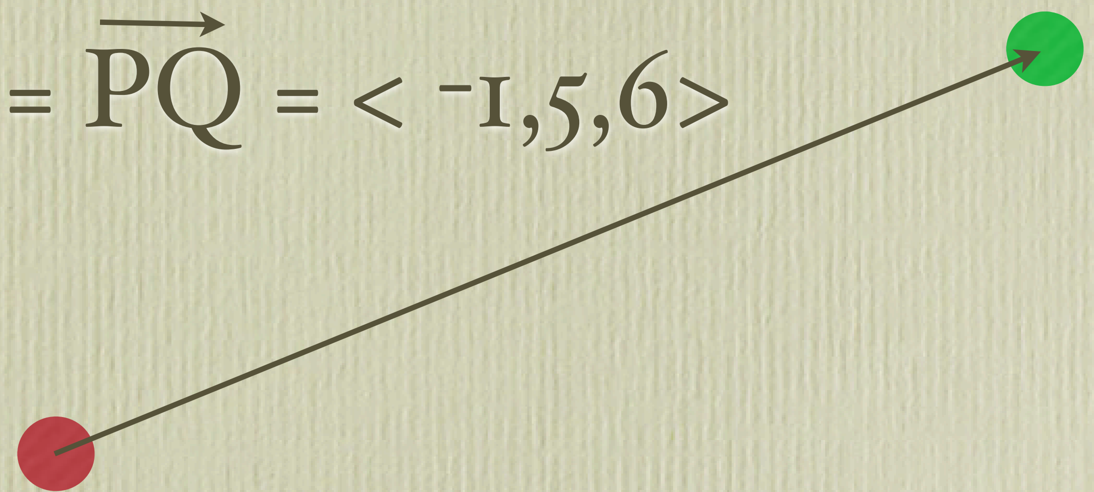
P = (3,0,1)

The components of $\vec{v}$
are the differences between
the coordinates of Q and P.

Points and Vectors

$\vec{v} = \overrightarrow{PQ} = \langle {}^{-}1,5,6 \rangle$

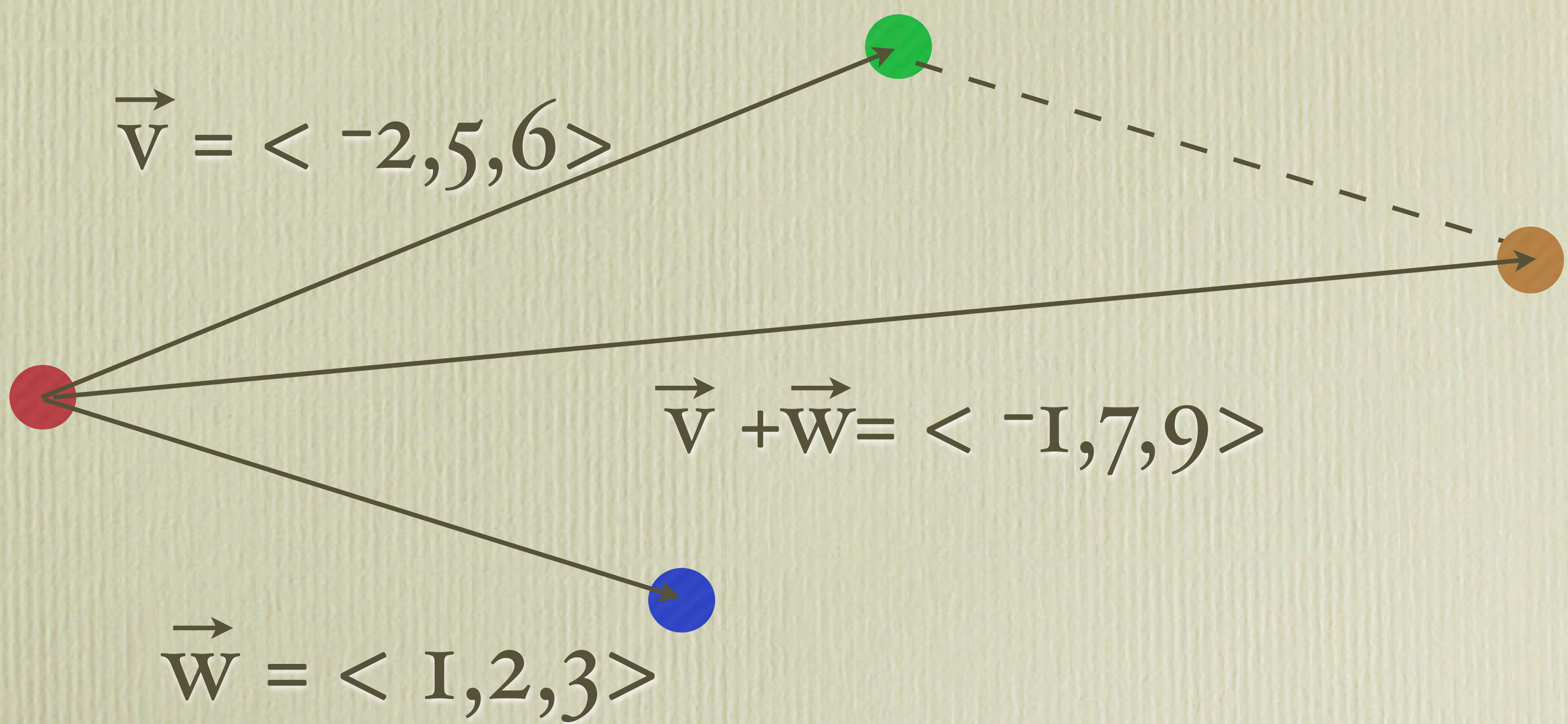
$Q = (2,5,7)$



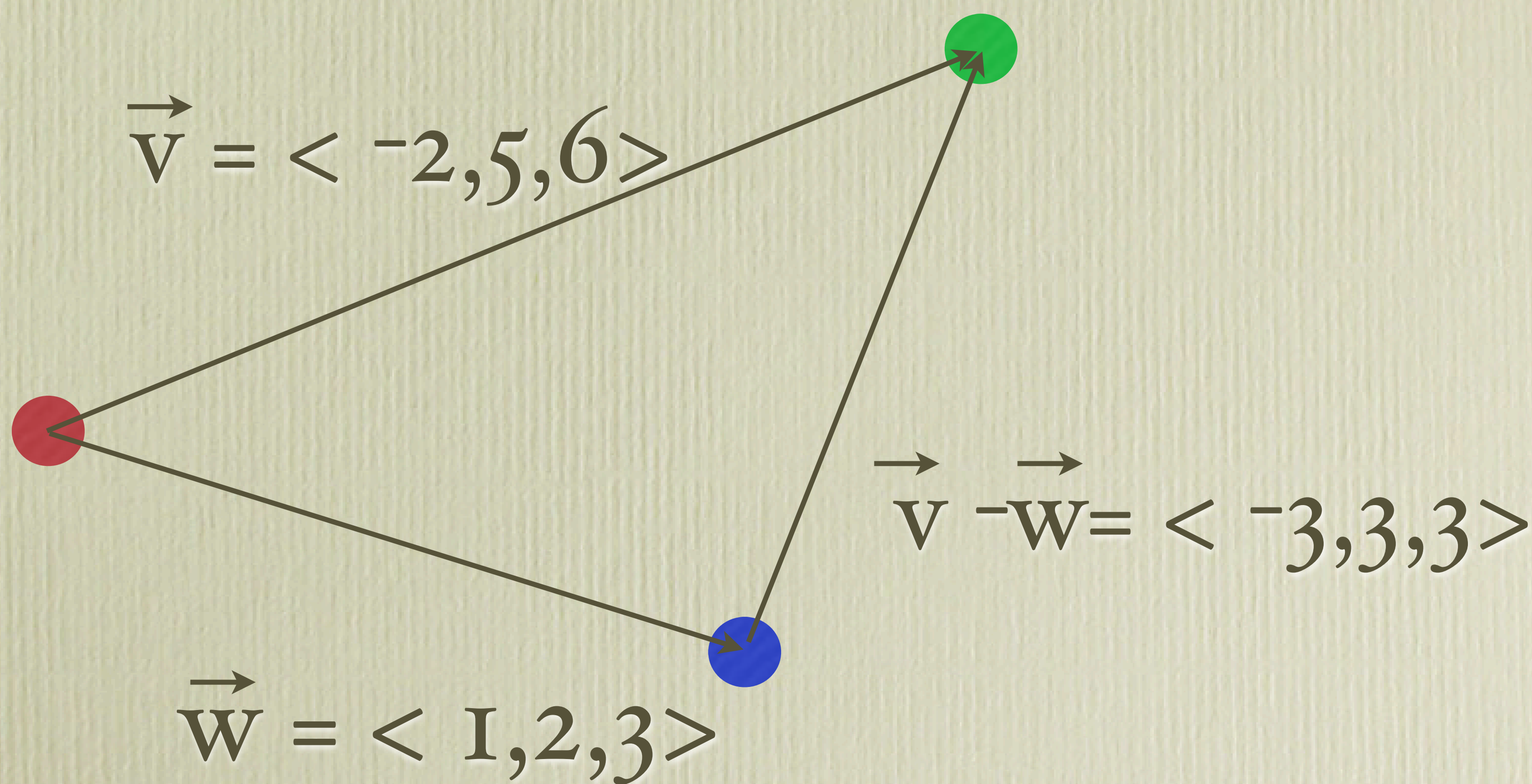
$P = (3,0,1)$

The components of $\vec{v}$ are the differences between the coordinates of Q and P.

Addition

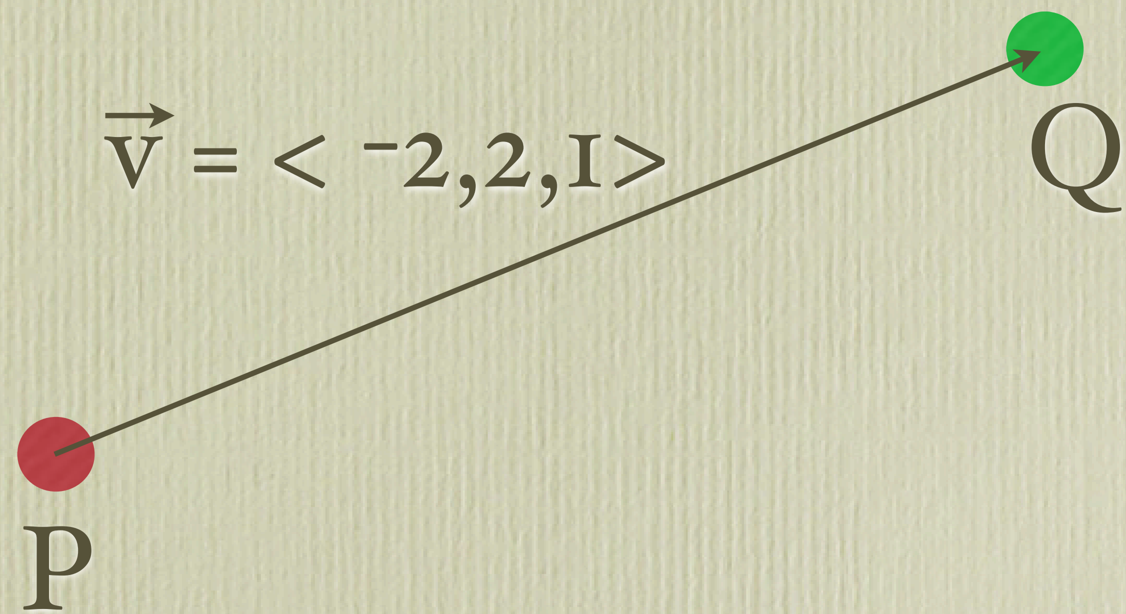

$\vec{v} = \langle -2,5,6 \rangle$
$\vec{v} + \vec{w} = \langle -1,7,9 \rangle$
$\vec{w} = \langle 1,2,3 \rangle$

Subtraction



Distances

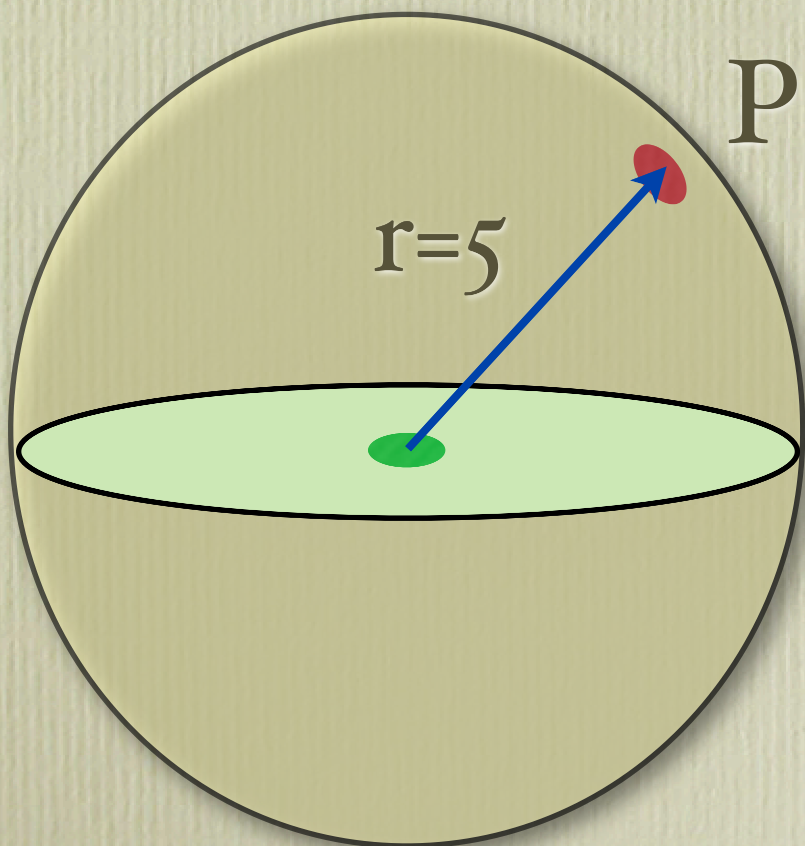
$\vec{v} = < -2, 2, 1 >$



$d(P,Q) = |\vec{v}|$

Spheres

$$(x-3)^2 + (y+4)^2 + (z-2)^2 = 25$$

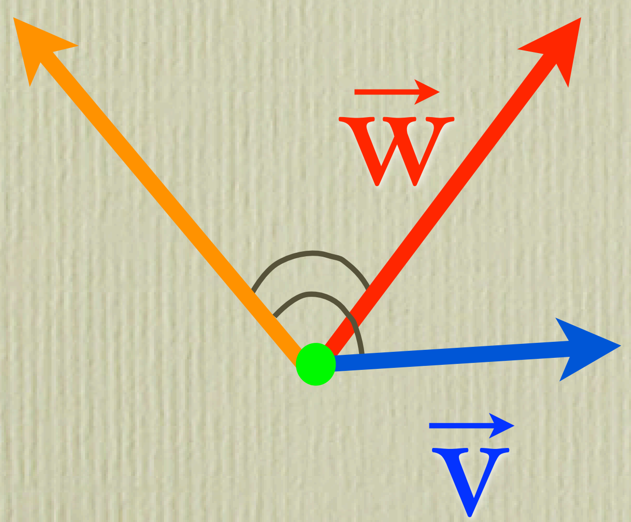


Q=(3,-4,2)

center

Dot and Cross product

$\vec{v} = \langle 3,4,1 \rangle$ $\vec{w} = \langle 2,-1,1 \rangle$



$\vec{v} \cdot \vec{w} = 6-4+1 = 3$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

$= \langle 9,-4,-11 \rangle$

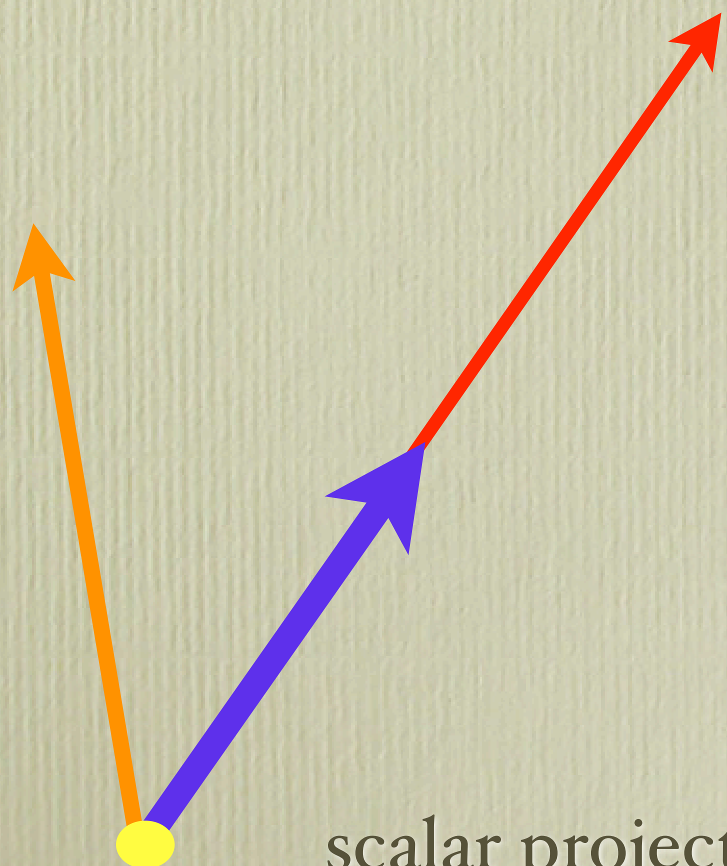
Two important formulas

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\cos(\alpha)$$

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}|\sin(\alpha)$$

Projection

$\vec{v} = \langle 2, {}^{-}3, 4 \rangle$ $\vec{w} = \langle 4, 0, 1 \rangle$



Project $\vec{v}$ onto $\vec{w}$:

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} \frac{\vec{w}}{|\vec{w}|}$$

$$= \frac{12}{17} \langle 4, 0, 1 \rangle$$

scalar projection = component

Projection

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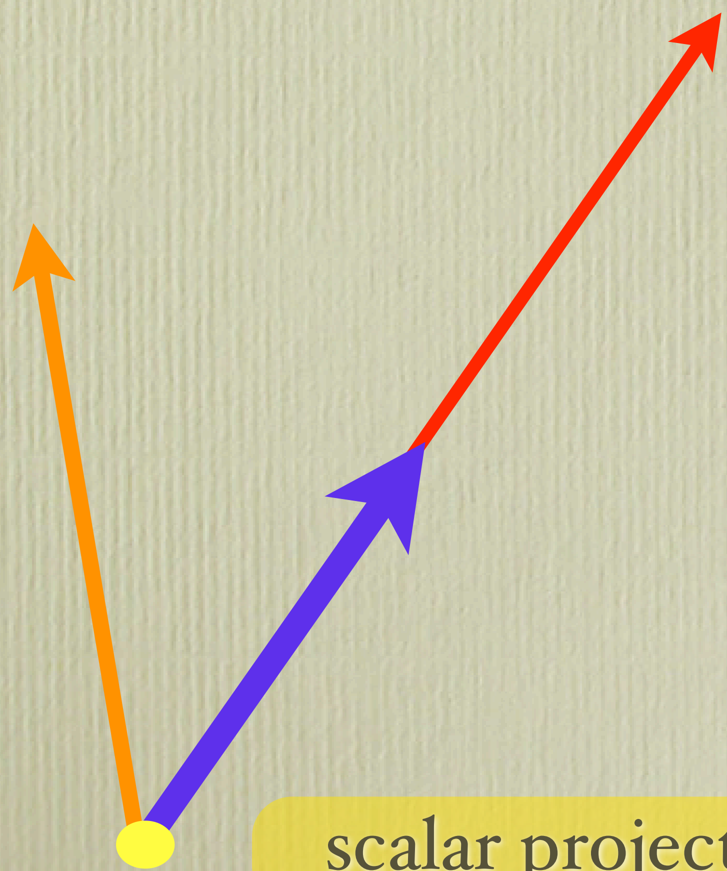

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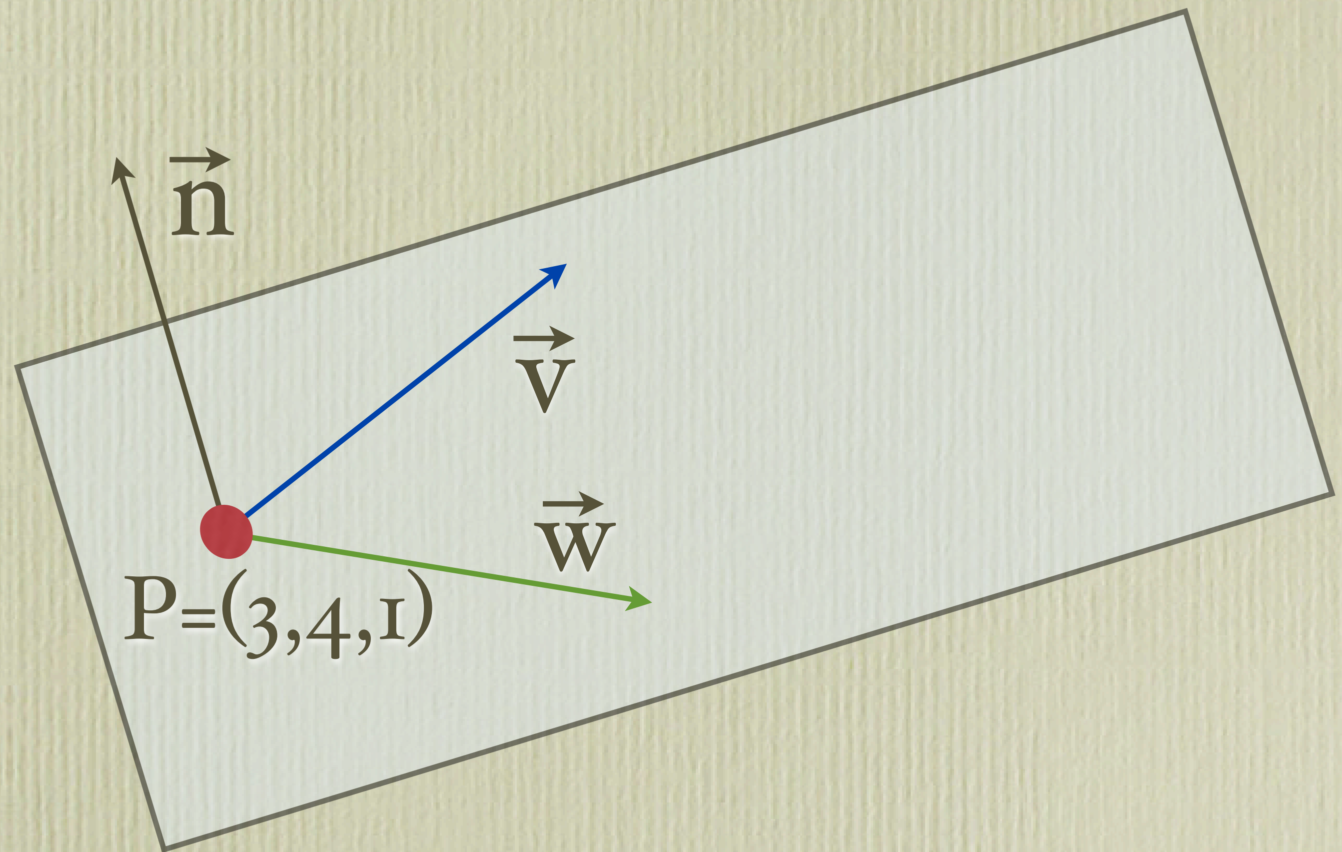
Lines and Planes



$$\overrightarrow{OP} = <3,4,1>$$

$$\vec{v} = <1,1,-3>$$

$$\vec{w} = <1,-2,1>$$

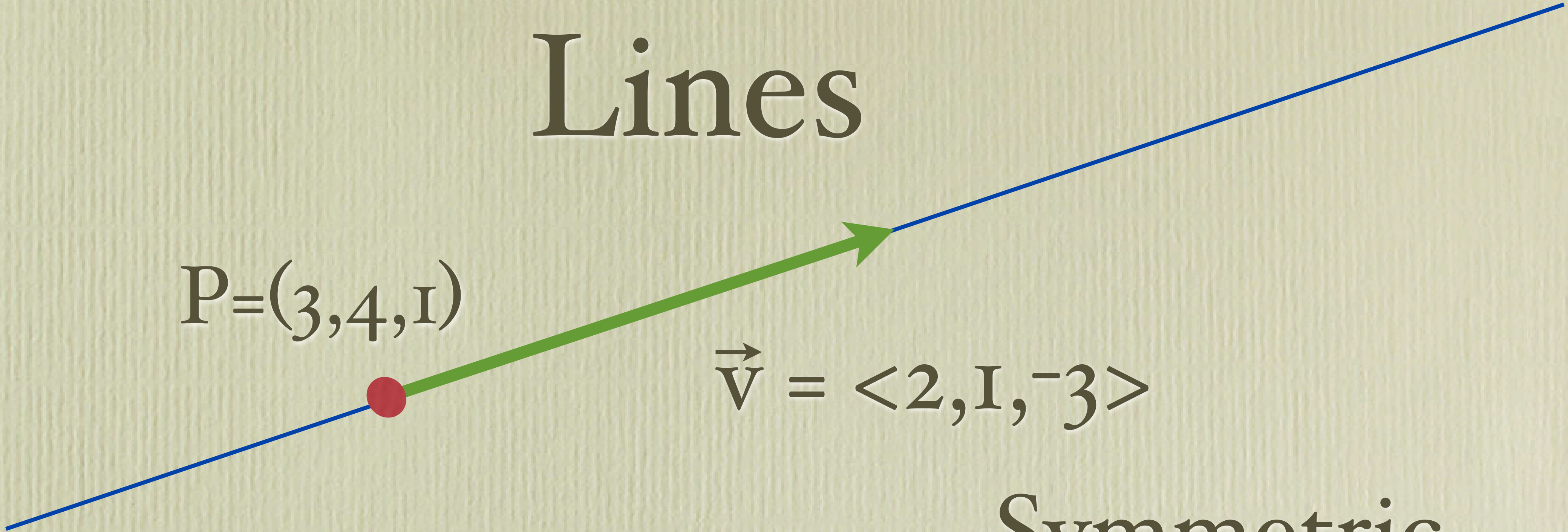
Parametrization

$$\vec{n} = <7,-4,-3>$$

$$\vec{r}(t,s) = < 3+t+s, 4+t-2s, 1-3t+s>$$

$$7x-4y-3z=2$$

Lines



Parametrization

$$\vec{r}(t) = \langle 3+2t, 4+t, 1-3t \rangle$$
$$= \langle x, y, z \rangle$$

Symmetric equations

$$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z-1}{-3}$$

(solve for t)

Problem

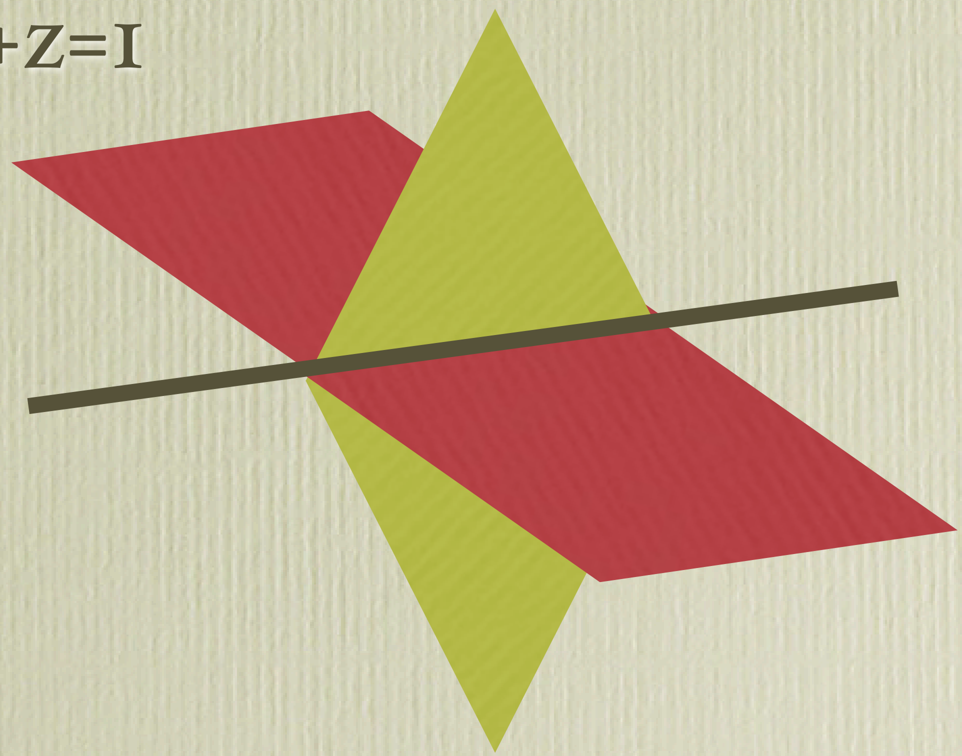
Find the equation of the plane passing through the points

$A=(0,1,1), B=(2,2,2), C=(5,5,4)$

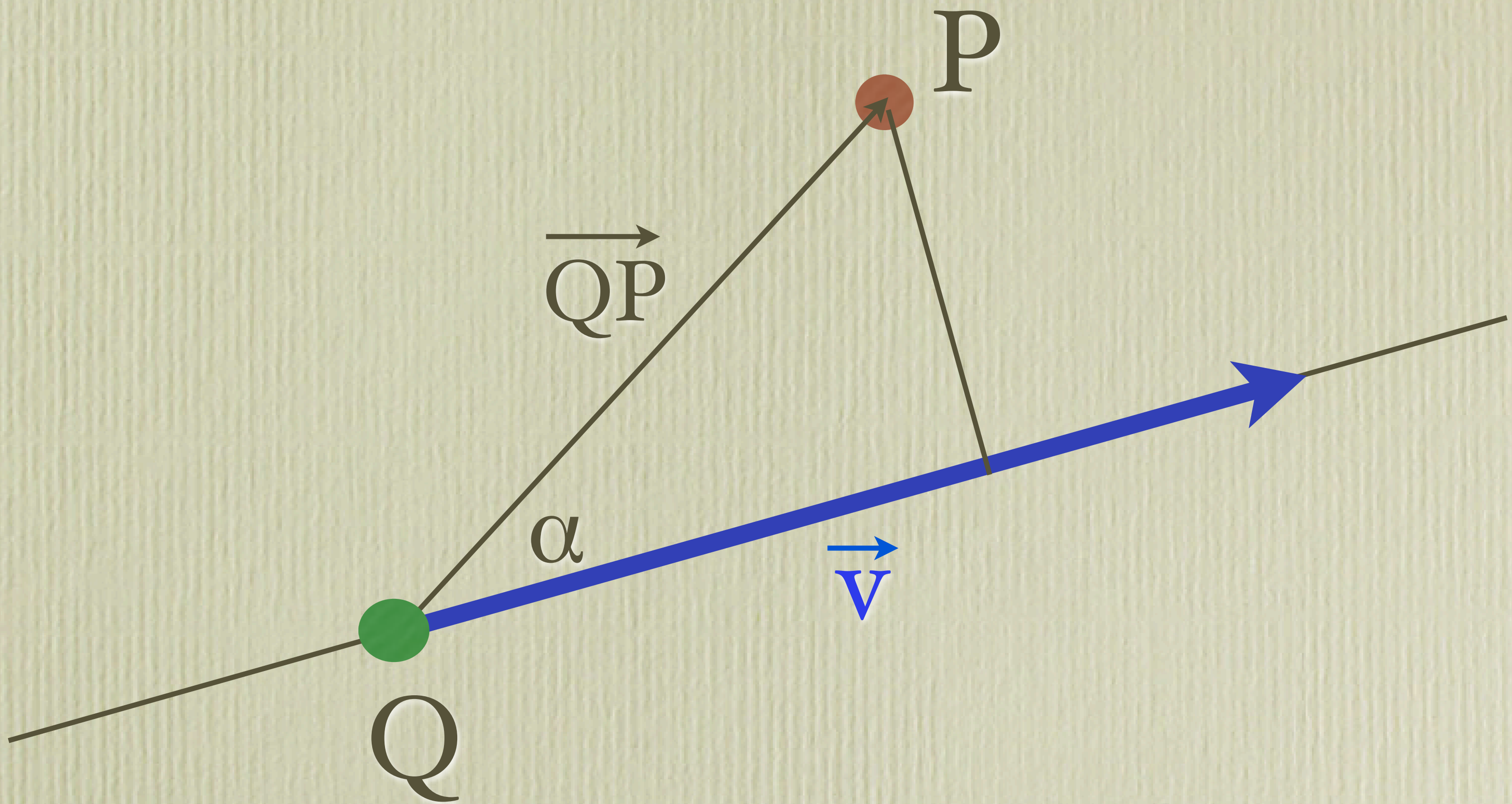
the symmetric equation of the normal line through A and the area of the triangle ABC.

Problem

Find the line of intersection of the plane computed before and the line plane x+y+z=1

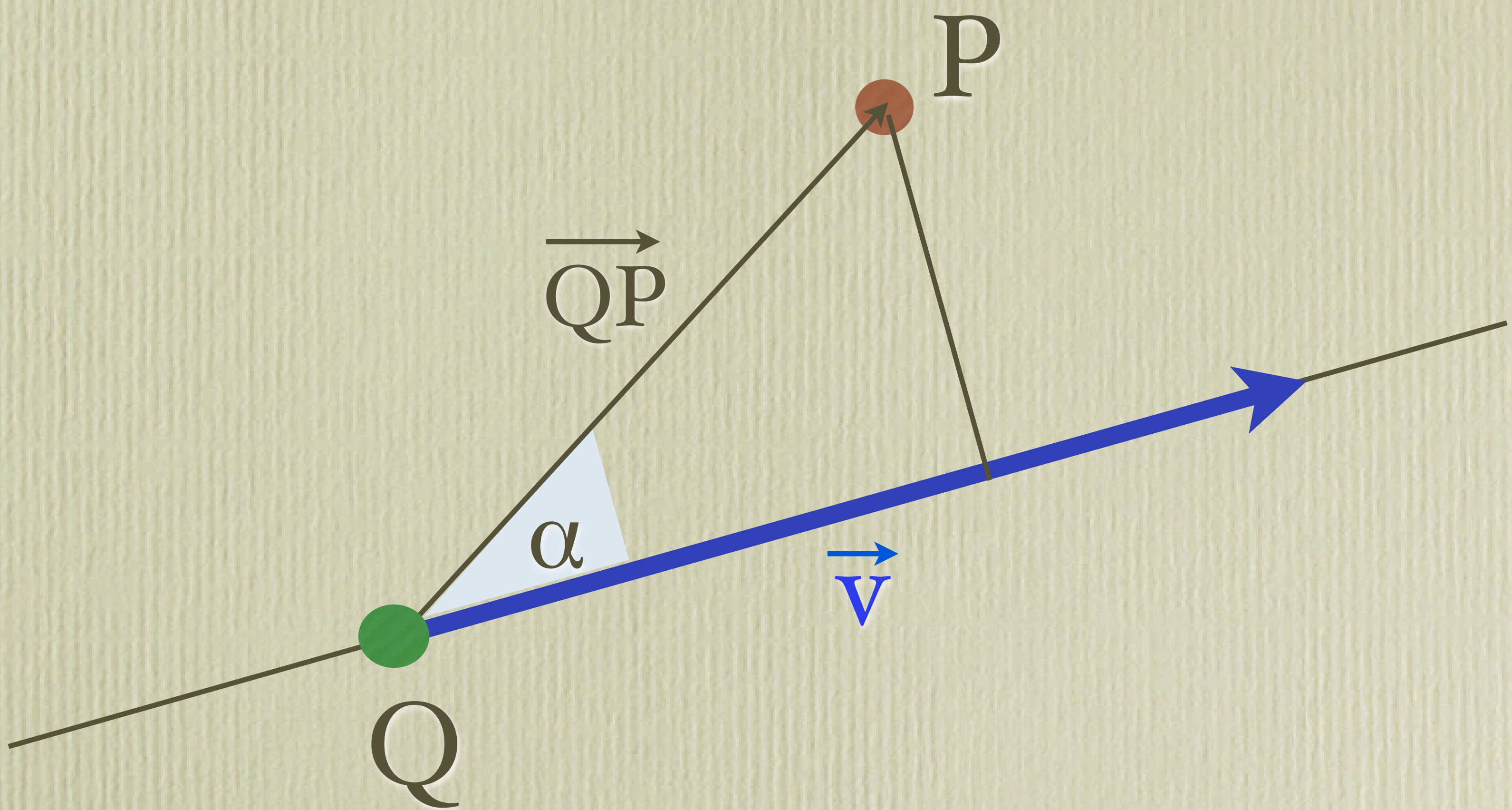


Distance Point/Line



$$\frac{|\overrightarrow{QP}| \sin(\alpha)}{}$$

Distance Point/Line



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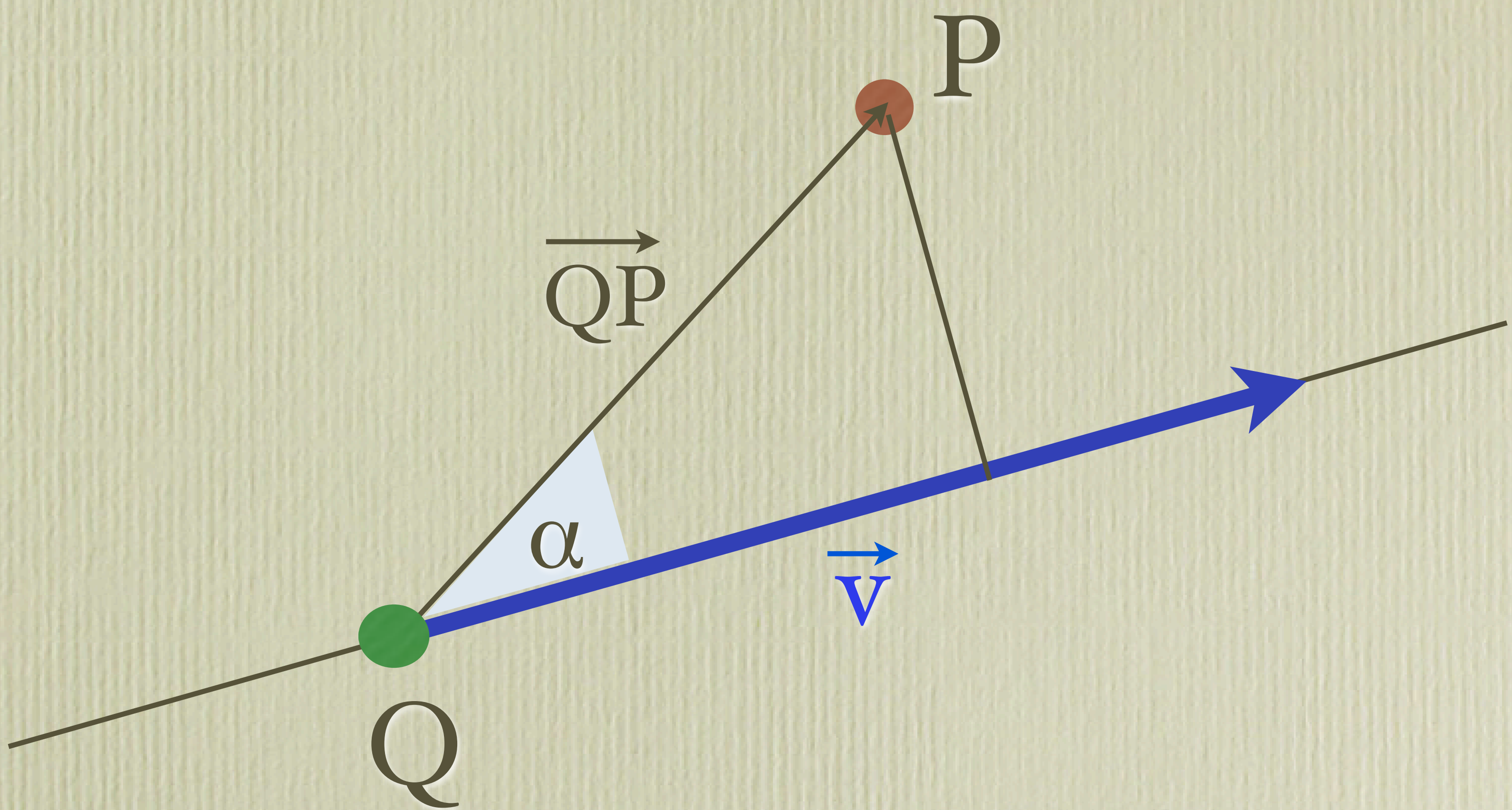
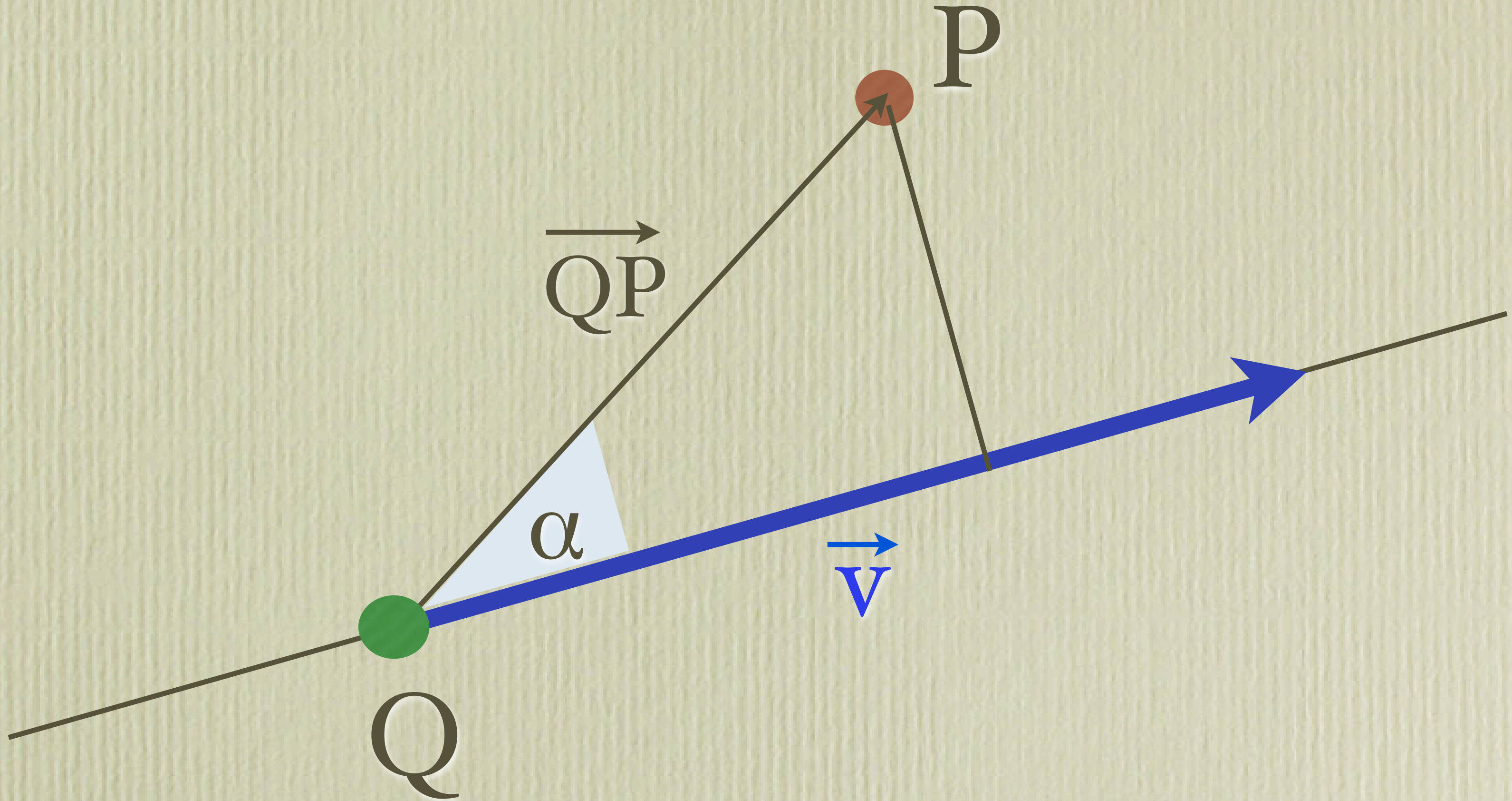


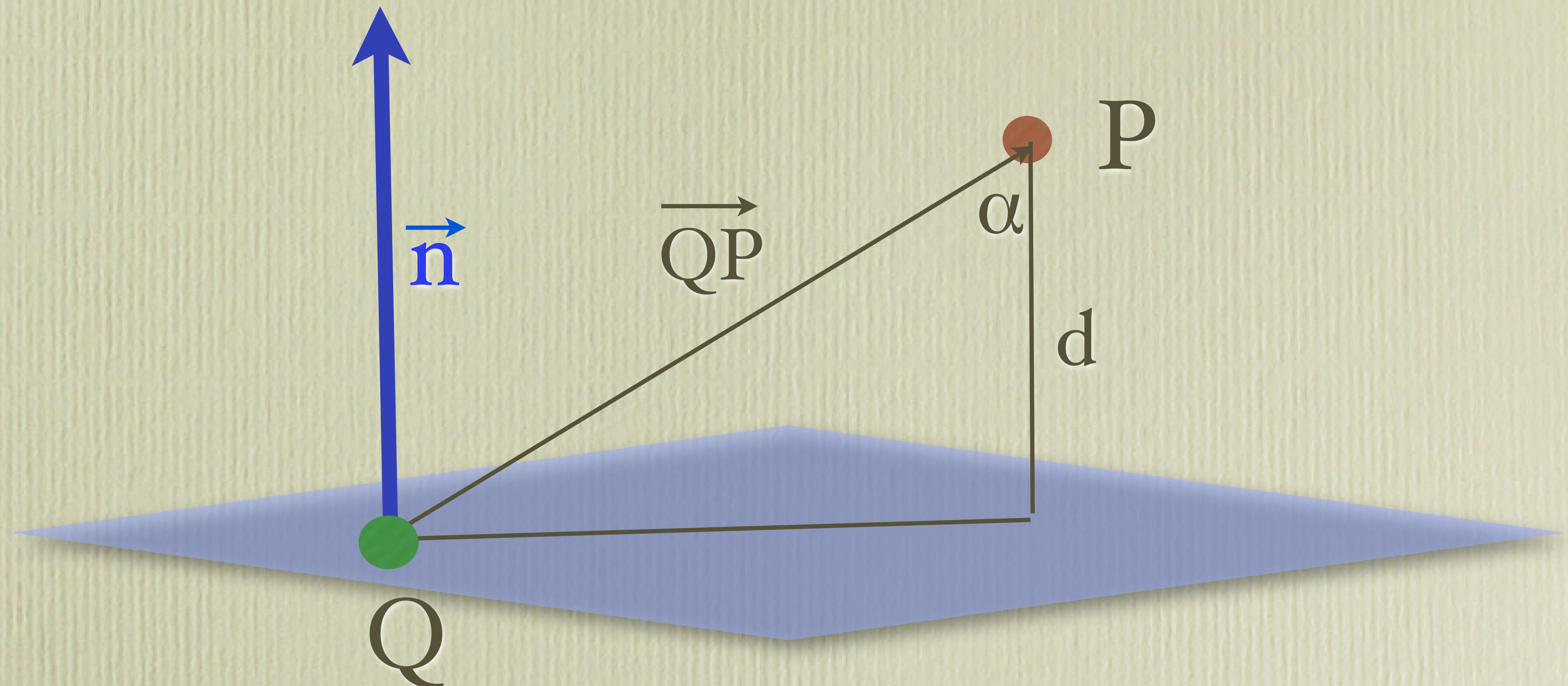
$$\frac{|\overrightarrow{QP}| \sin(\alpha)\ |\vec{v}|}{|\vec{v}|}$$

Distance Point/Line



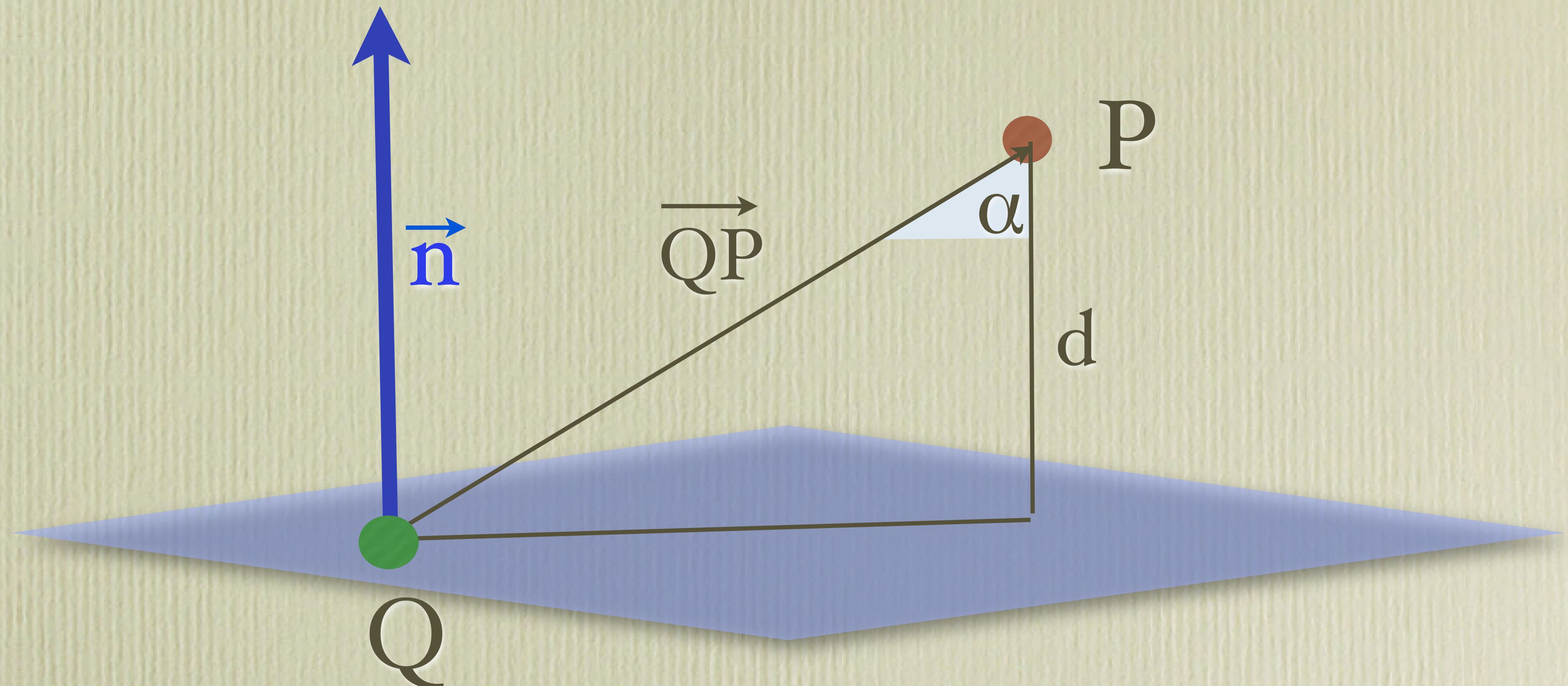
$$\frac{|\overrightarrow{QP}|\,\sin(\alpha)\;|\vec{v}|}{|\vec{v}|} = \frac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|}$$

Distance Point-Plane



$$d = \frac{|\overrightarrow{PQ}| \cos(\alpha)\ |\vec{n}|}{|\vec{n}|} = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

Distance Point-Plane



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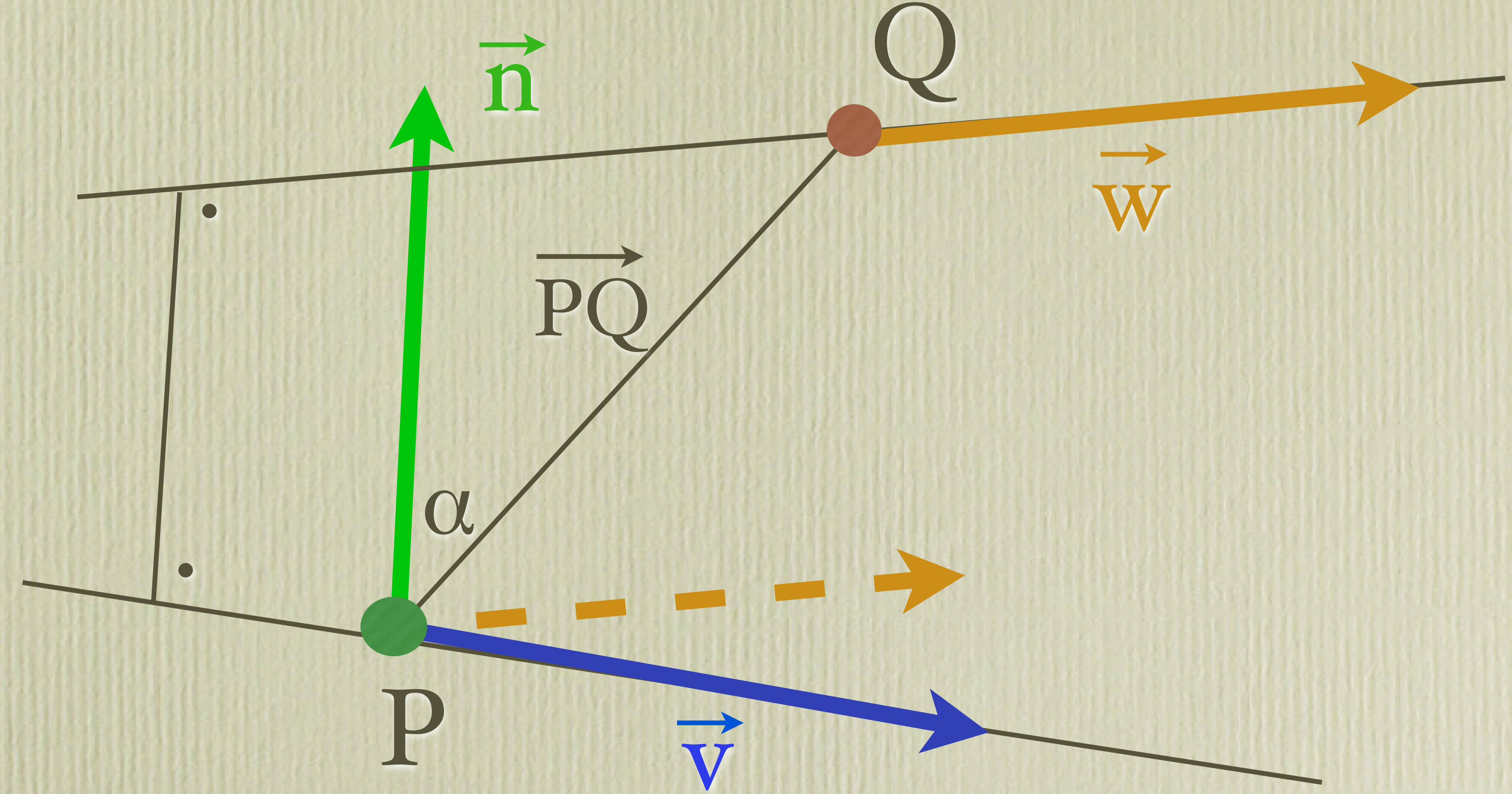
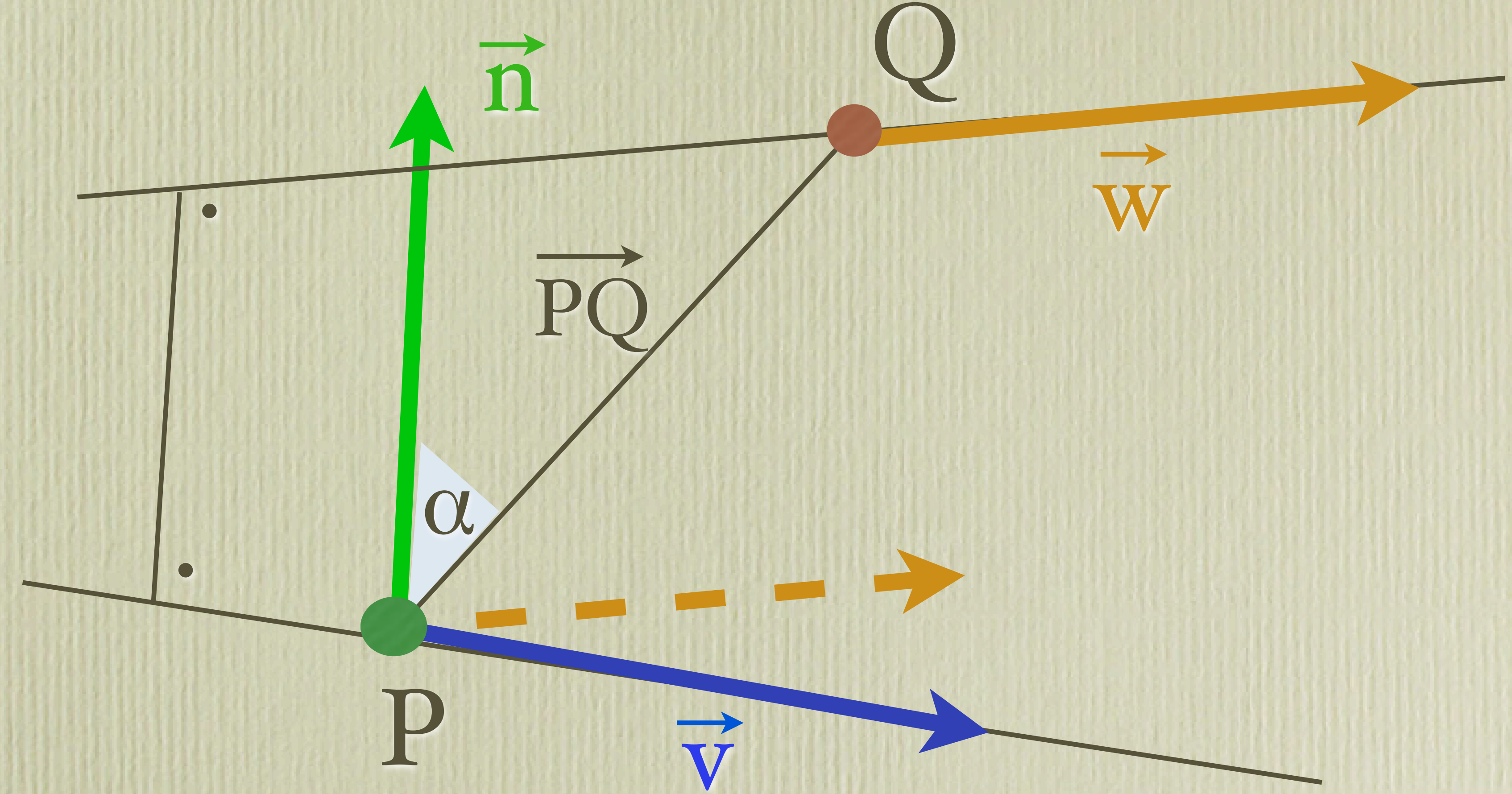
Distance Line/Line



$$\frac{|\vec{PQ}| \cos(\alpha)\ |\vec{n}|}{|\vec{n}|} = \frac{|\vec{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

Distance Line/Line



$$\frac{|\vec{PQ}|\cos(\alpha)\,|\vec{n}|}{|\vec{n}|} = \frac{|\,\vec{QP}\cdot\vec{v}\times\vec{w}\,|}{|\vec{v}\times\vec{w}|}$$

Problem

Starship "enterprise" officer Spock, beams from (1,1,1) to (3,4,5). The clingons can modify everything in distance 1 from the x axes. Will Spock be safe?

(1,1,1)

Area and Volume

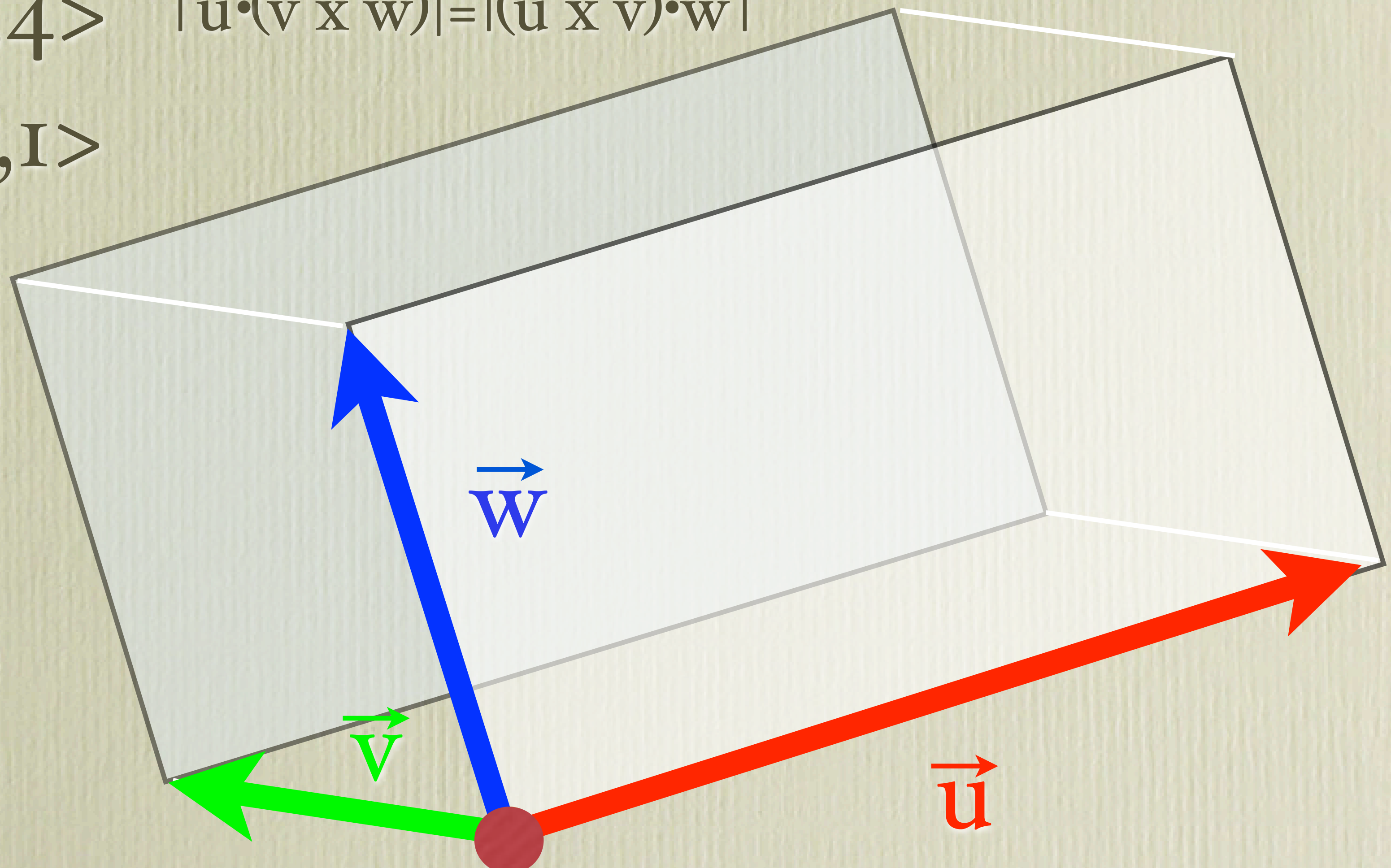
$\vec{u} = \langle 1,2,3 \rangle$

$\vec{v} = \langle 3,1,4 \rangle$

$\vec{w} = \langle 1,1,1 \rangle$

The volume of the parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ is

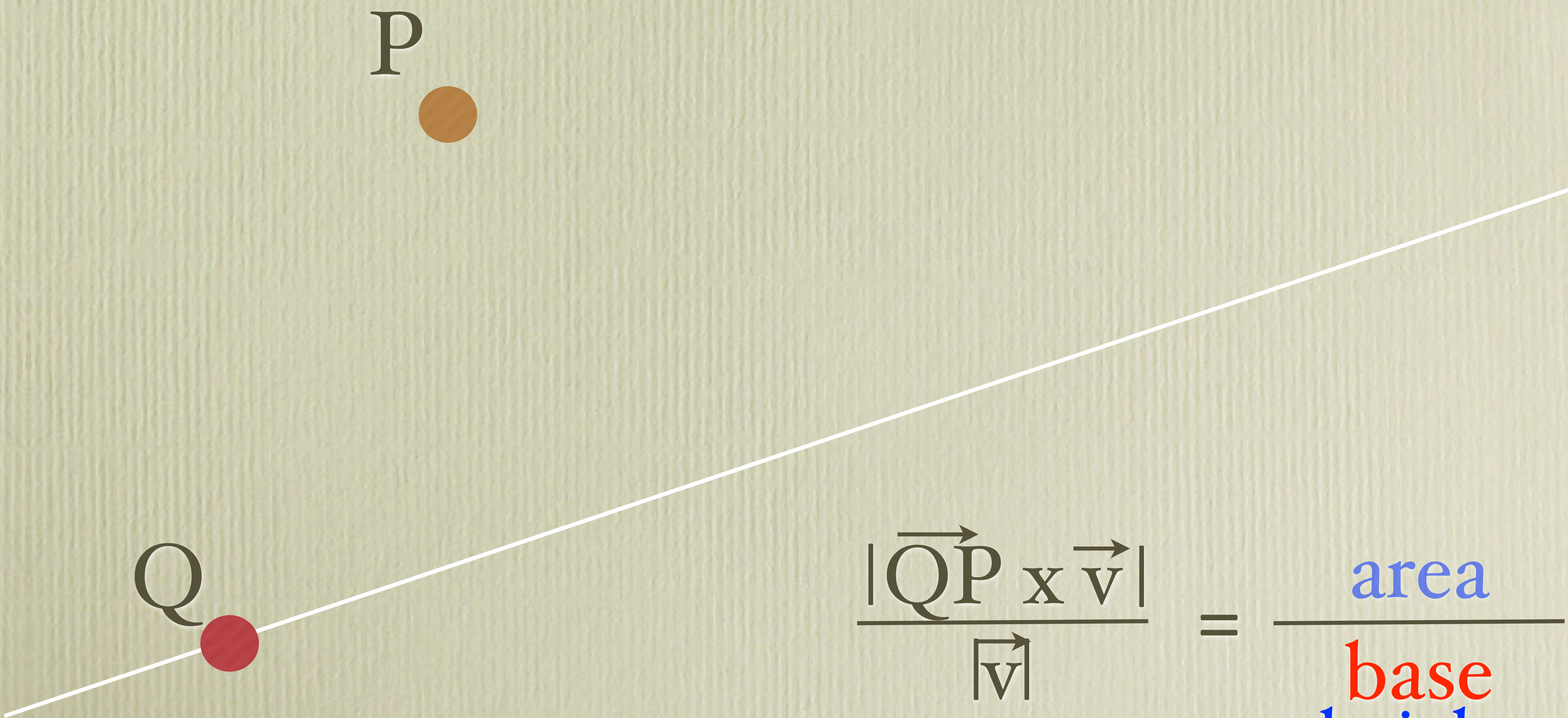
$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$$



Distance Point-Line again

$$\frac{|\overrightarrow{QP} \times \vec{v}|}{|\vec{v}|} = \frac{\text{area}}{\text{base}} = \text{height}$$

Distance Point-Line again



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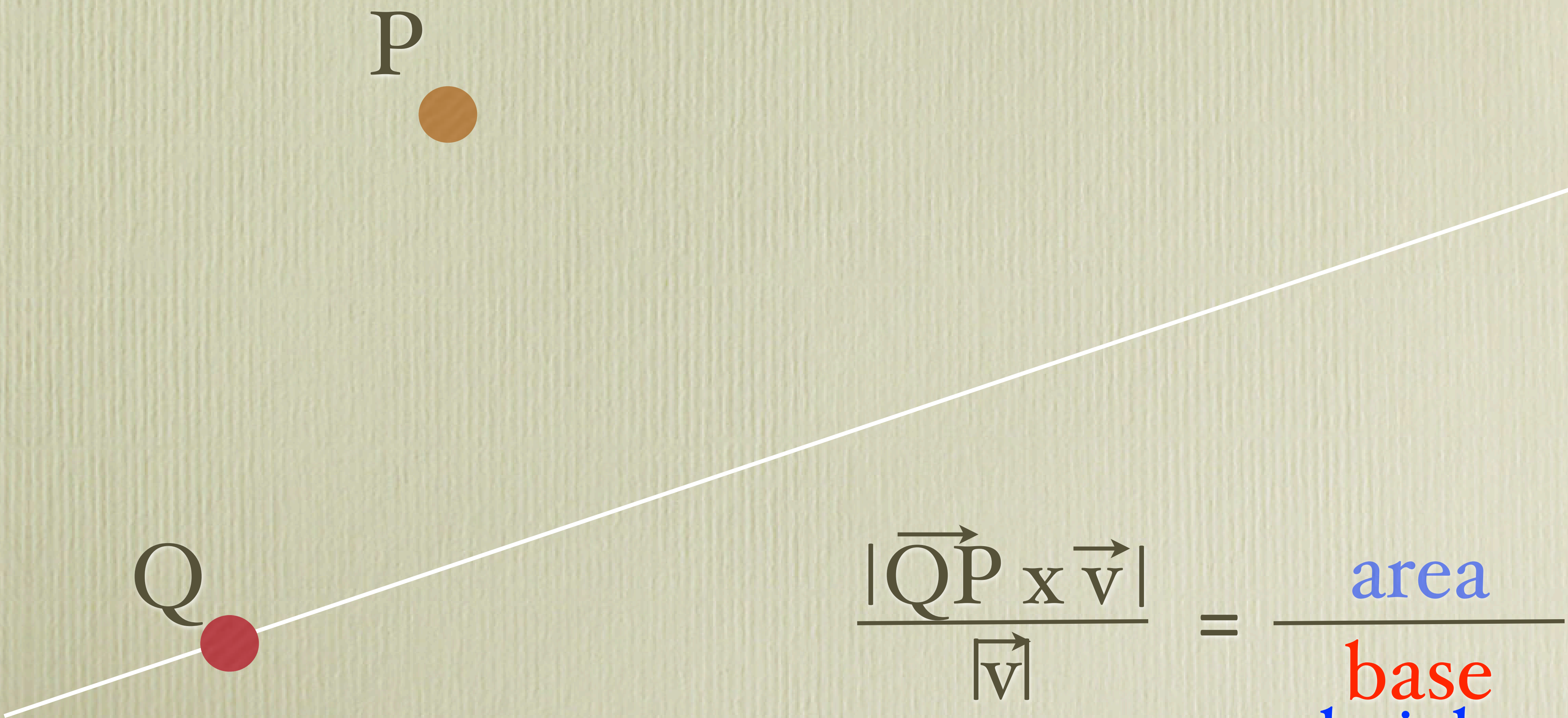

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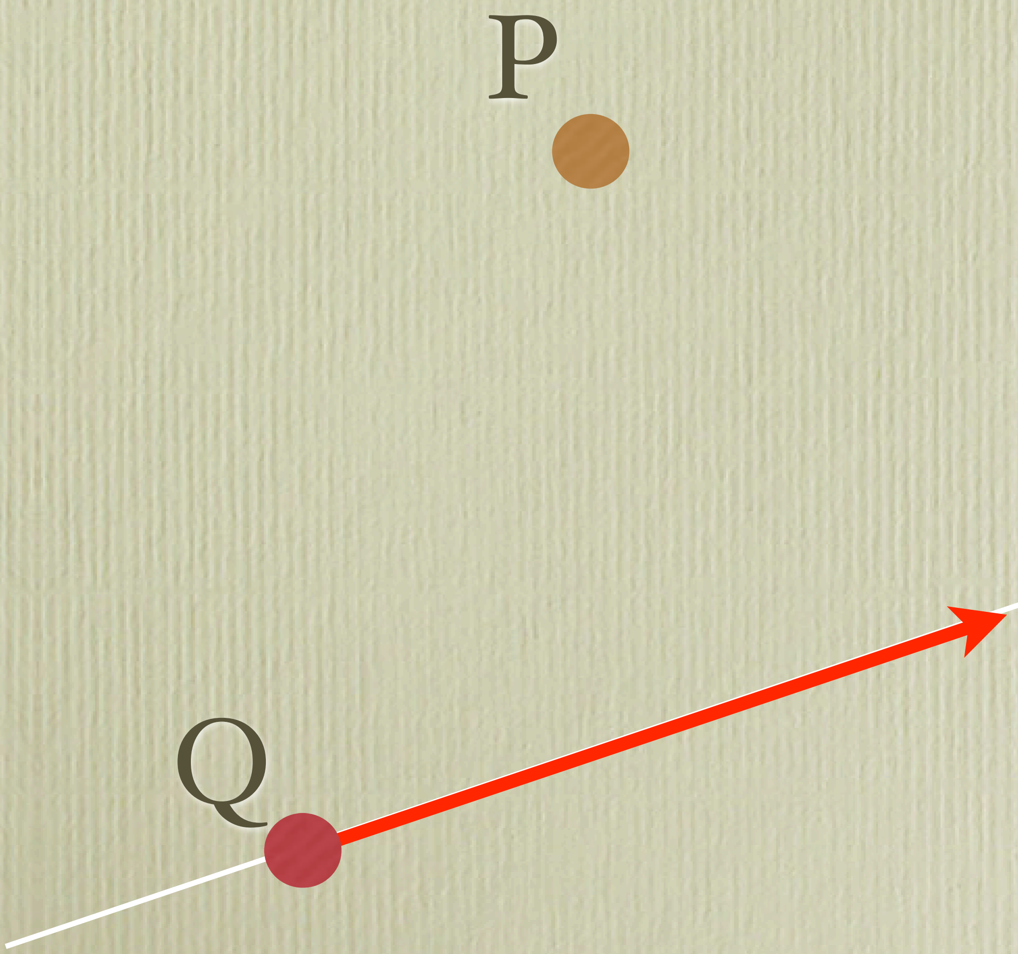

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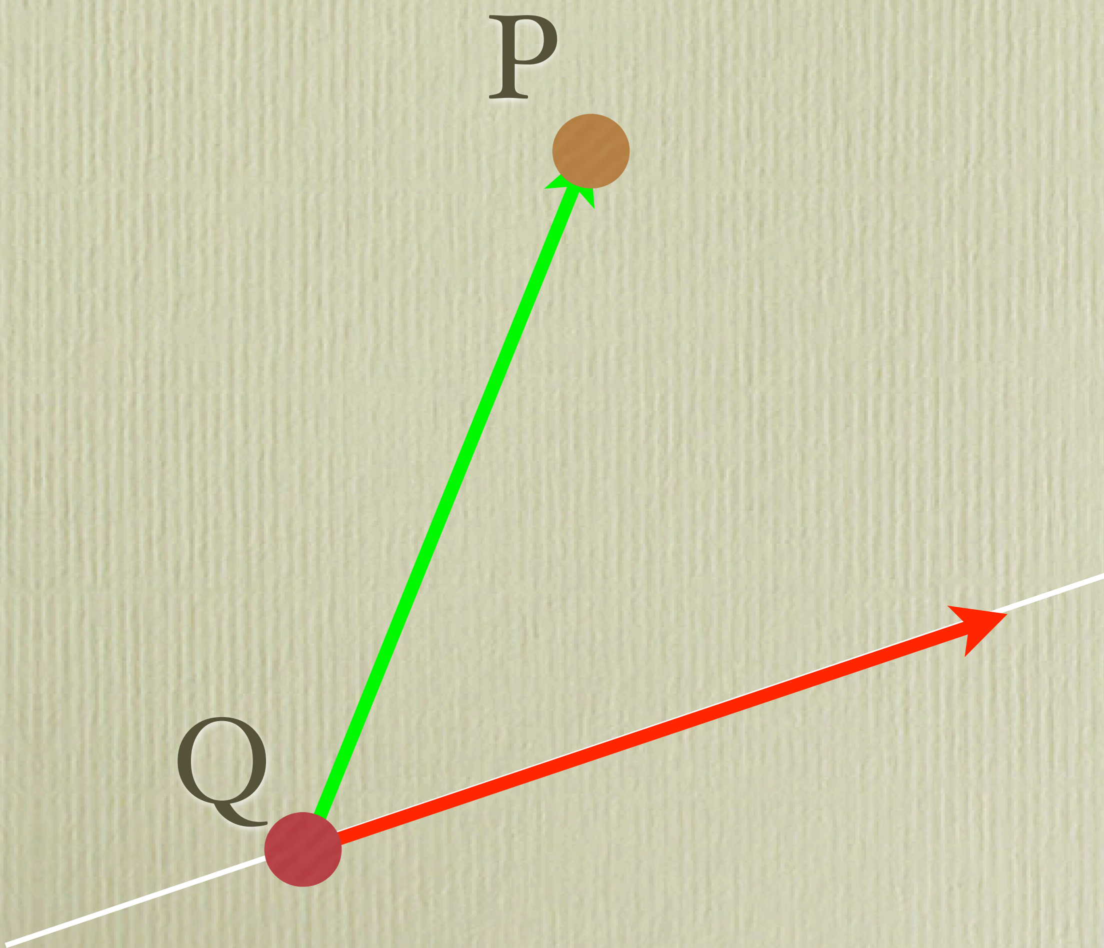

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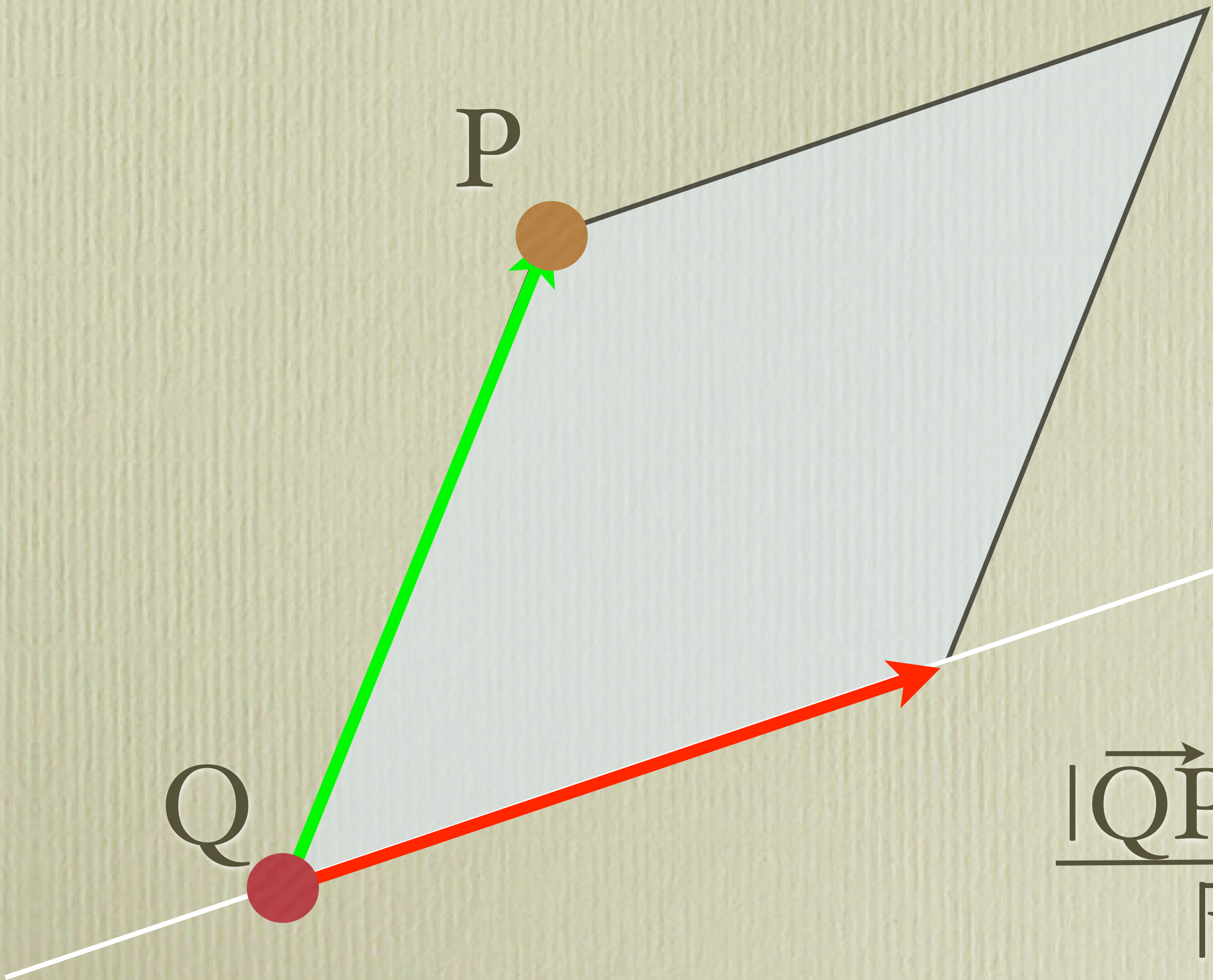

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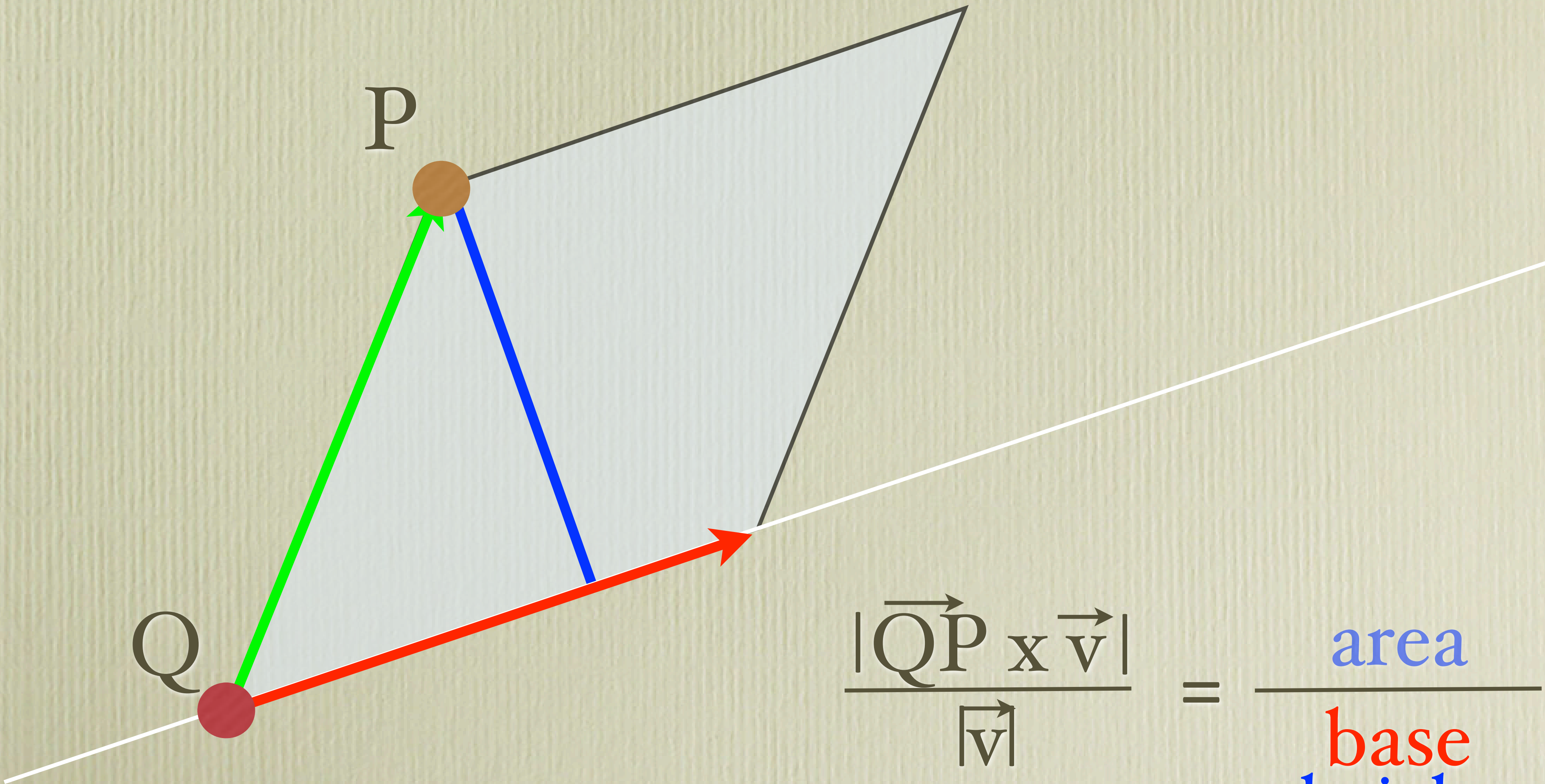

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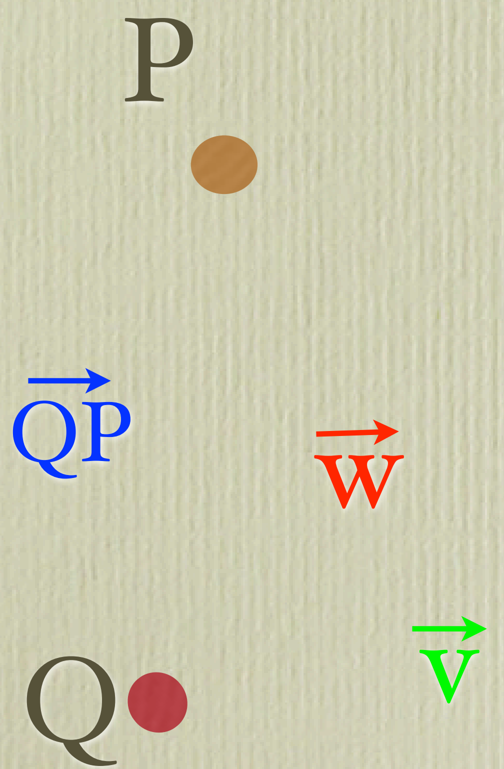
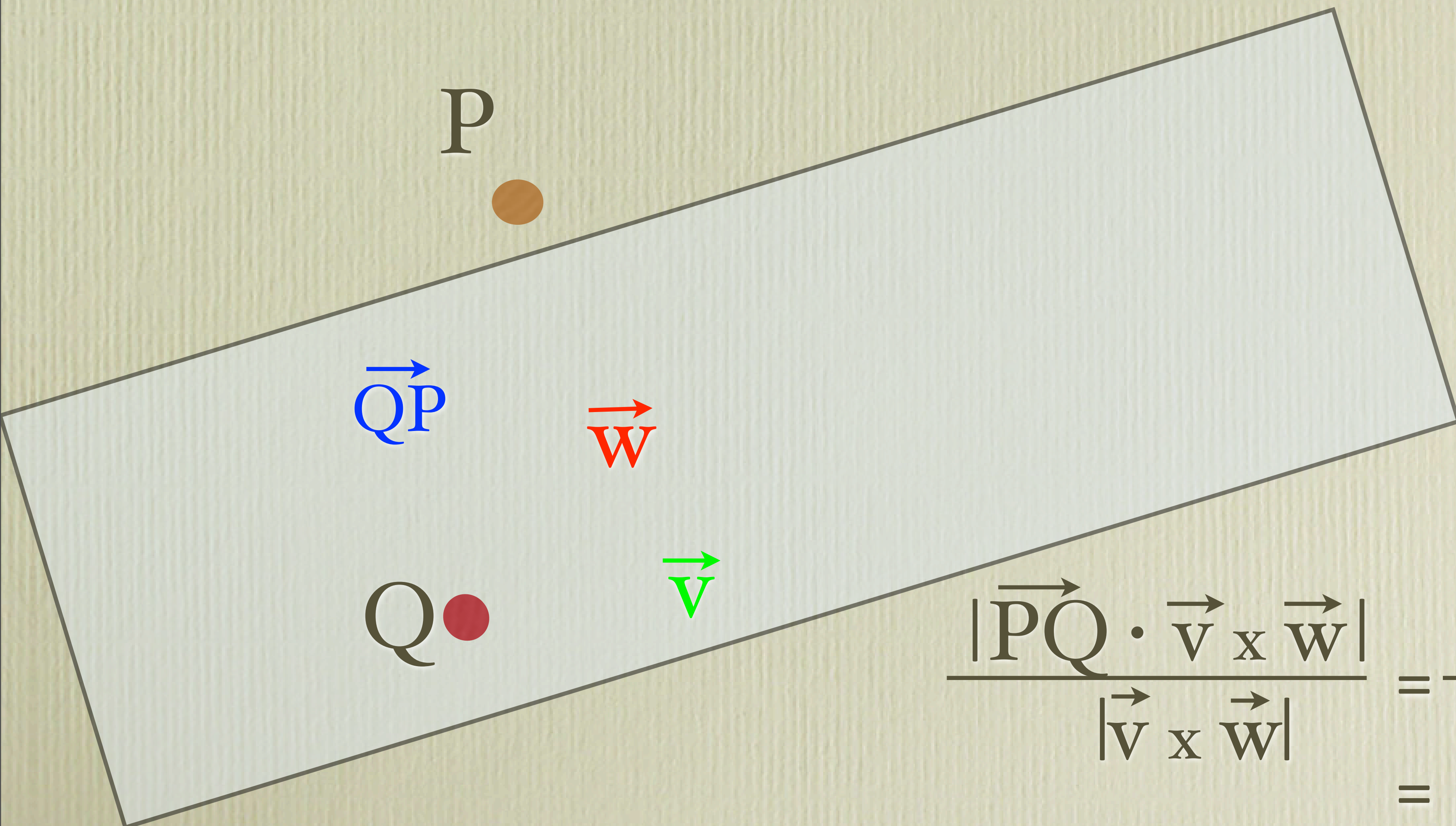
Distance Point-Plane again



$$\frac{|\overrightarrow{PQ} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|} = \frac{\text{volume}}{\text{area}}$$

$$= \text{height}$$

Distance Point-Plane again



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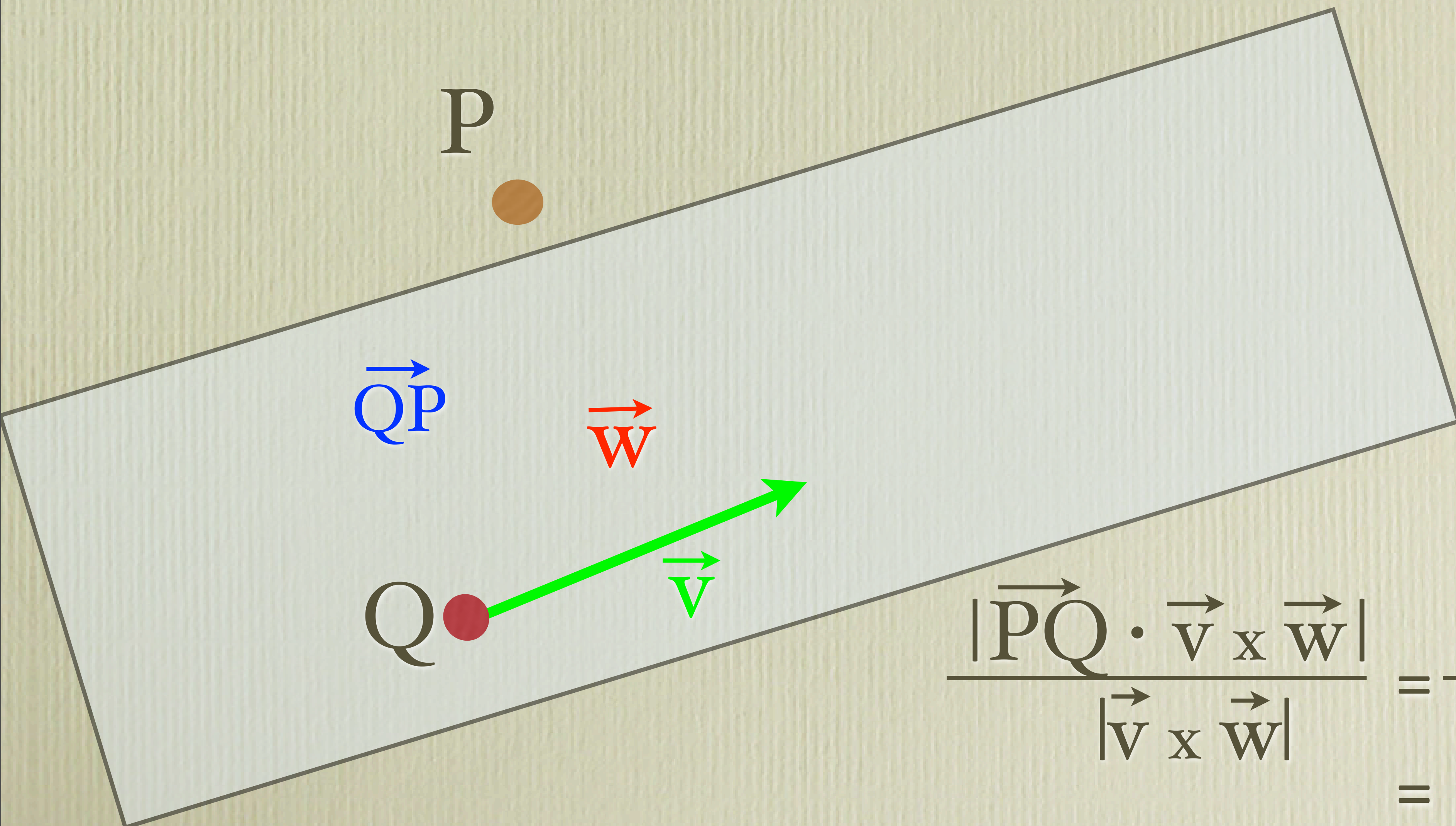
Distance Point-Plane again



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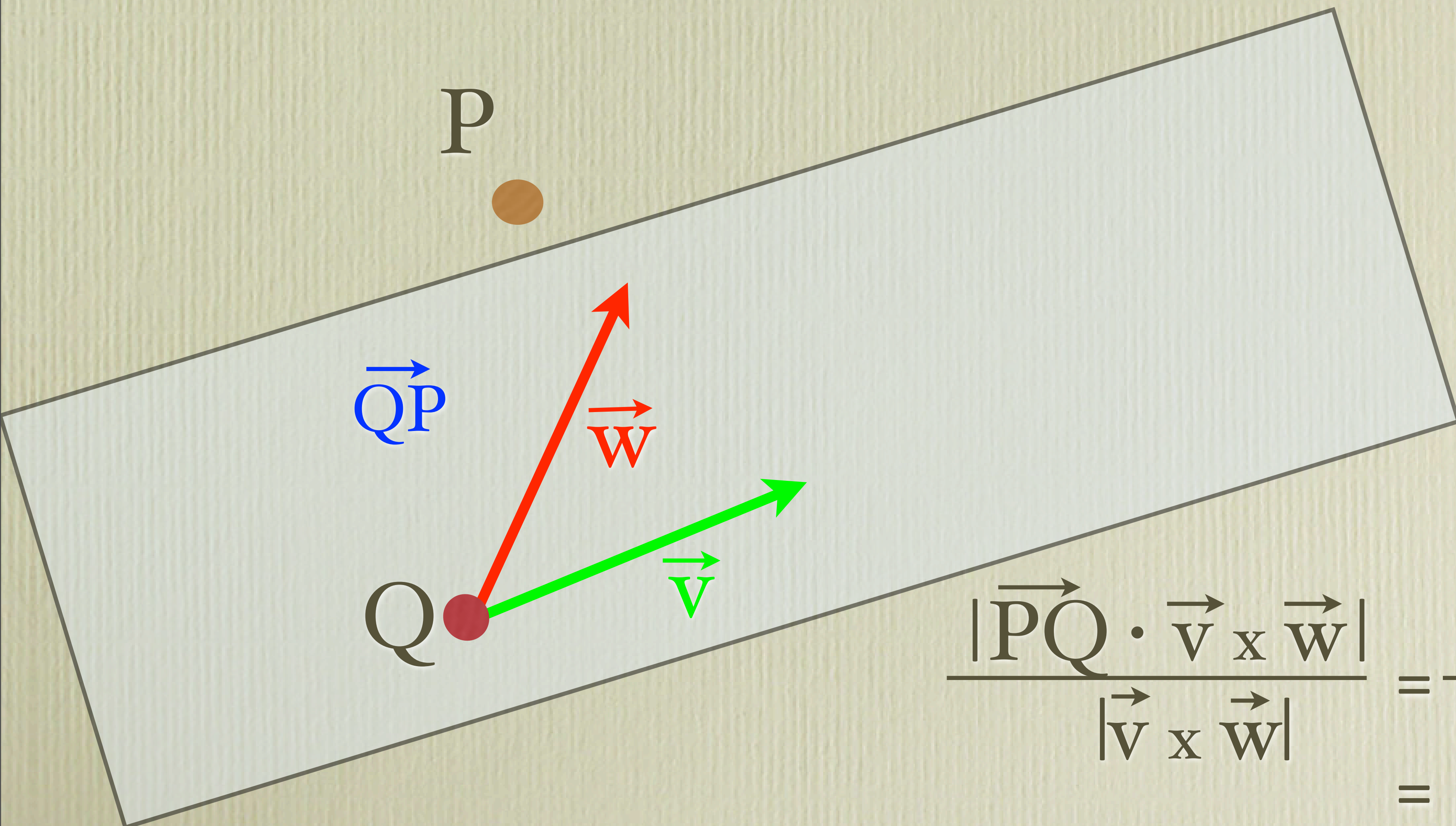
$= \text{height}$

Distance Point-Plane again



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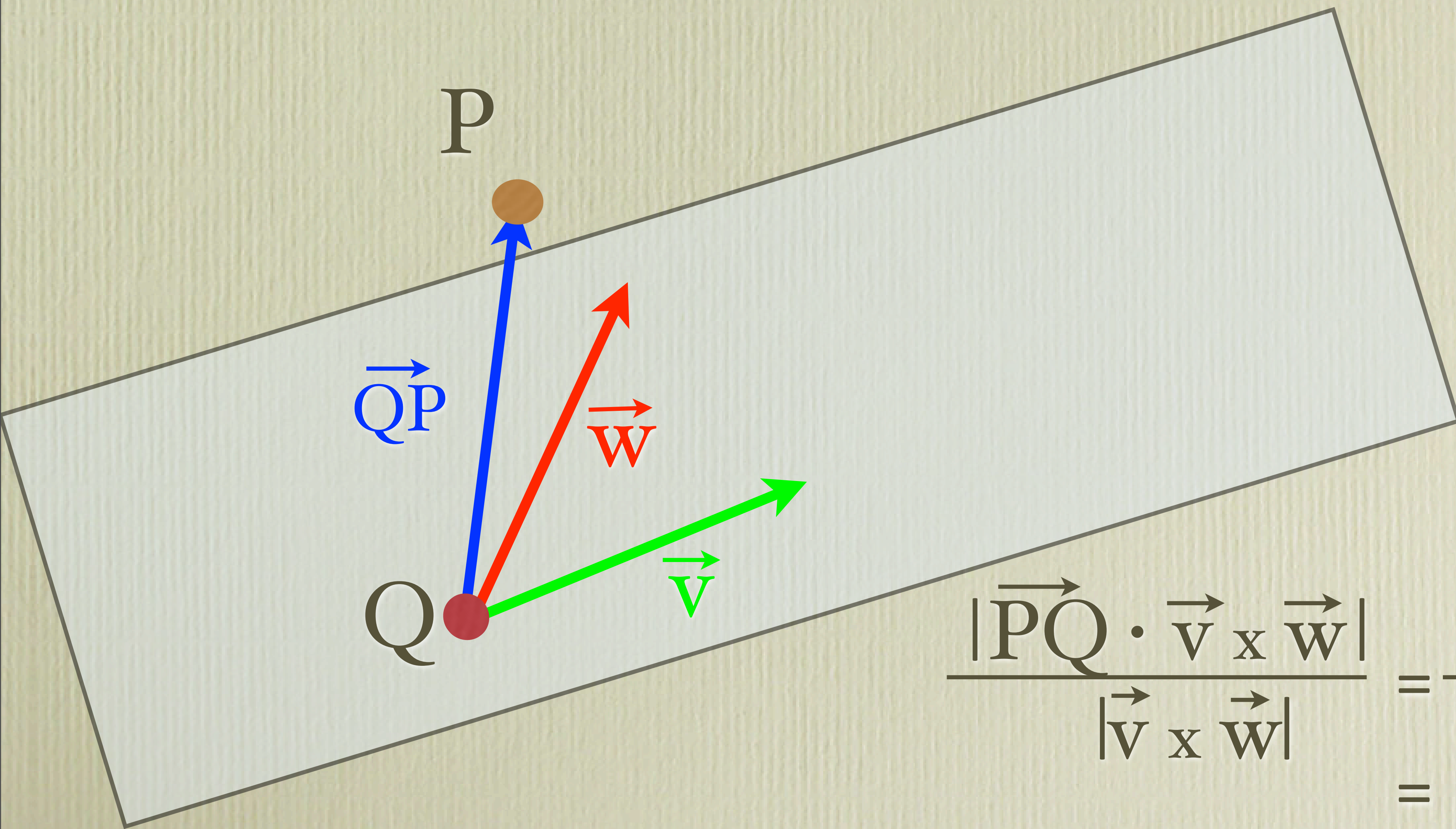

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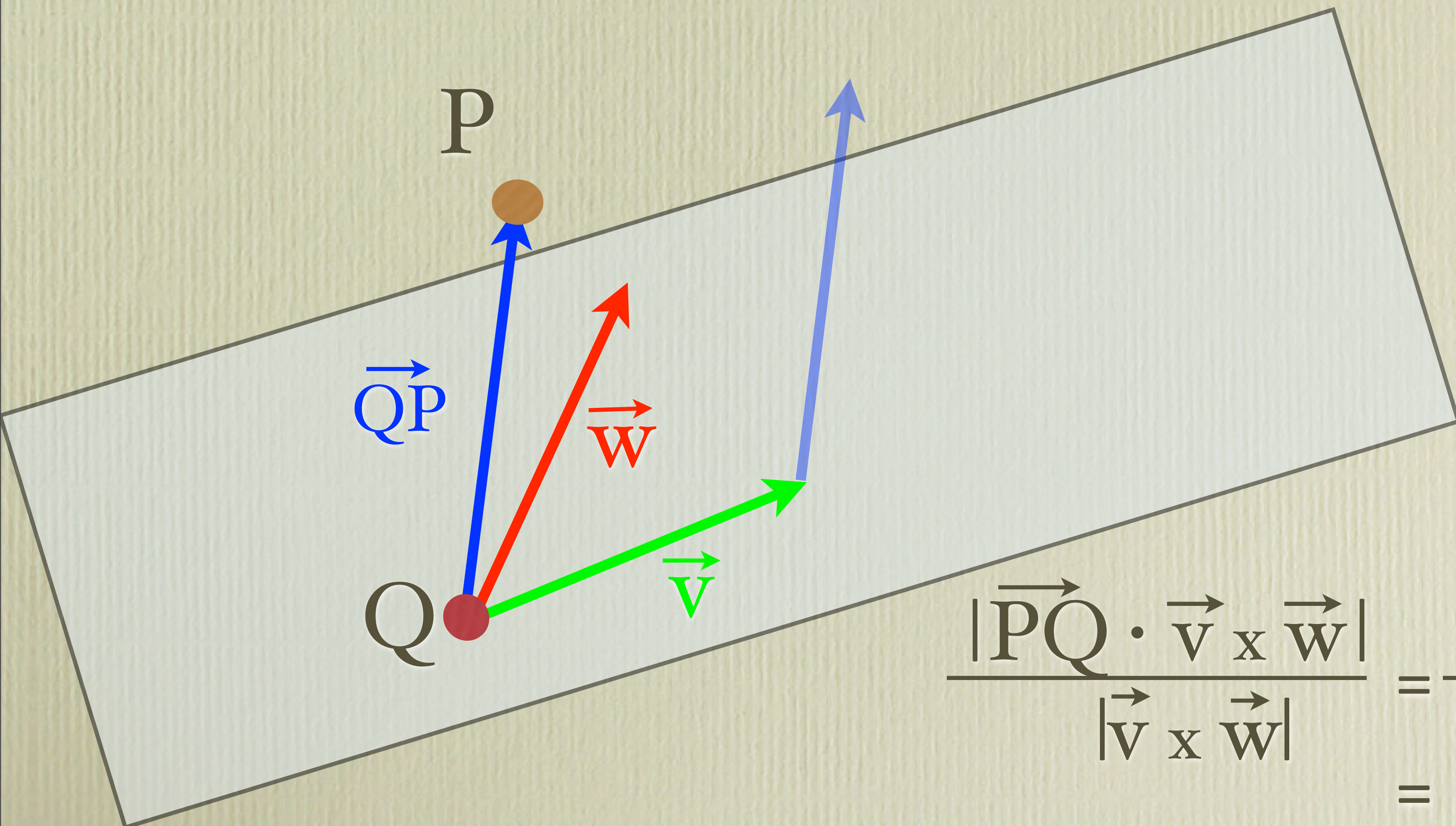

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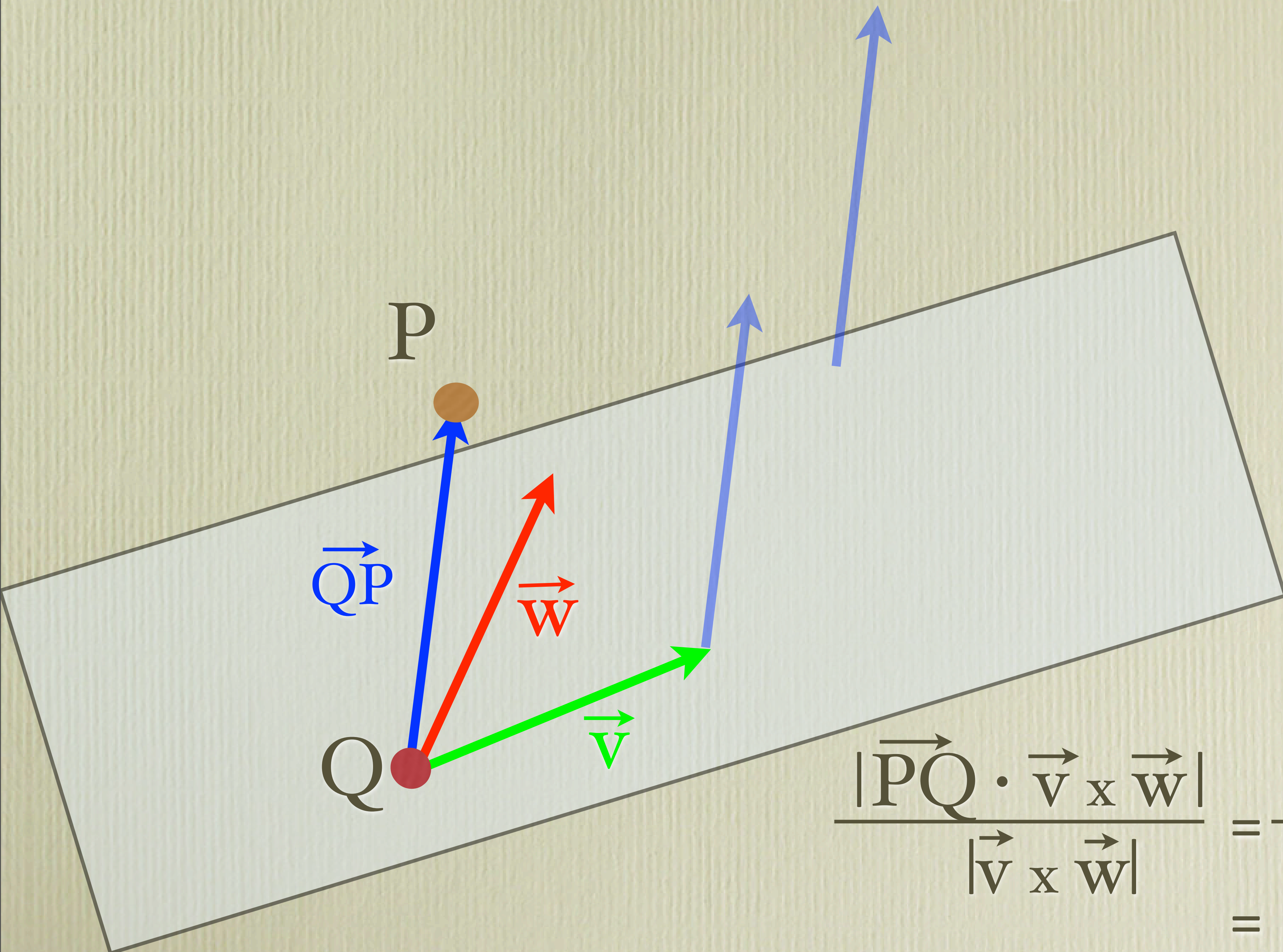

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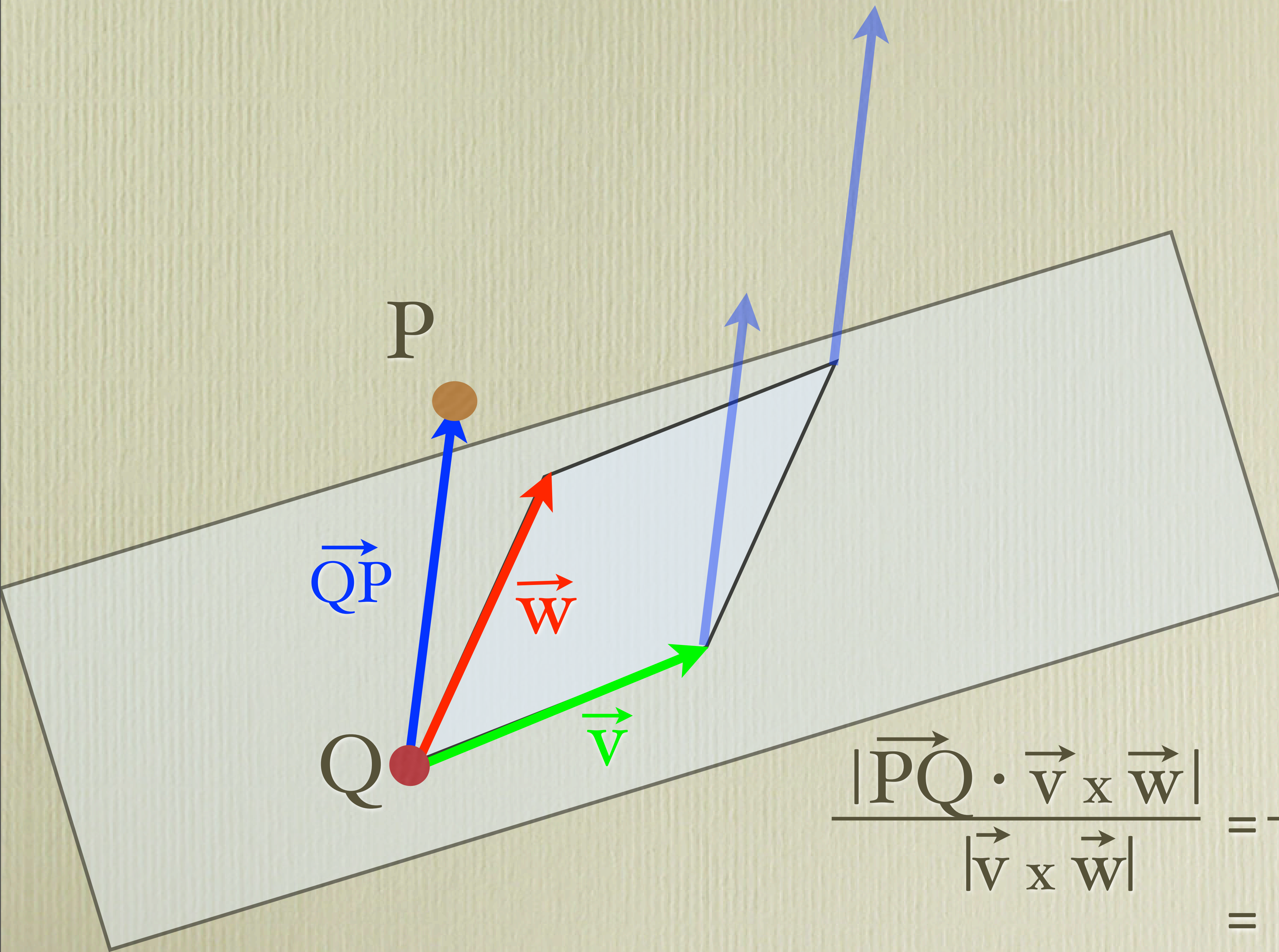

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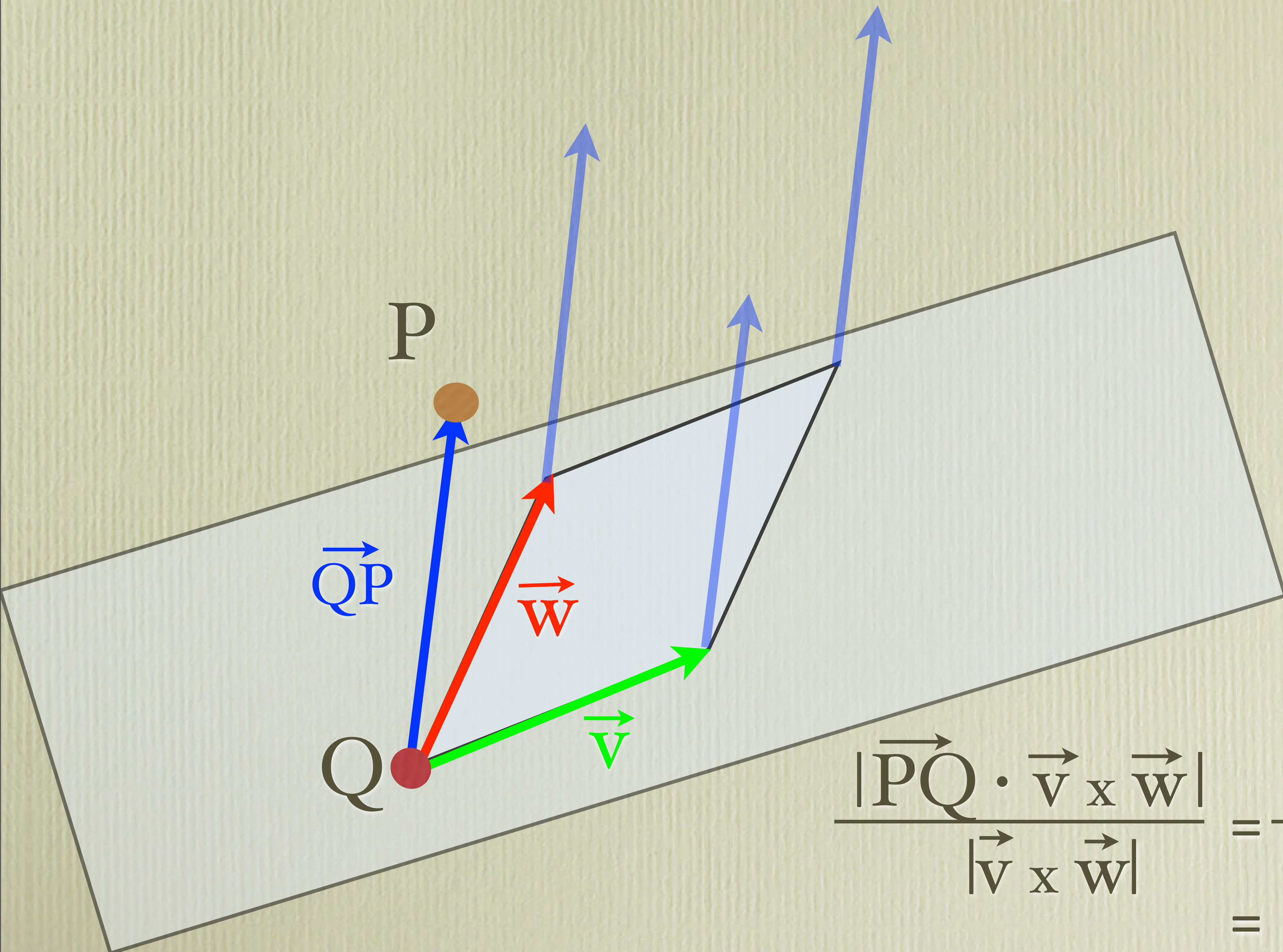

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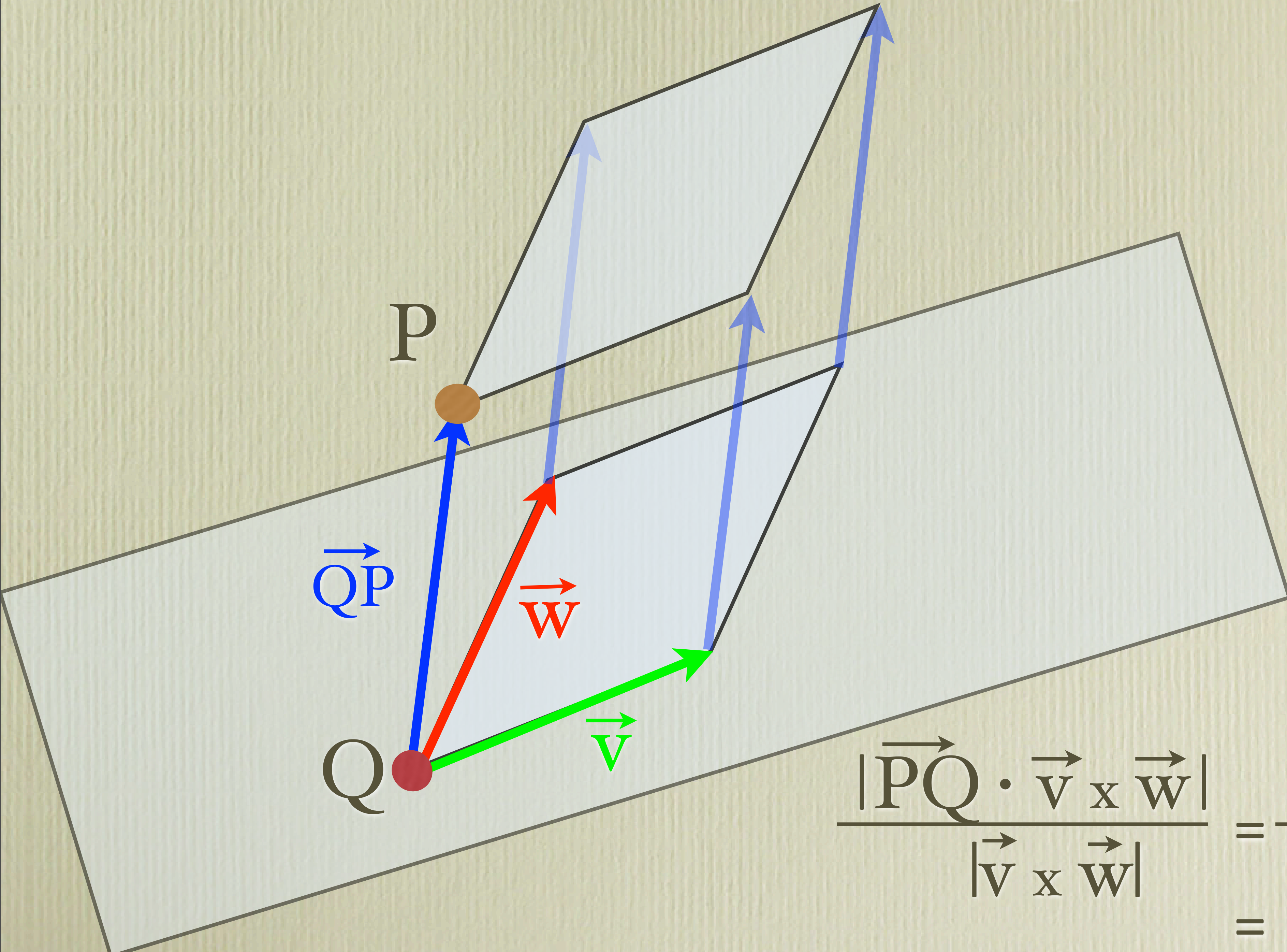

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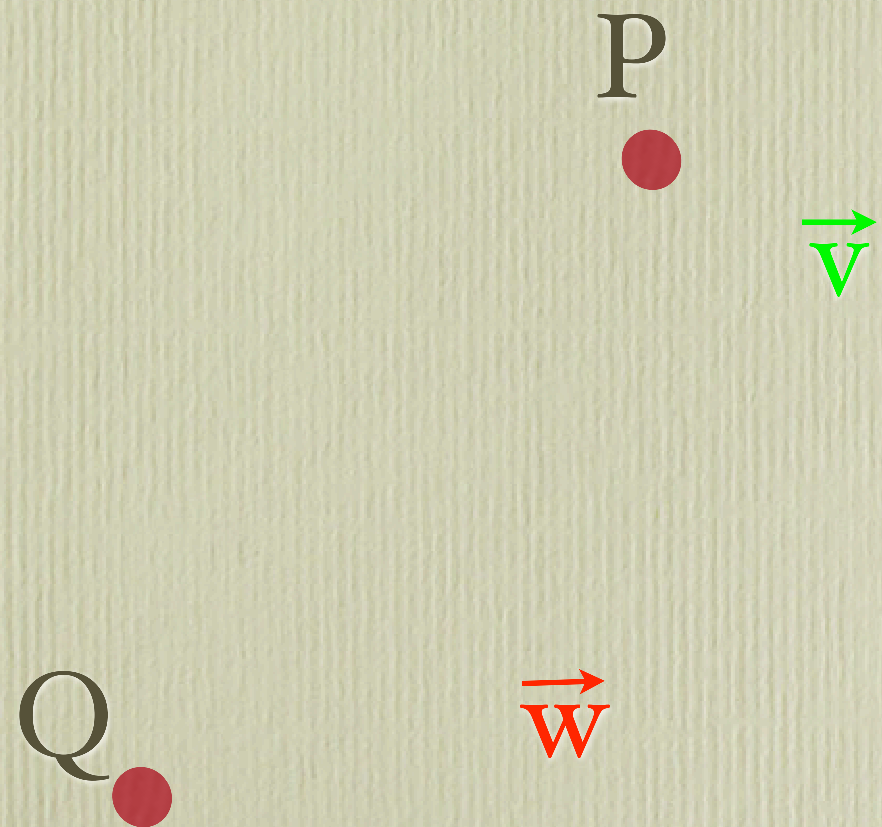
Distance Line-Line



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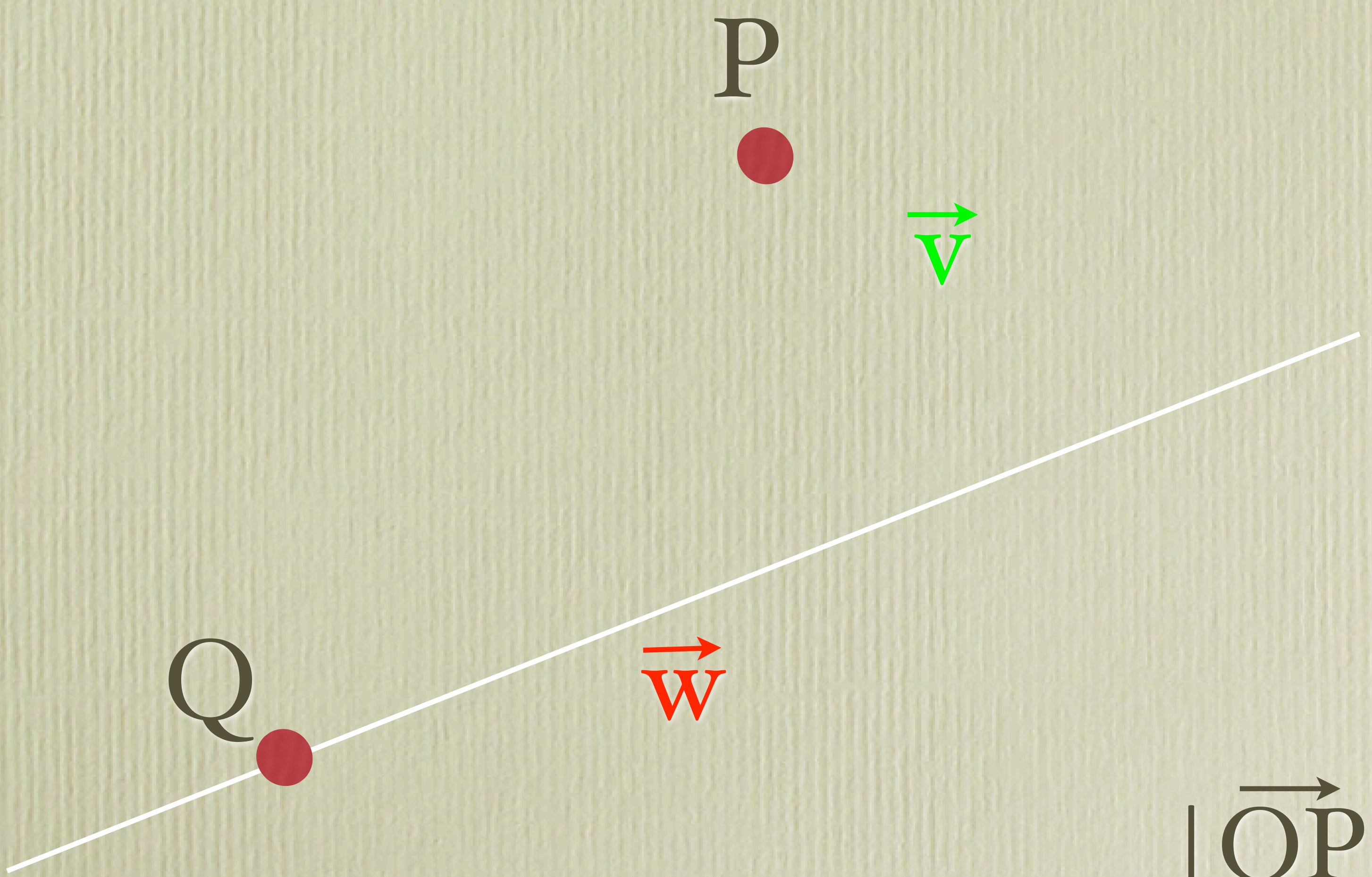

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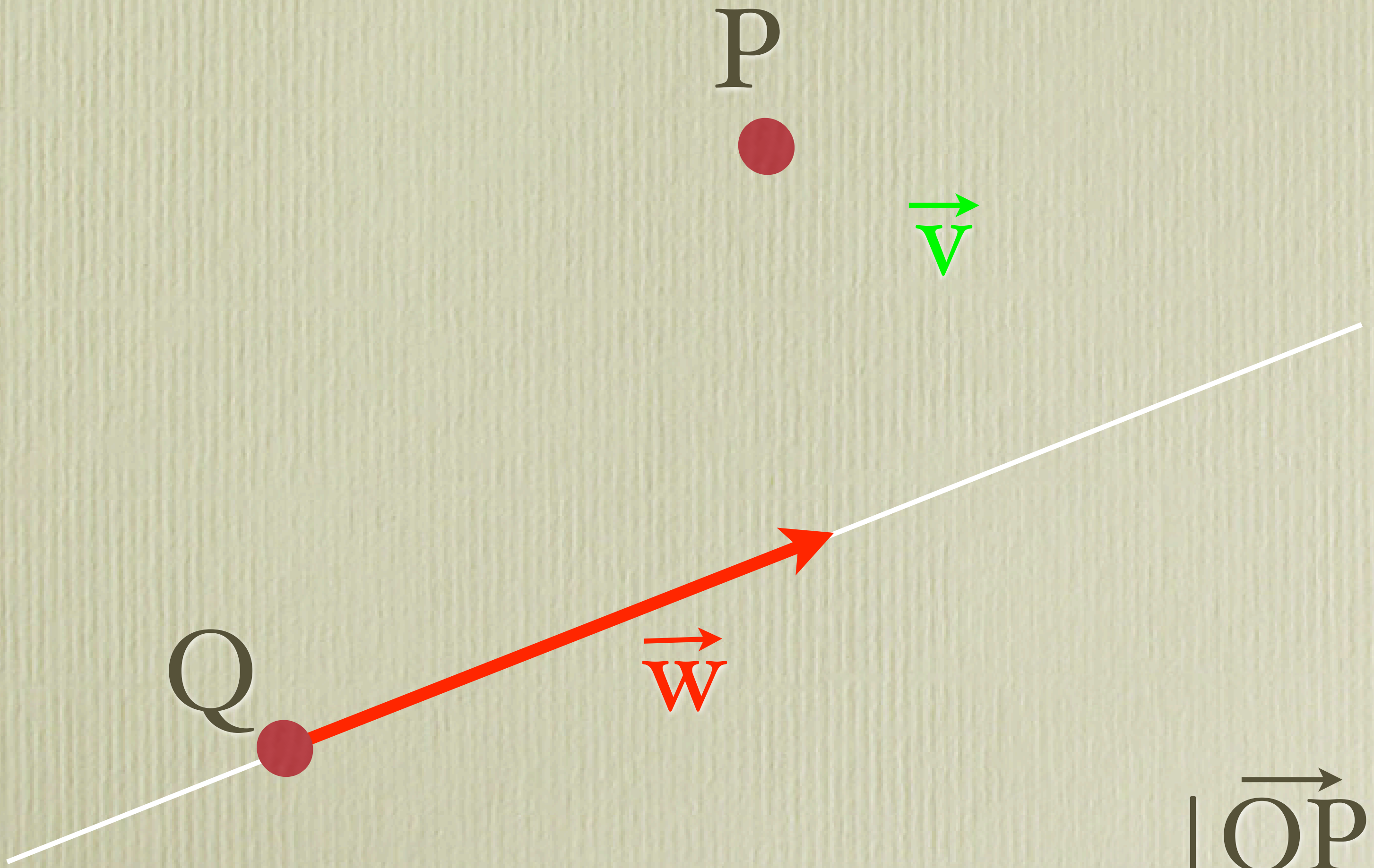

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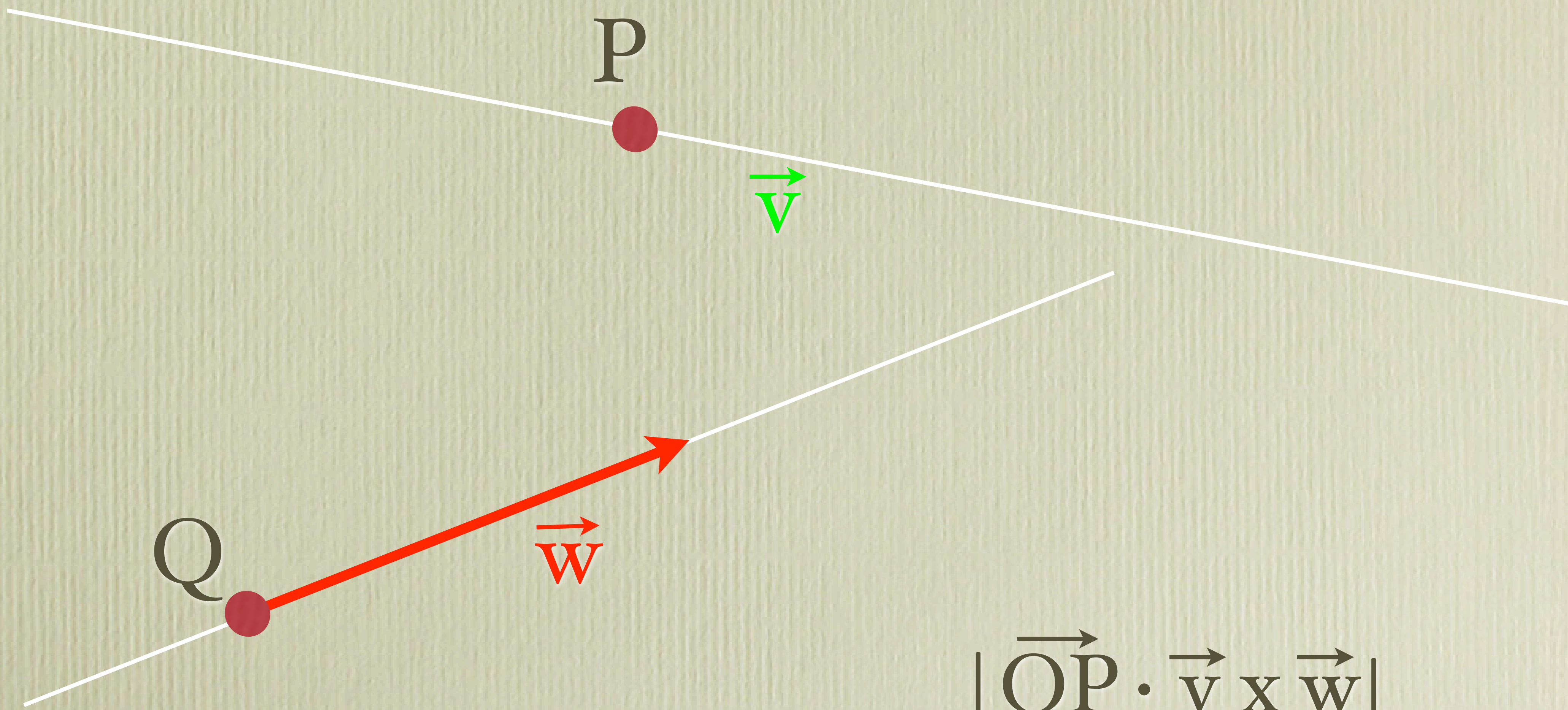

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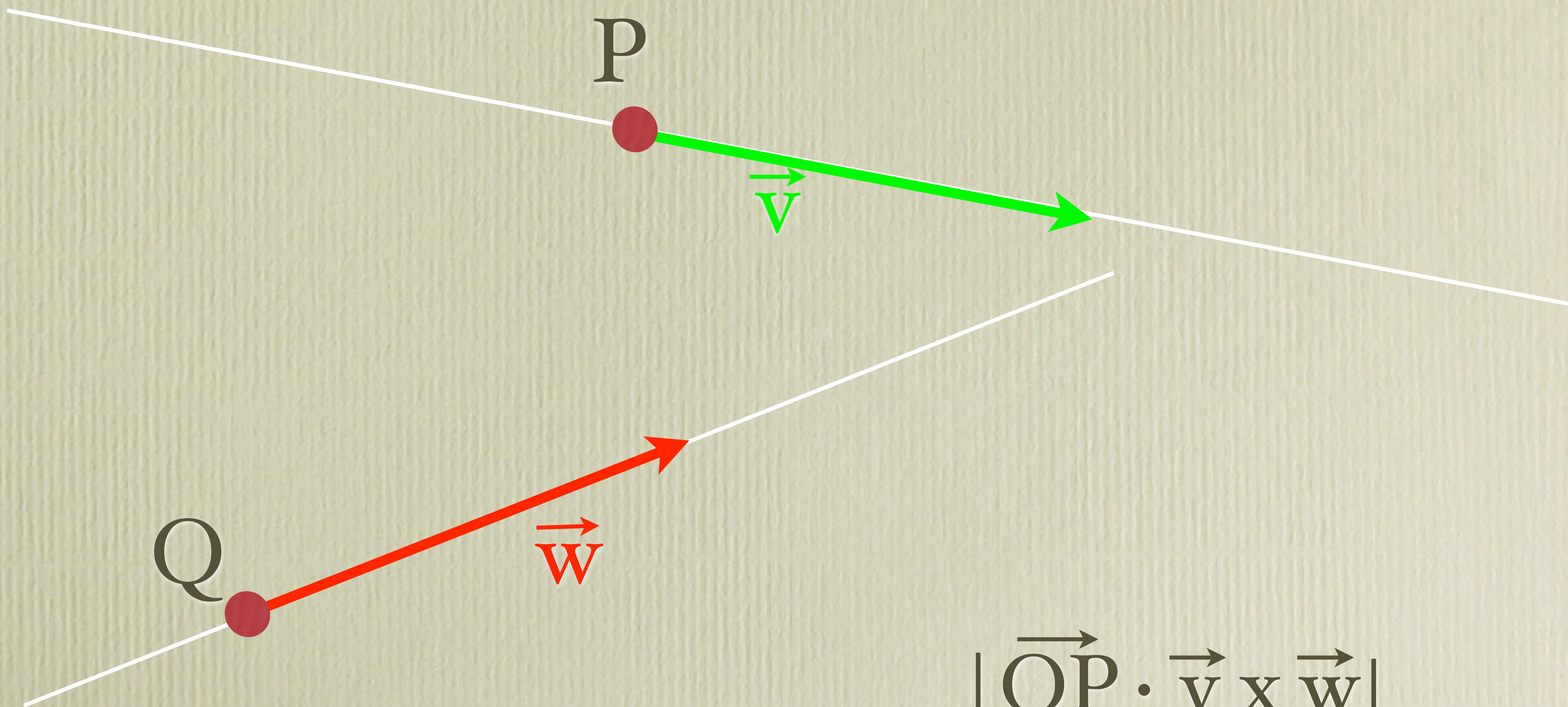

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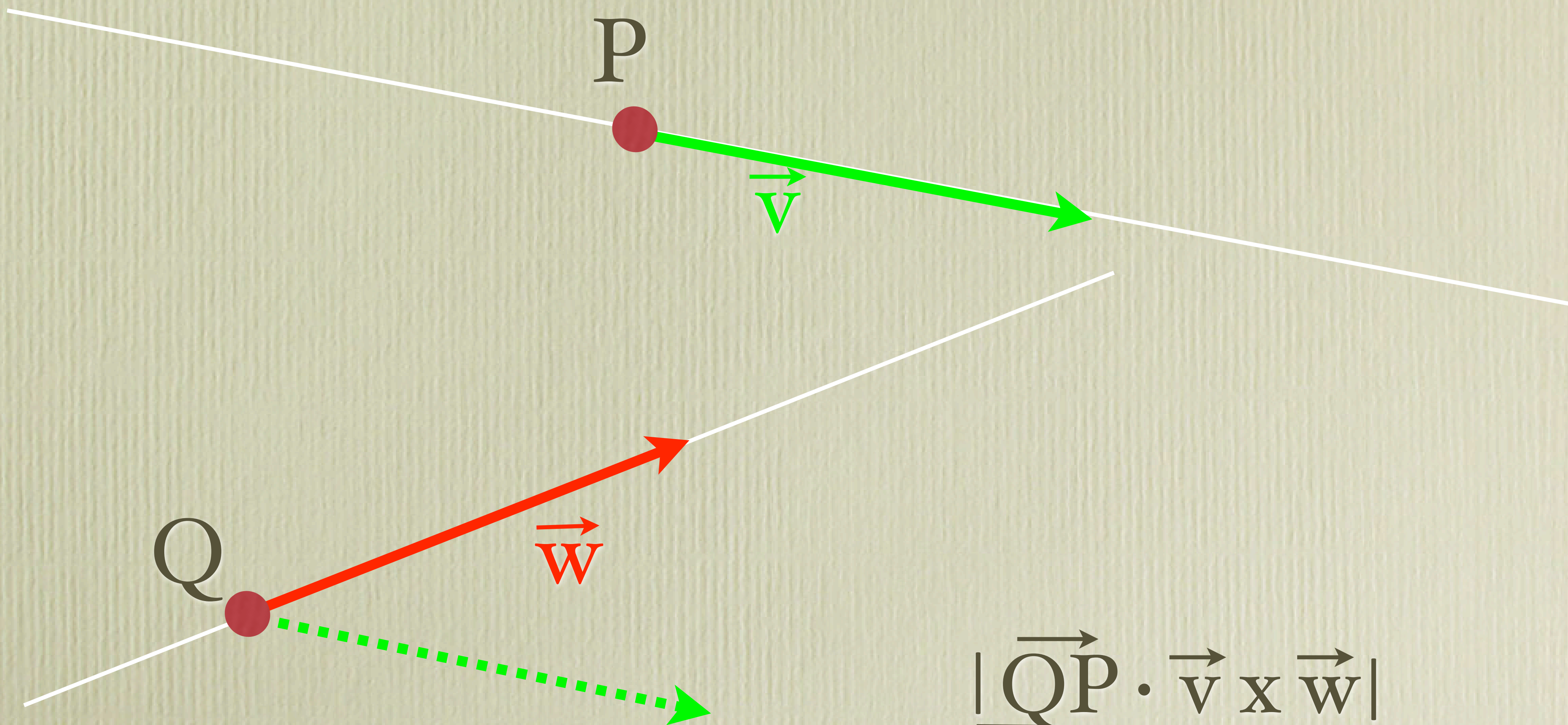

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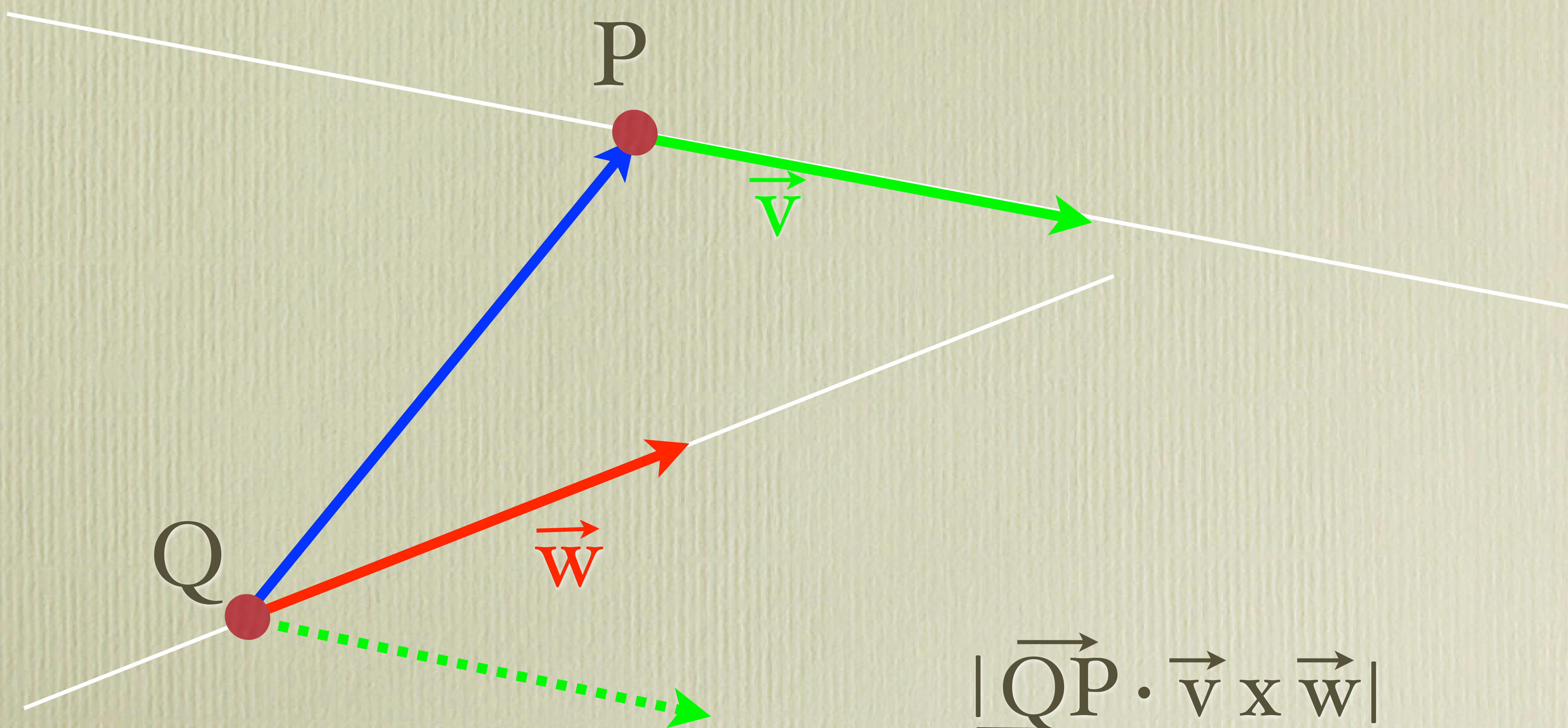

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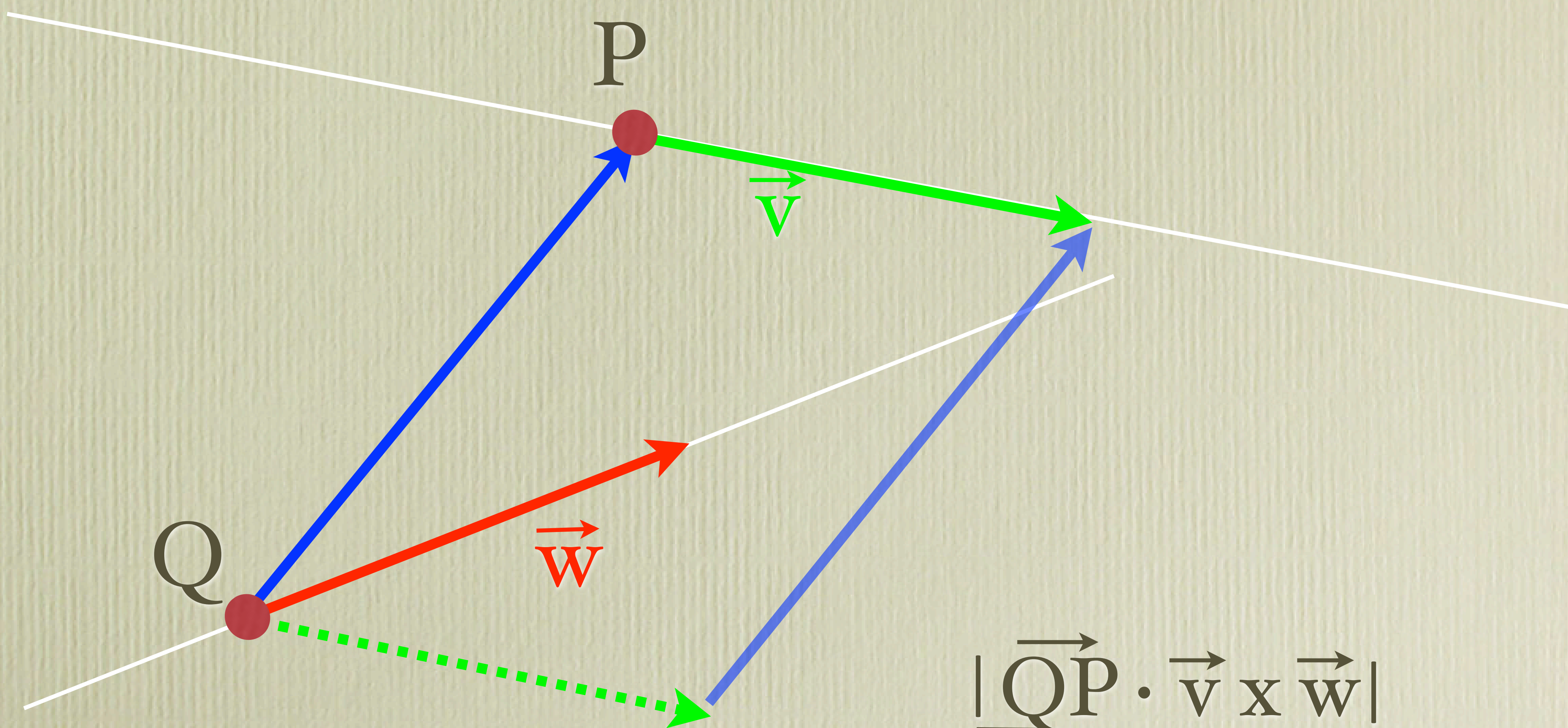

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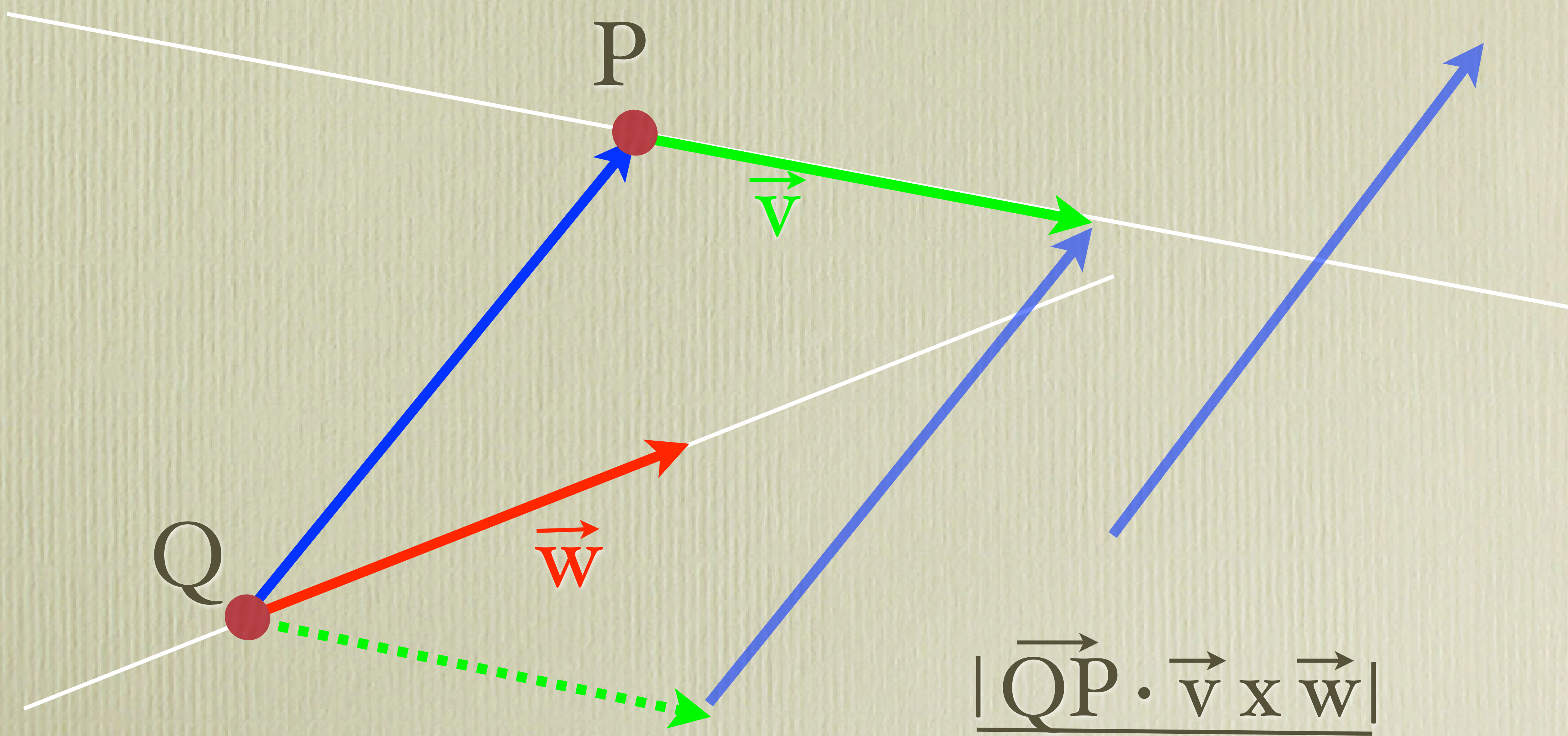

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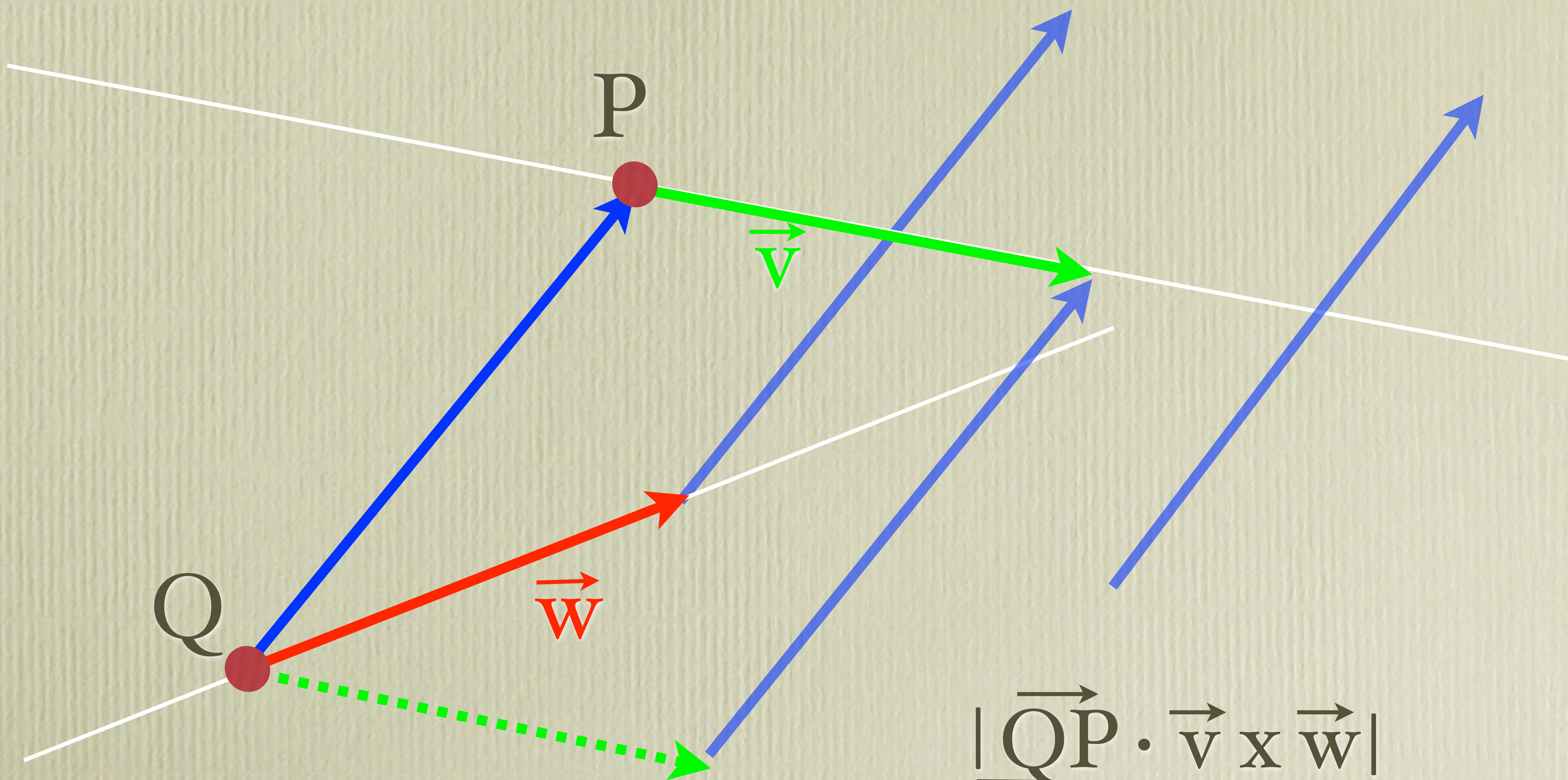
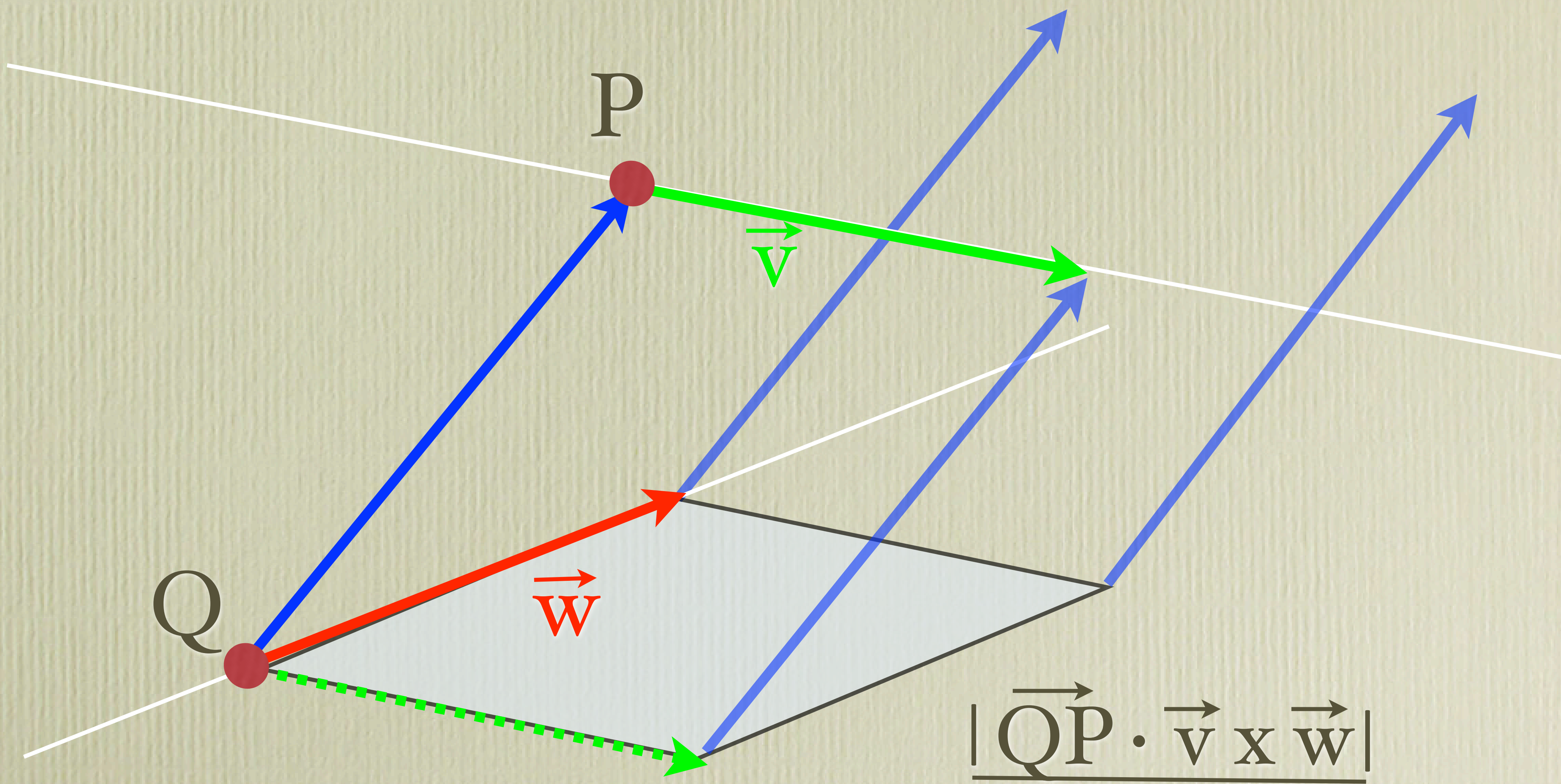
Distance Line-Line



$$= \frac{\text{Volume}}{\text{Area}}$$

$$\frac{|\overrightarrow{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

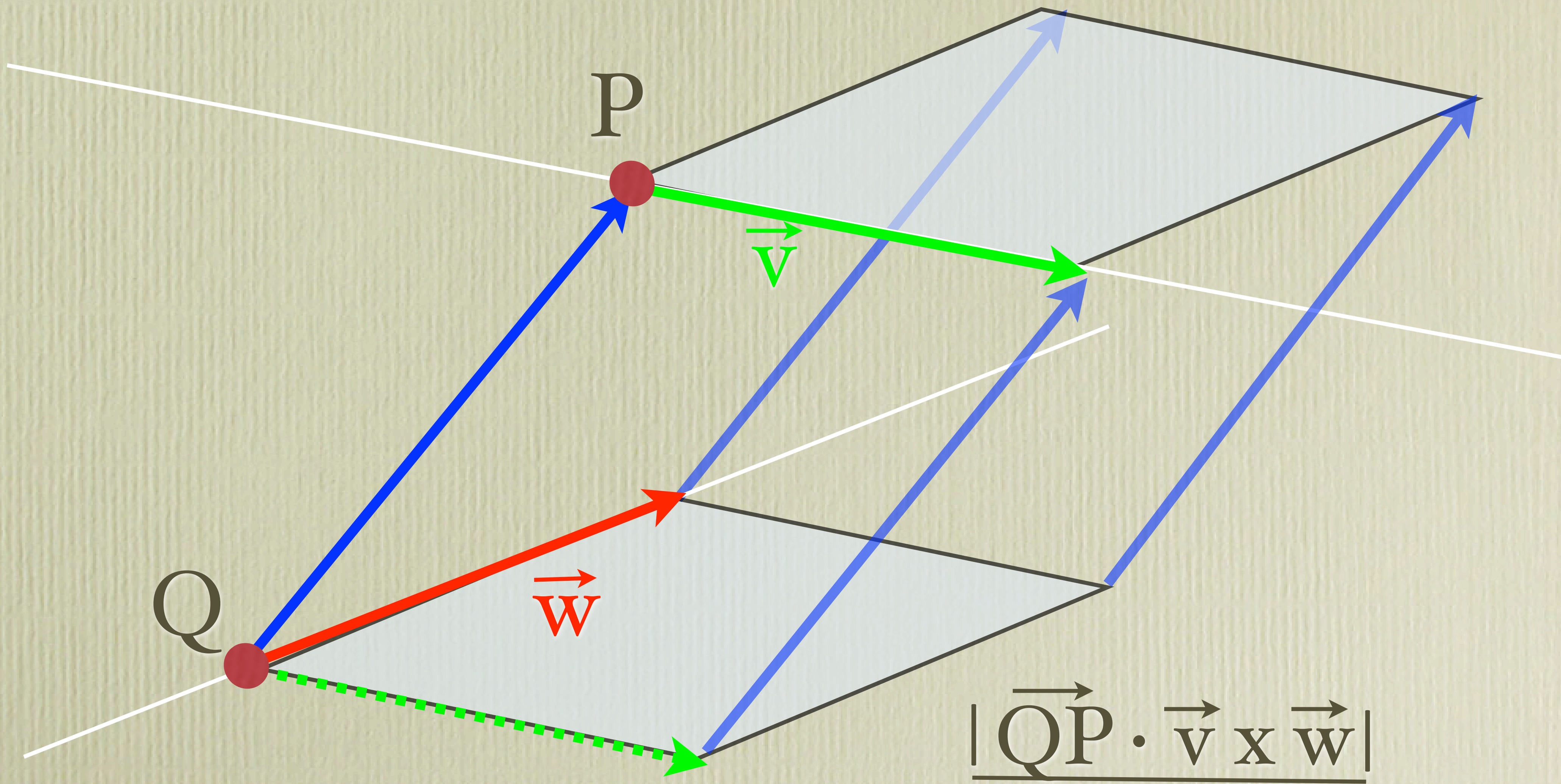
Distance Line-Line



$$= \frac{\text{Volume}}{\text{Area}}$$

$$\frac{|\overrightarrow{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

Distance Line-Line



$$\frac{|\overrightarrow{QP} \cdot \vec{v} \times \vec{w}|}{|\vec{v} \times \vec{w}|}$$

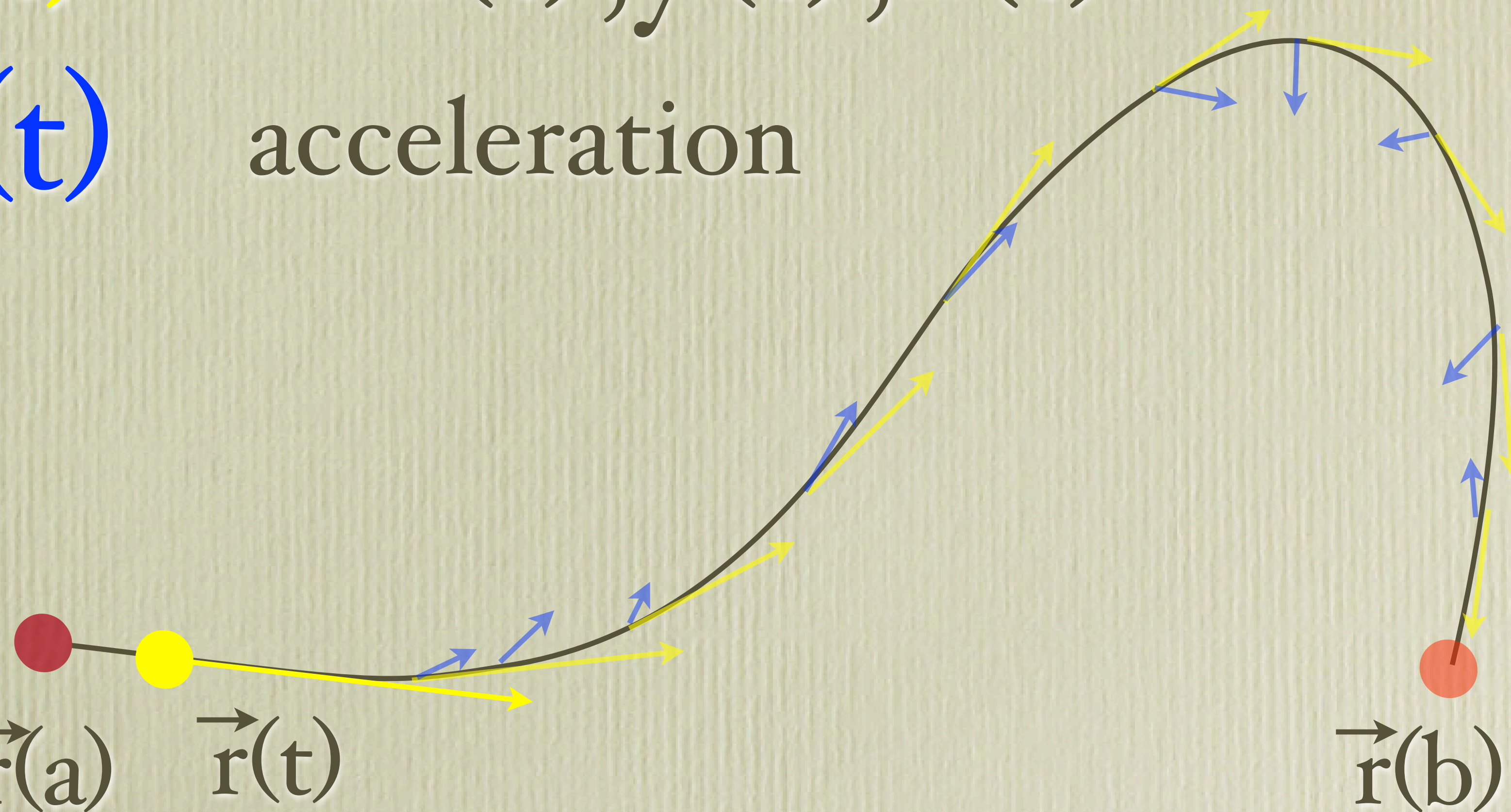
$$= \frac{\text{Volume}}{\text{Area}}$$

Parametrized Curves

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$ velocity

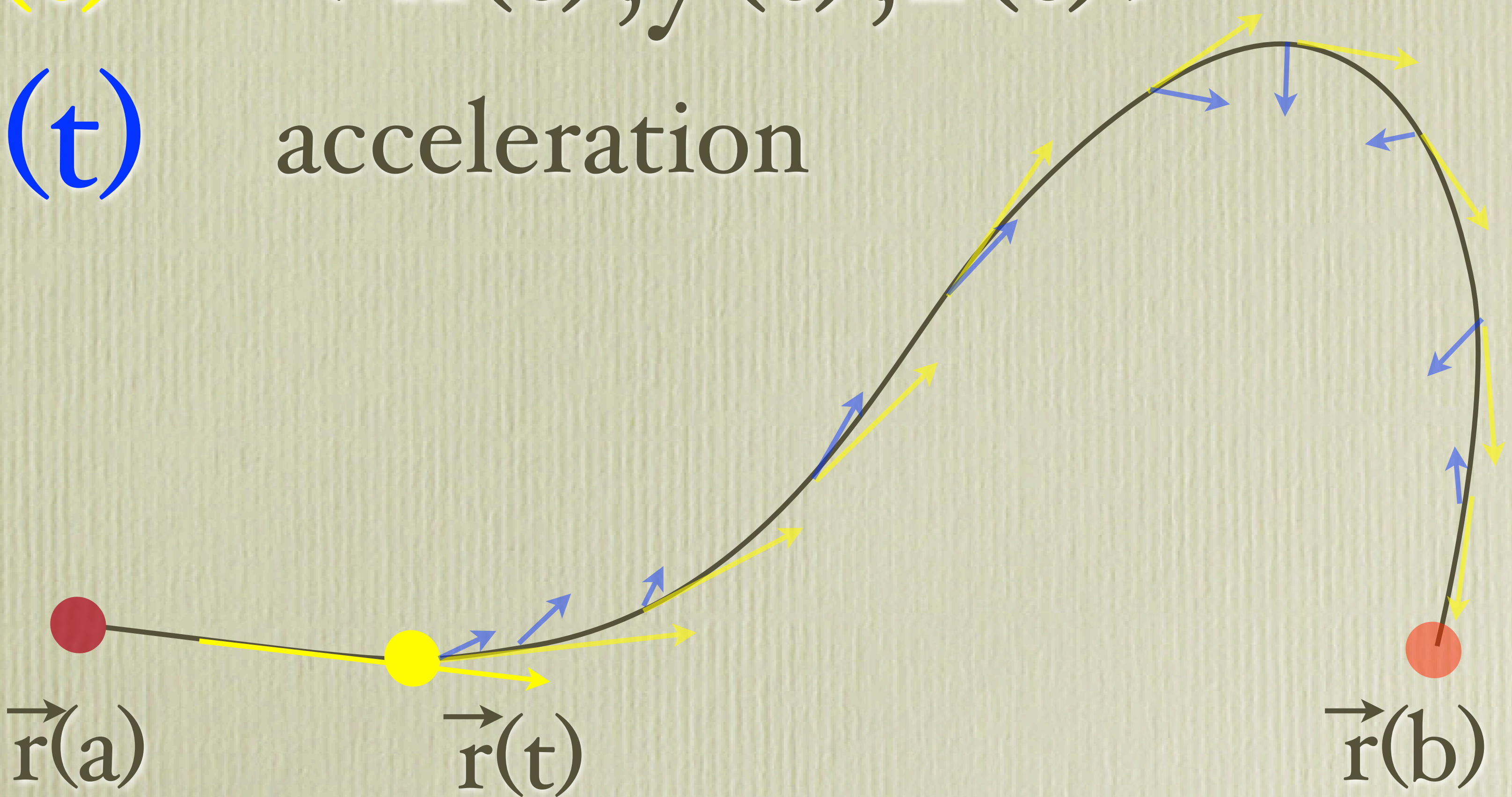
$$\vec{r}''(t)$$ acceleration



$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

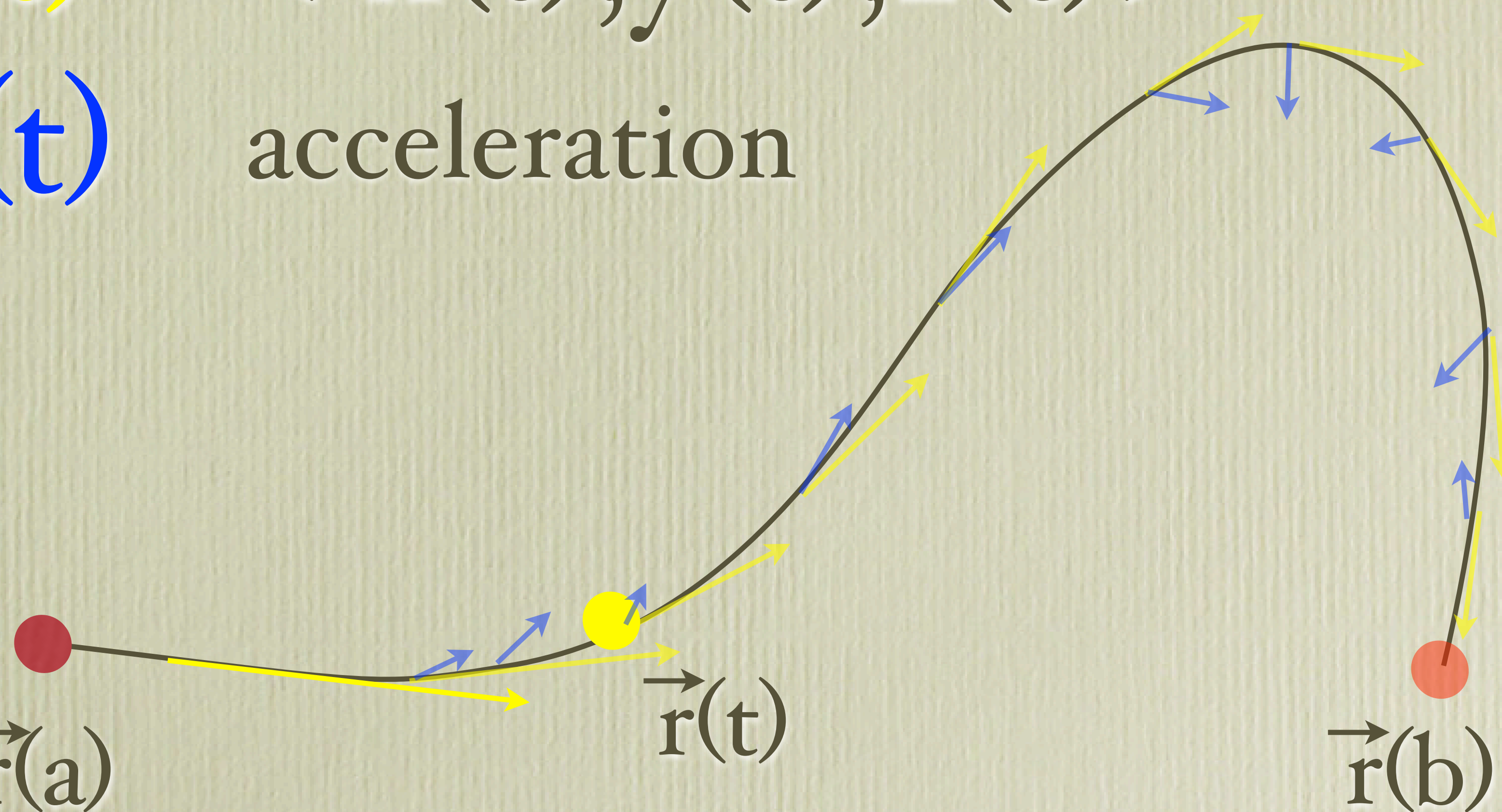
$$\vec{r}''(t)$$ acceleration



$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

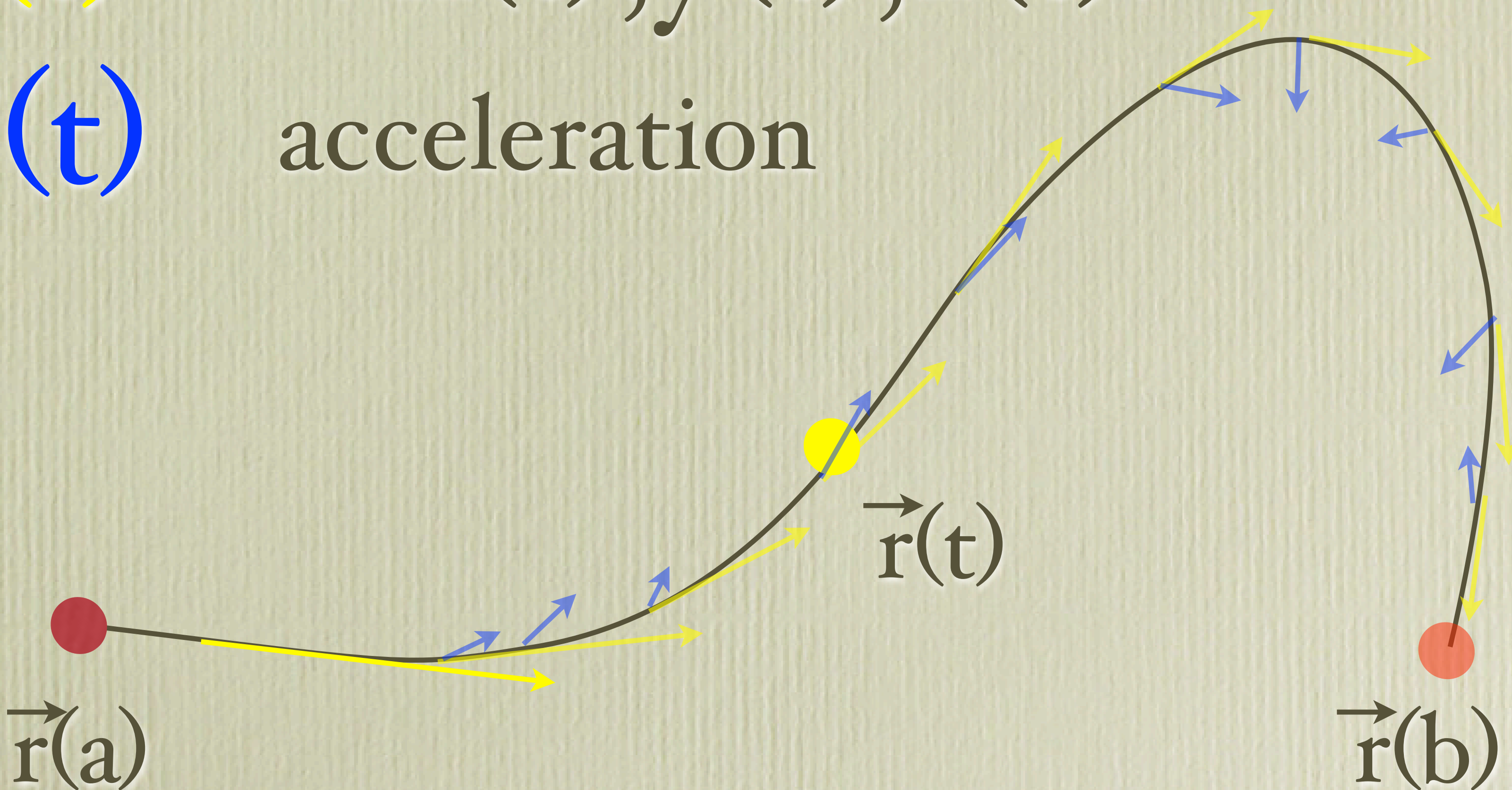
$\vec{r}''(t)$ acceleration



$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

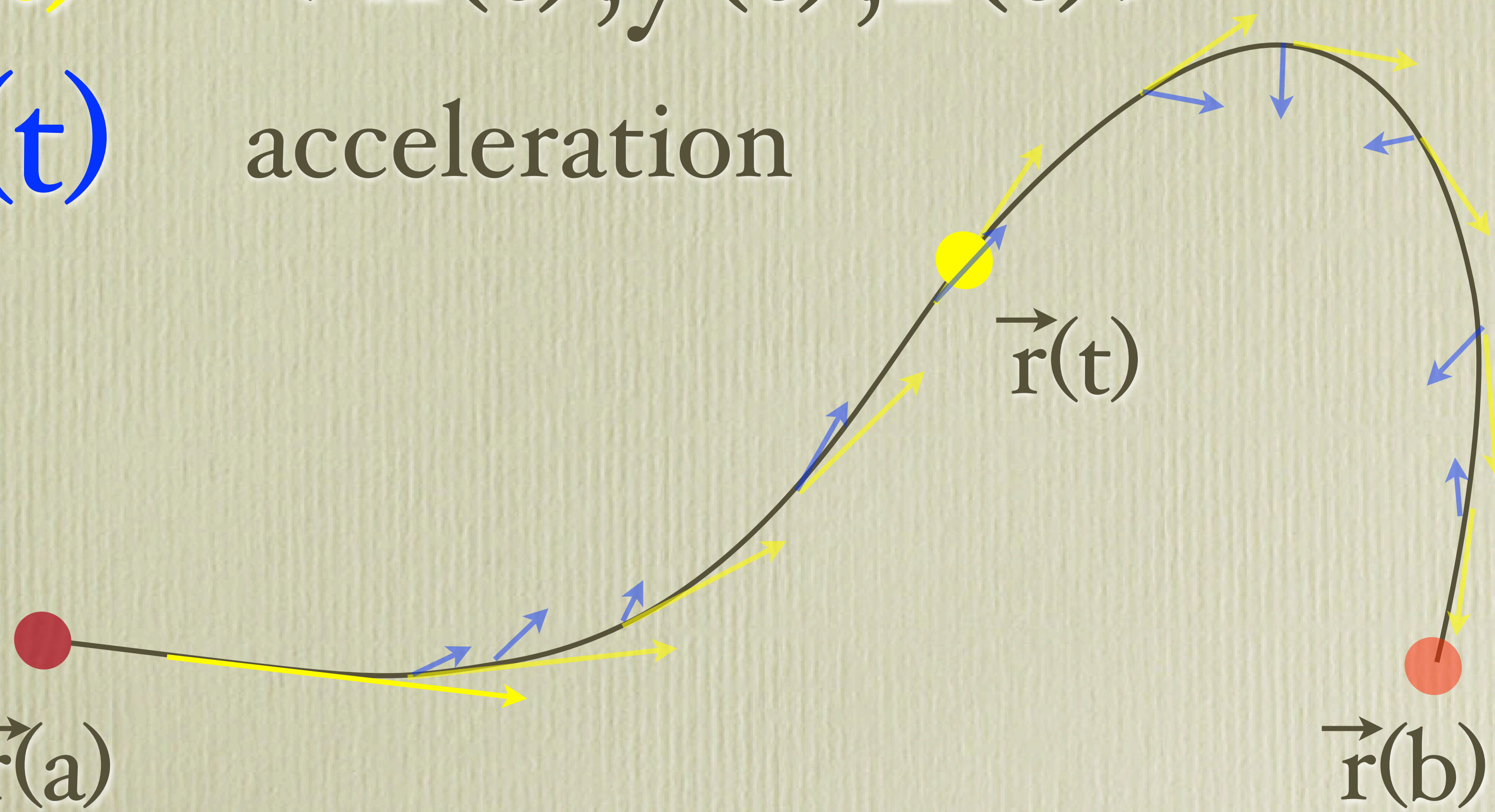
$$\vec{r}''(t)$$ acceleration



$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

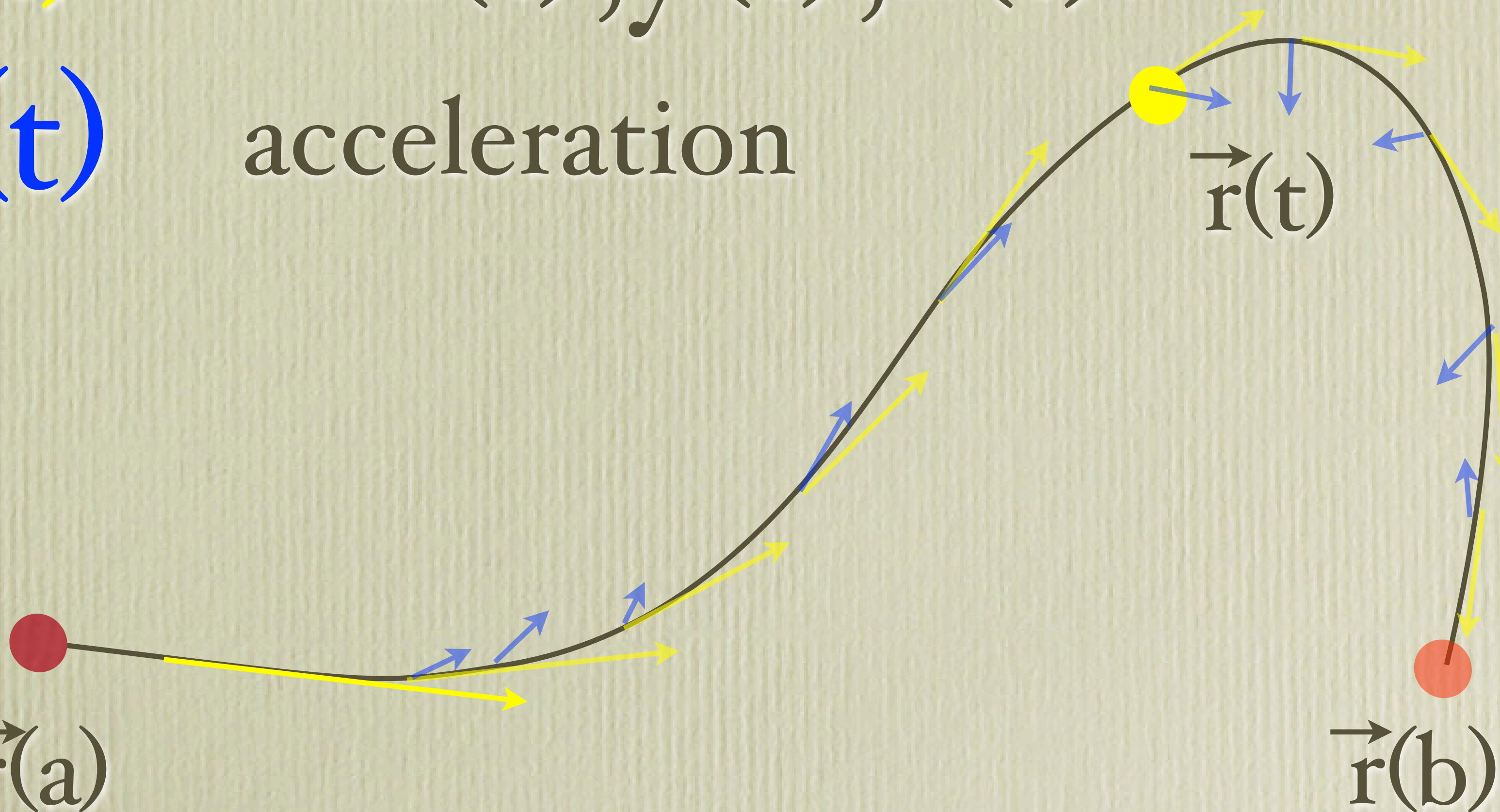
$$\vec{r}''(t)$$ acceleration



$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \quad \text{velocity}$$

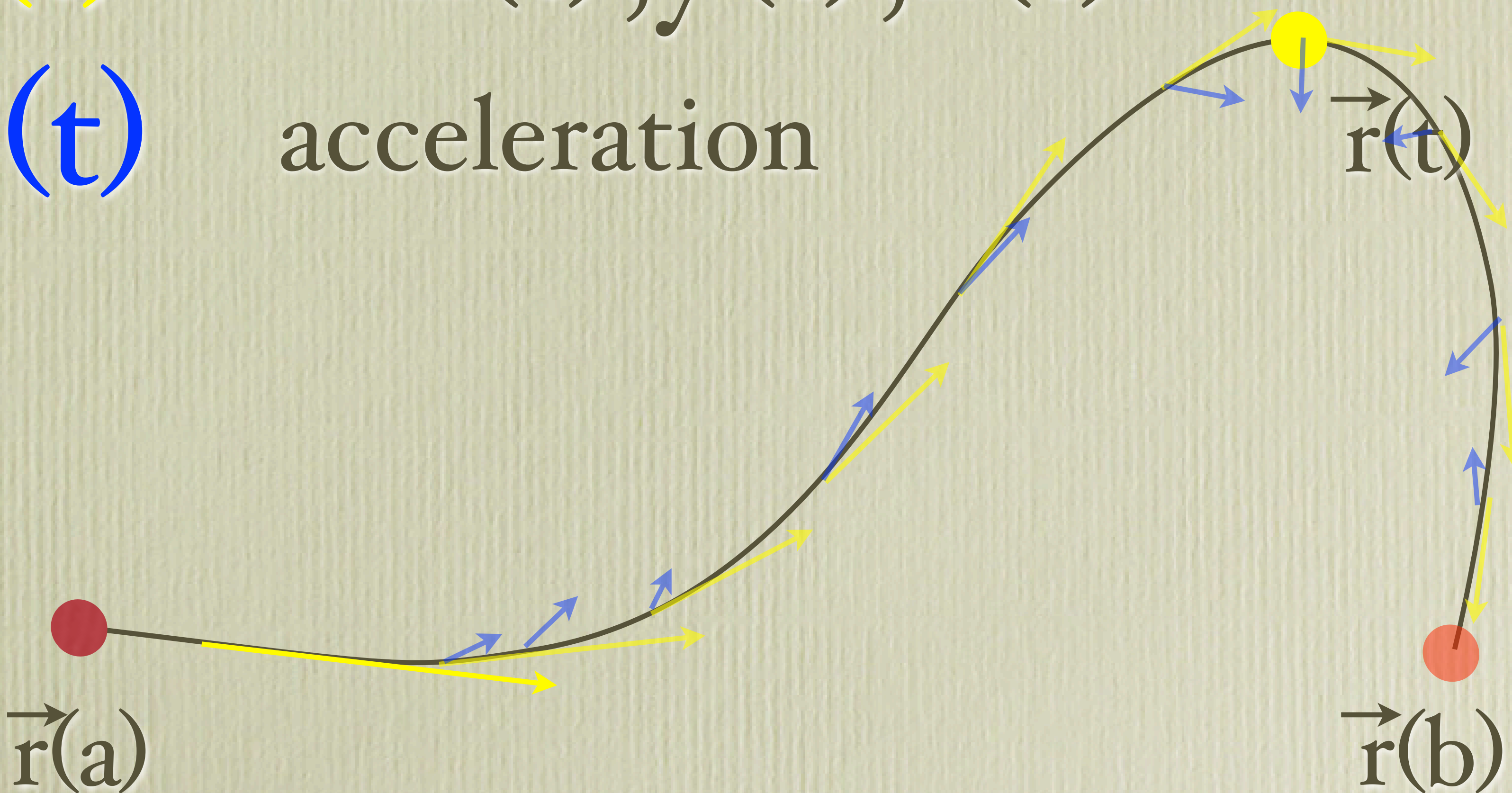
$$\vec{r}''(t) \quad \text{acceleration}$$



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ velocity

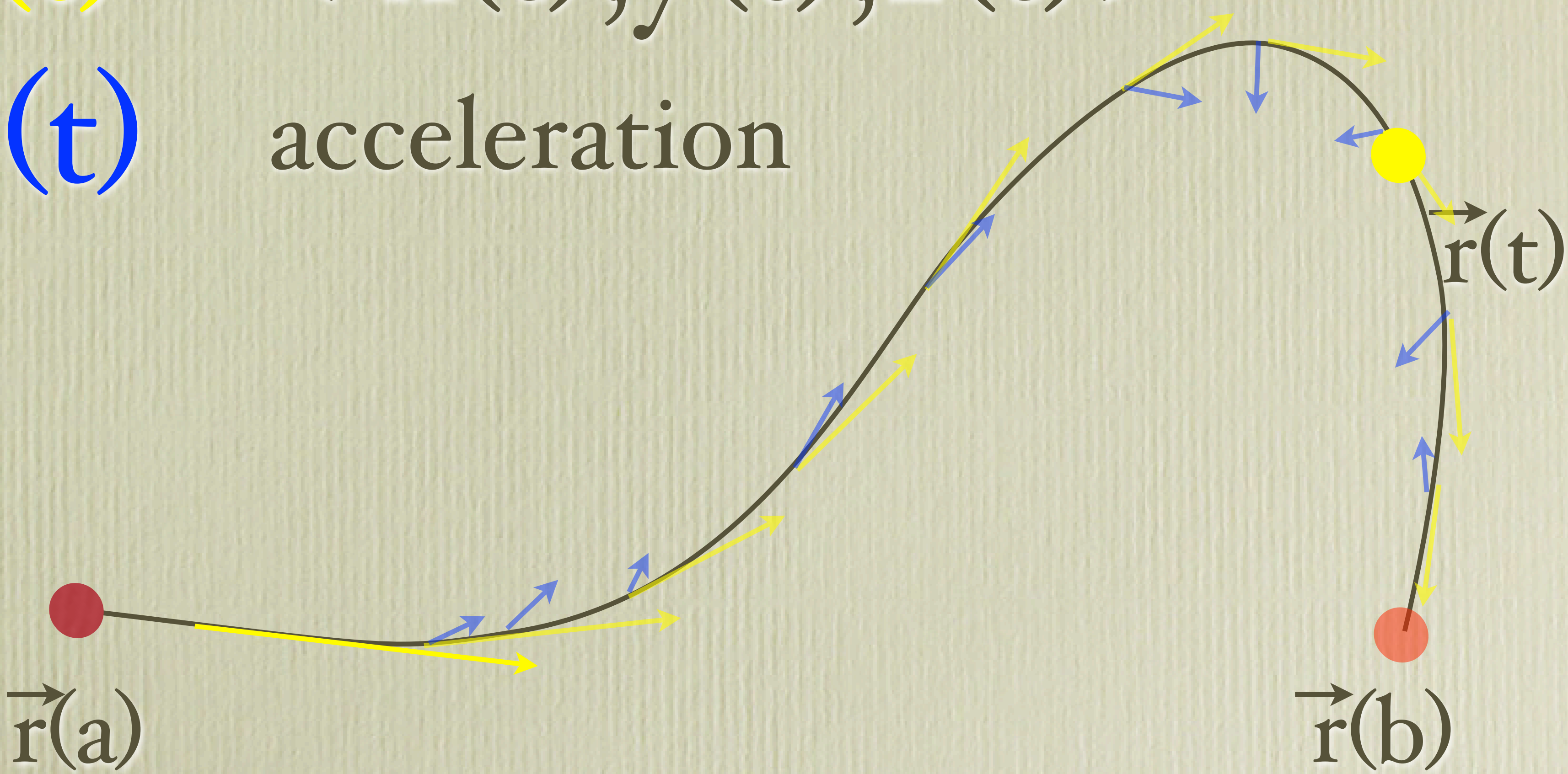
$\vec{r}''(t)$ acceleration



$\vec{r}(t) = < x(t), y(t), z(t) >$

$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$ velocity

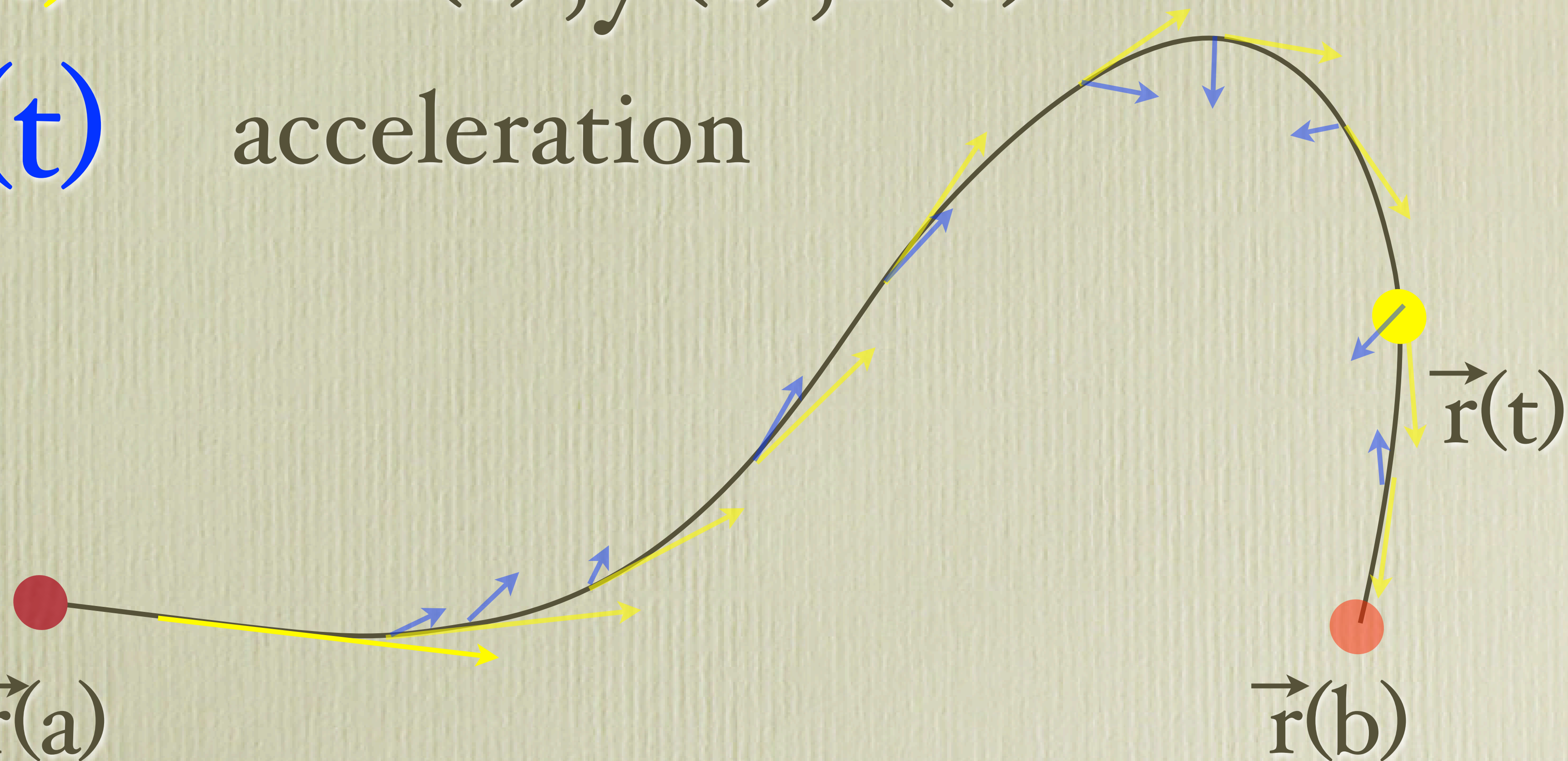
$\vec{r}''(t)$ acceleration



$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

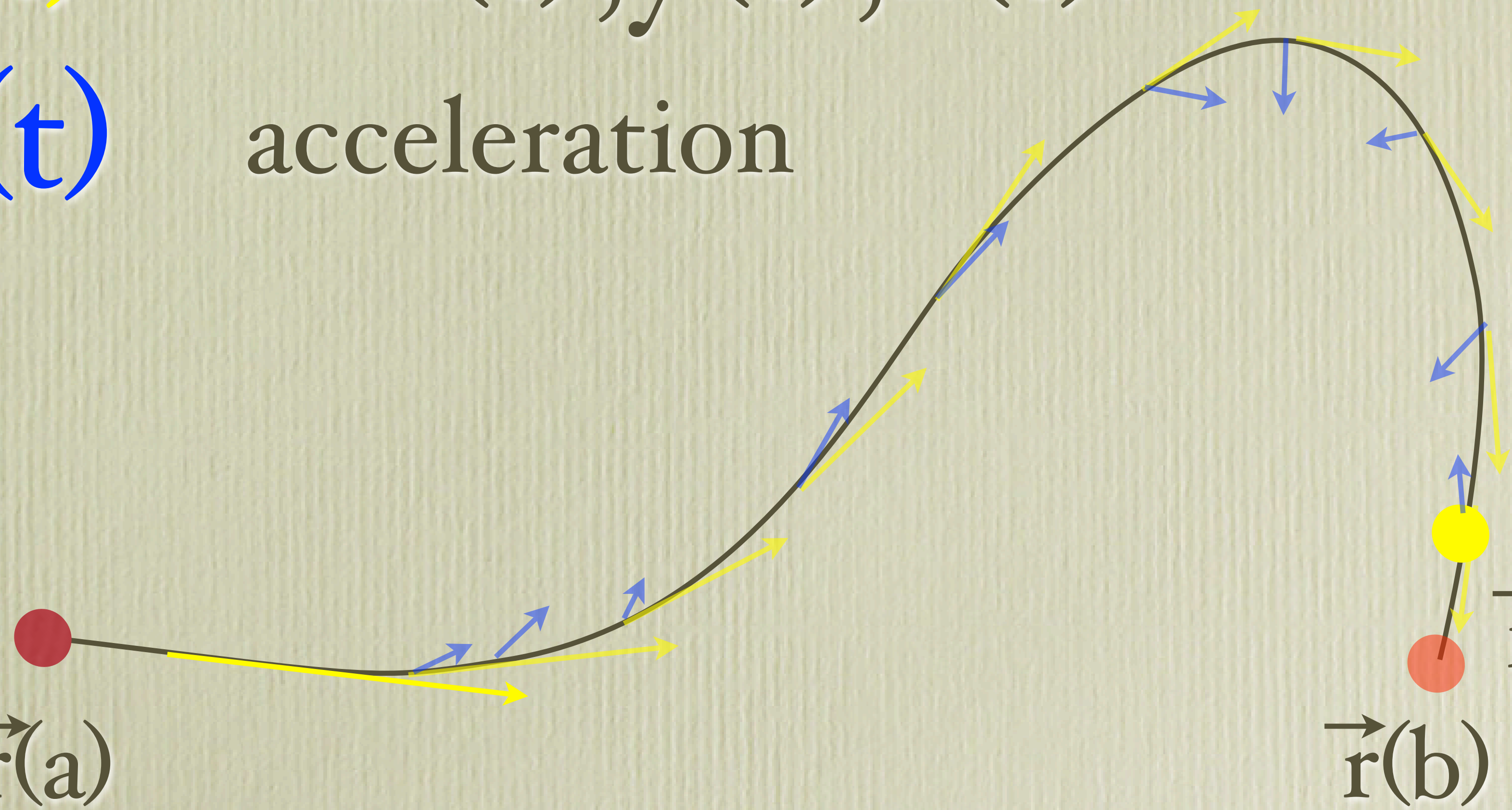
$$\vec{r}''(t)$$ acceleration

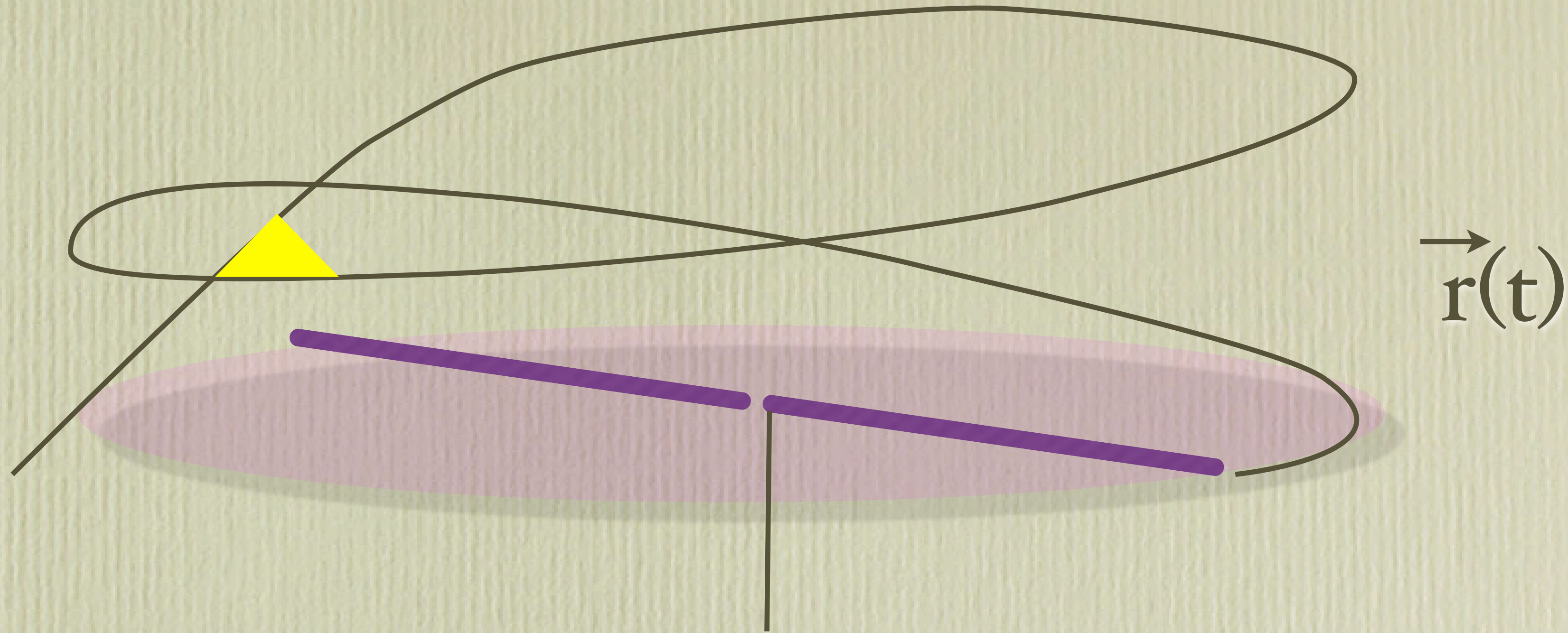


$$\vec{r}(t) = < x(t), y(t), z(t) >$$

$$\vec{r}'(t) = < x'(t), y'(t), z'(t) >$$ velocity

$\vec{r}''(t)$ acceleration





Problem: a helicopter blade moves on the curve

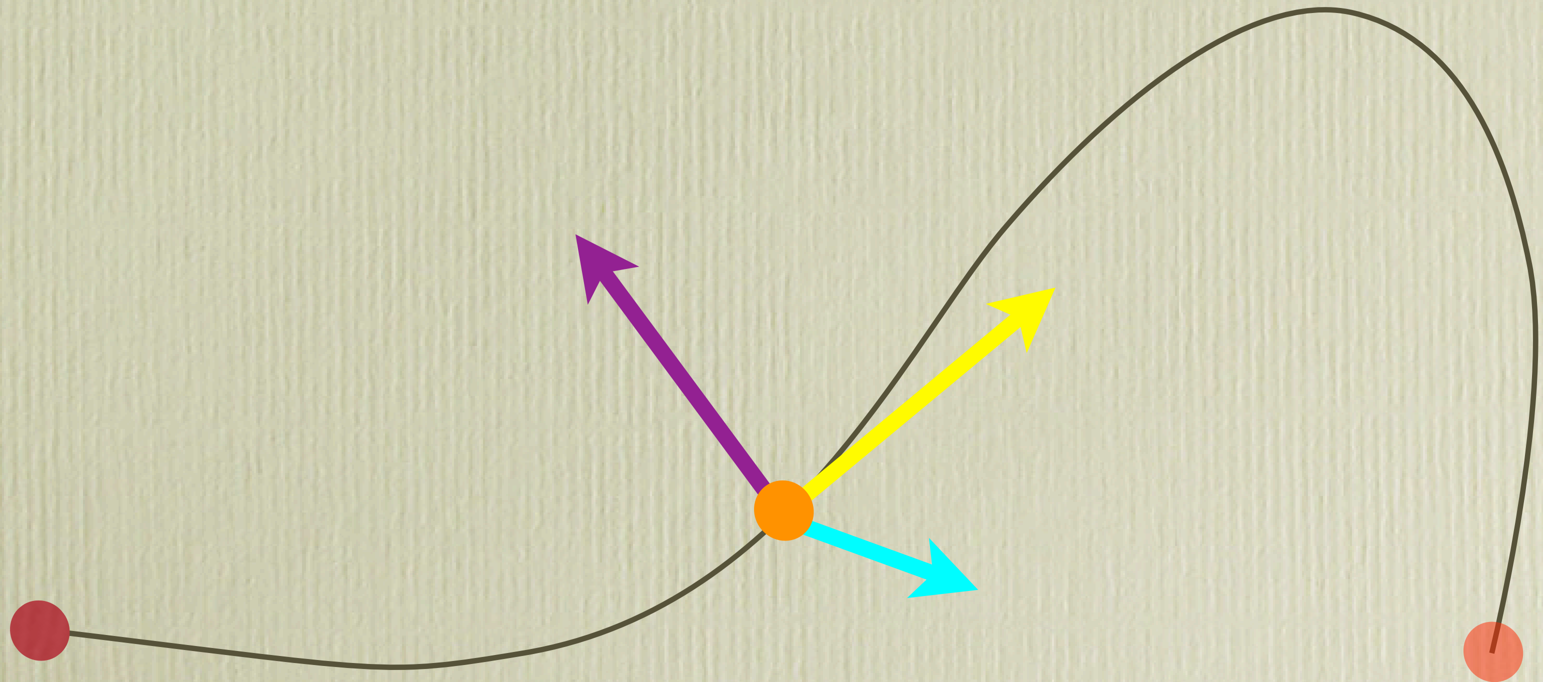
$\vec{r}(t) = < \cos(2t), \sin(2t), \sin(t) >$

Under which angle does the path hit itself at (1,0,0)

$$\vec{T}(t) = \vec{r}'(t)/|\vec{r}'(t)|$$

$$\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$



Question:

Oliver (virtually) flies over Harvard in the plane

$$4x + 2y - 4z = 12$$

What is the z component of the binormal vector B ?

Arc Length and Curvature

$$L = \int_a^b \left| \vec{r}'(t) \right| dt$$

$\vec{r}(b)$

$$\kappa(t) = \frac{\left| \vec{r}'(t) \times \vec{r}''(t) \right|}{\left| \vec{r}'(t) \right|^3}$$

$\vec{r}(a)$

Independent of parametrization

Problem 3

The helicopter crashes along the curve

$\vec{r}(t) = < t^2/2 , 1, -t^3/3 >$

From t=0 to t=1. Find the arc length of that path.

Problem

The helicopter is subject to a wind force:

$$\vec{r}''(t) = \langle t^2, 1, -t^3 \rangle$$

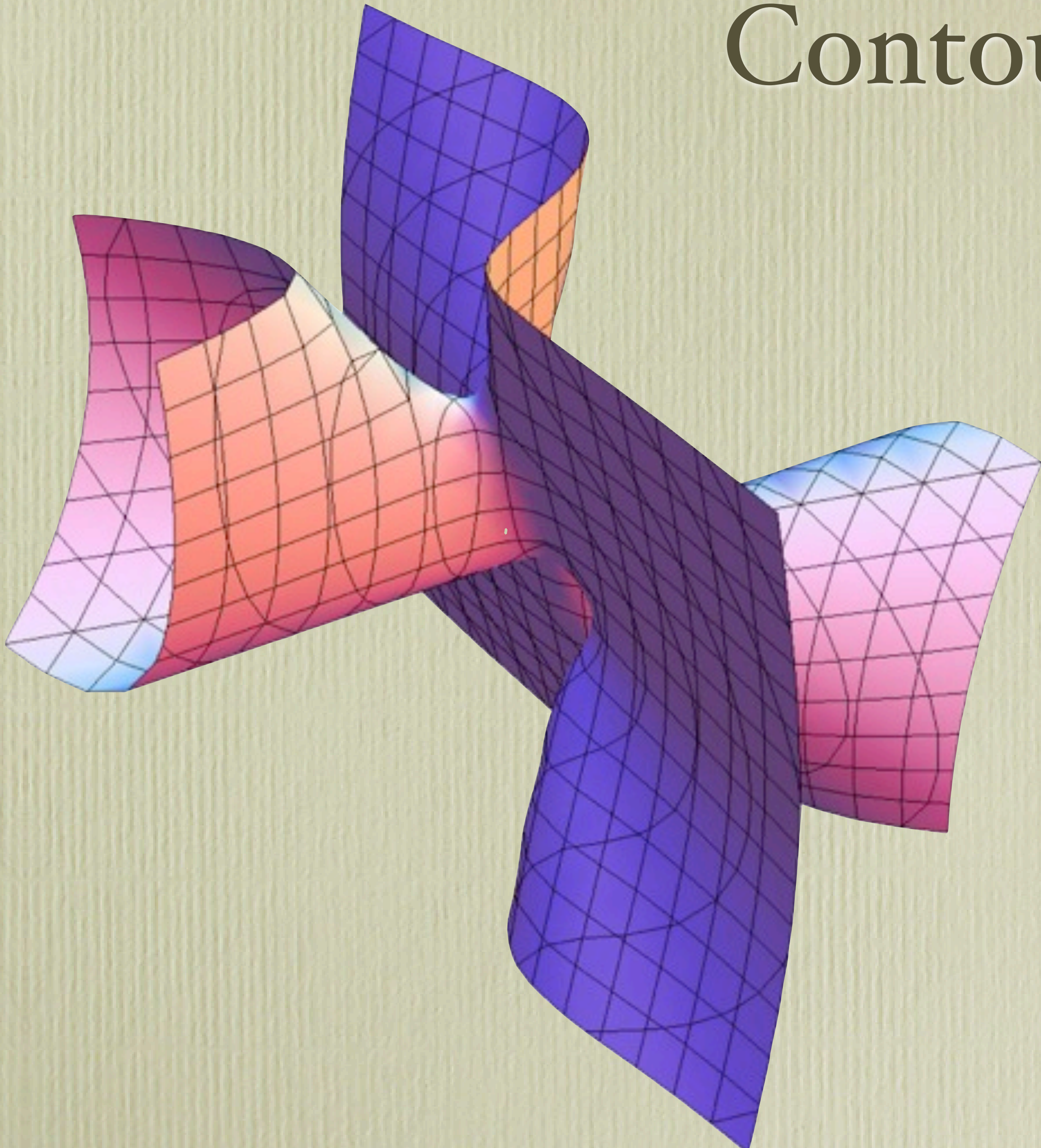
Where is it at time t=1 if

$$\vec{r}(0) = \langle 0,1,0 \rangle$$

$$\vec{r}'(0) = \langle 1,0,0 \rangle$$

(hit by the cat)

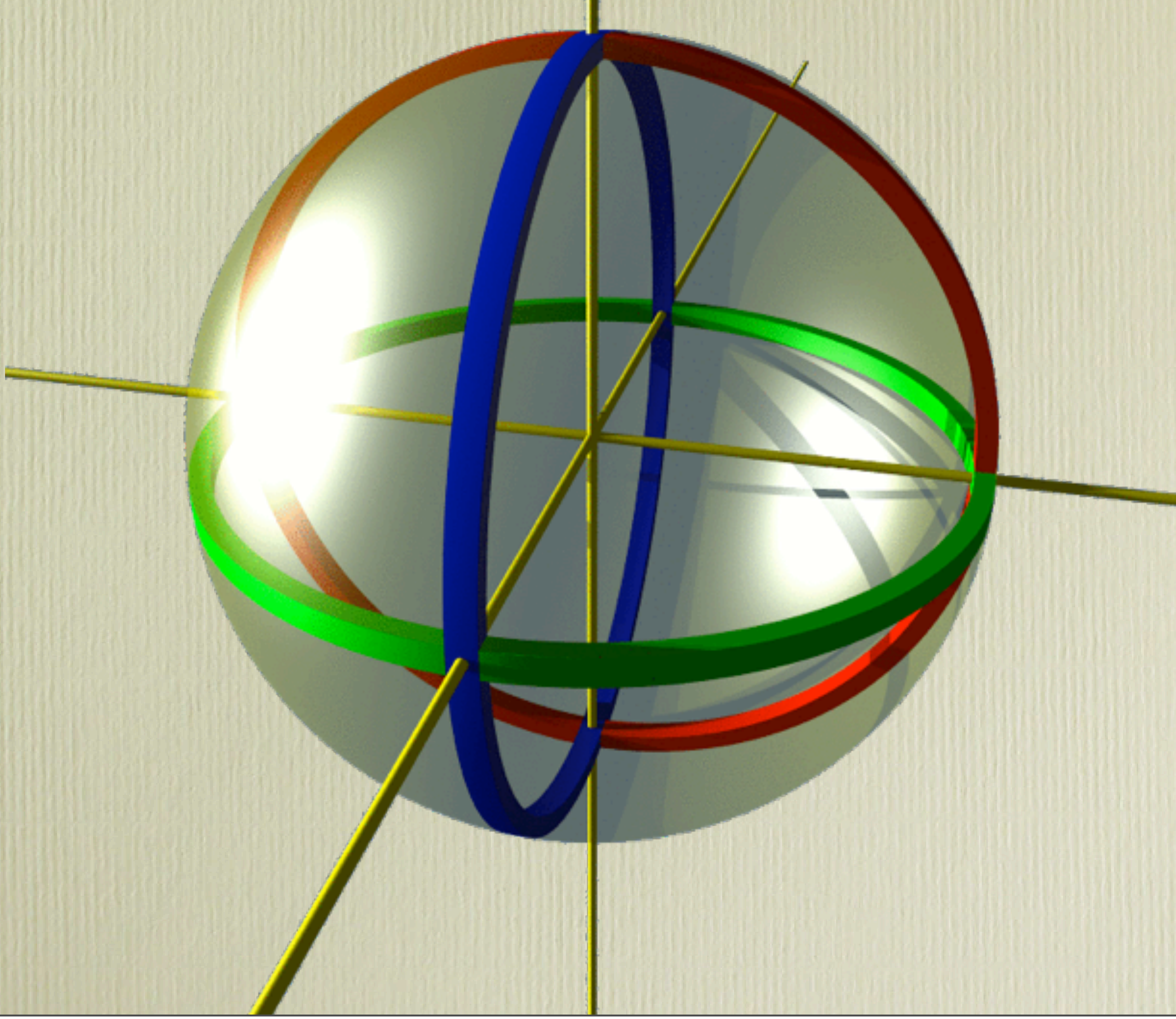
Contour Surfaces



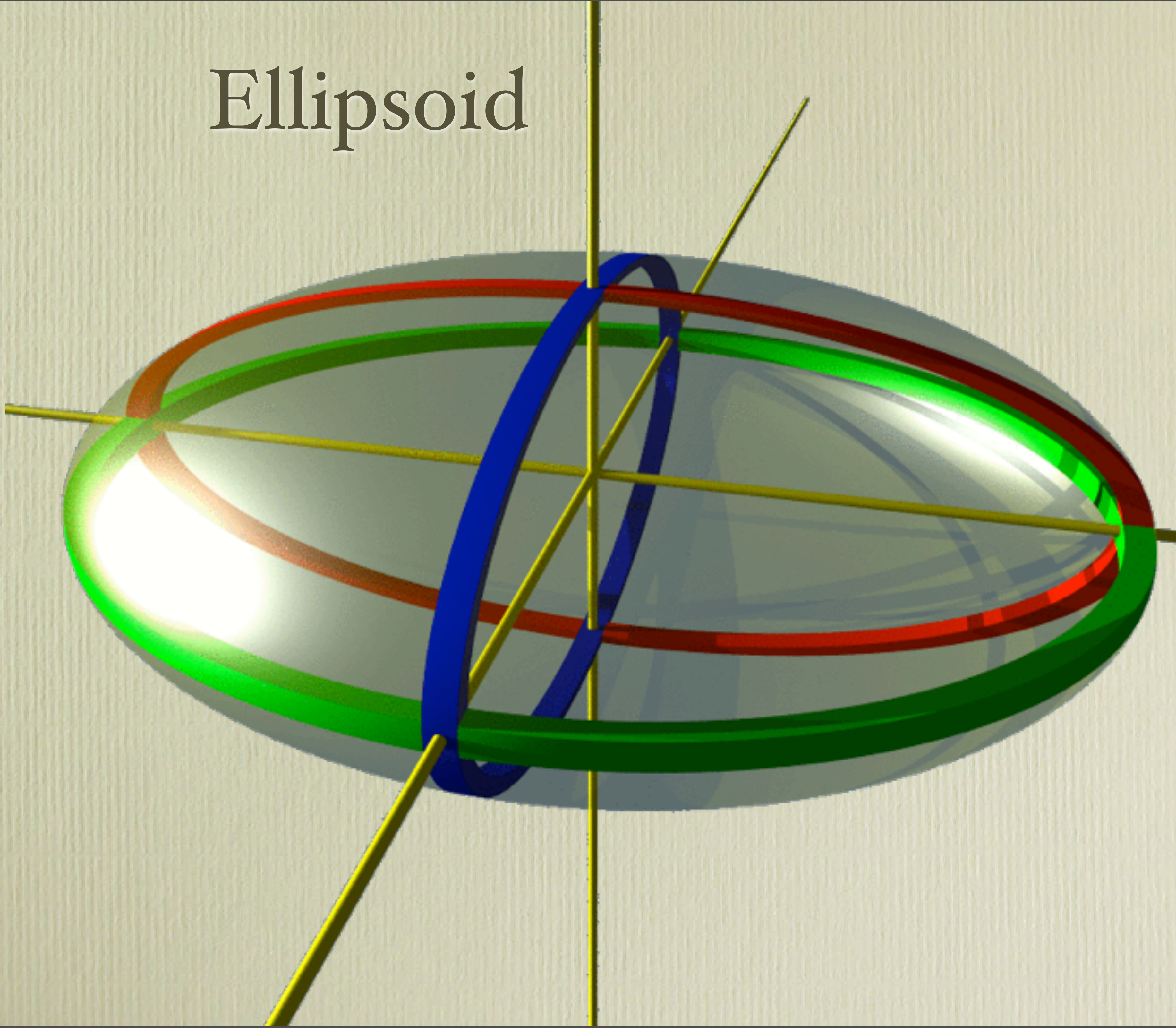
$g(x,y,z) = c$

ContourPlot3D[x^3+y^3+x z^2-2x y^2+z+x^2 z+x y z-3, {x,-6,6}, {y,-6, 6}, {z,-6, 6}]

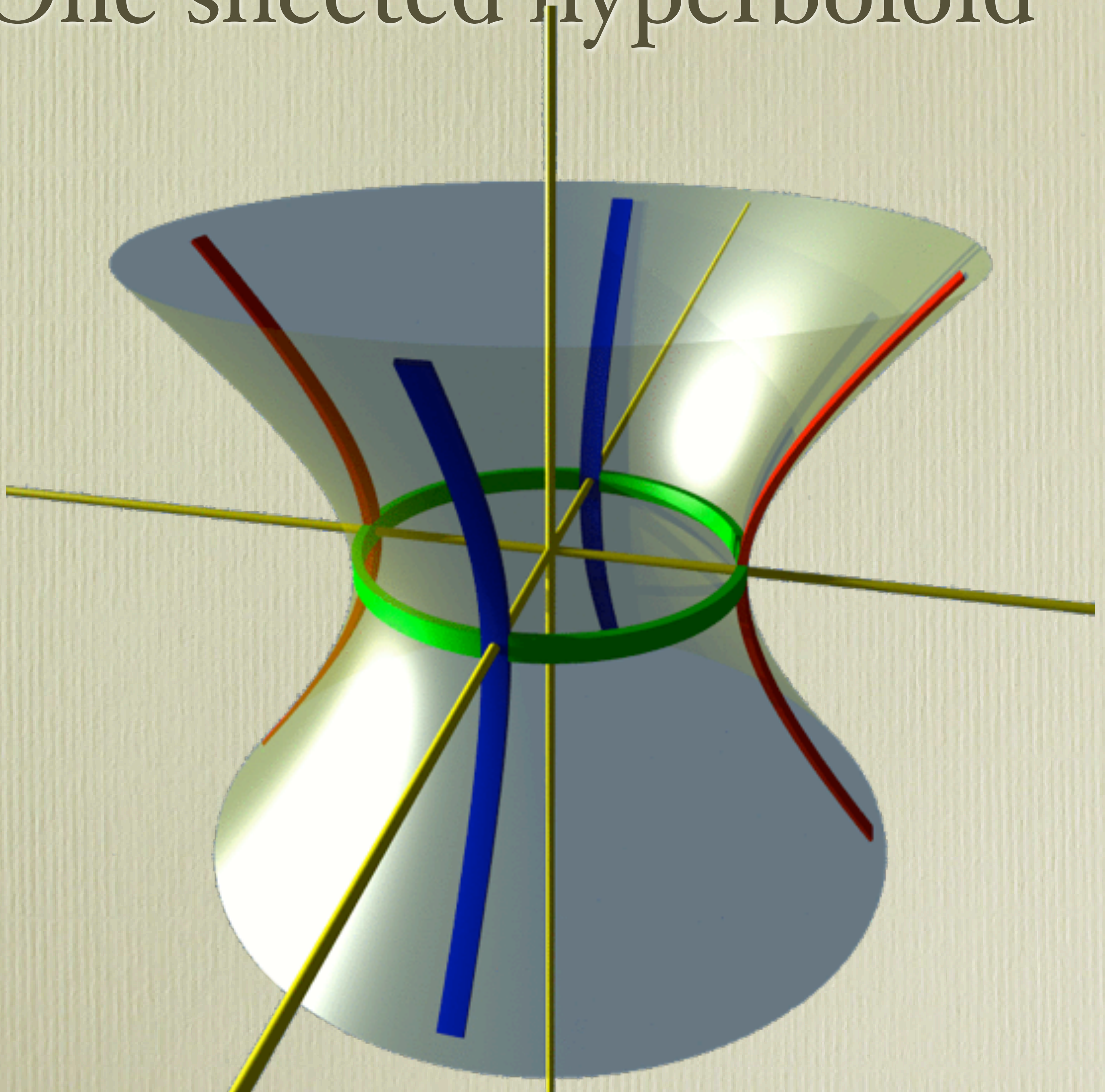
Spheres



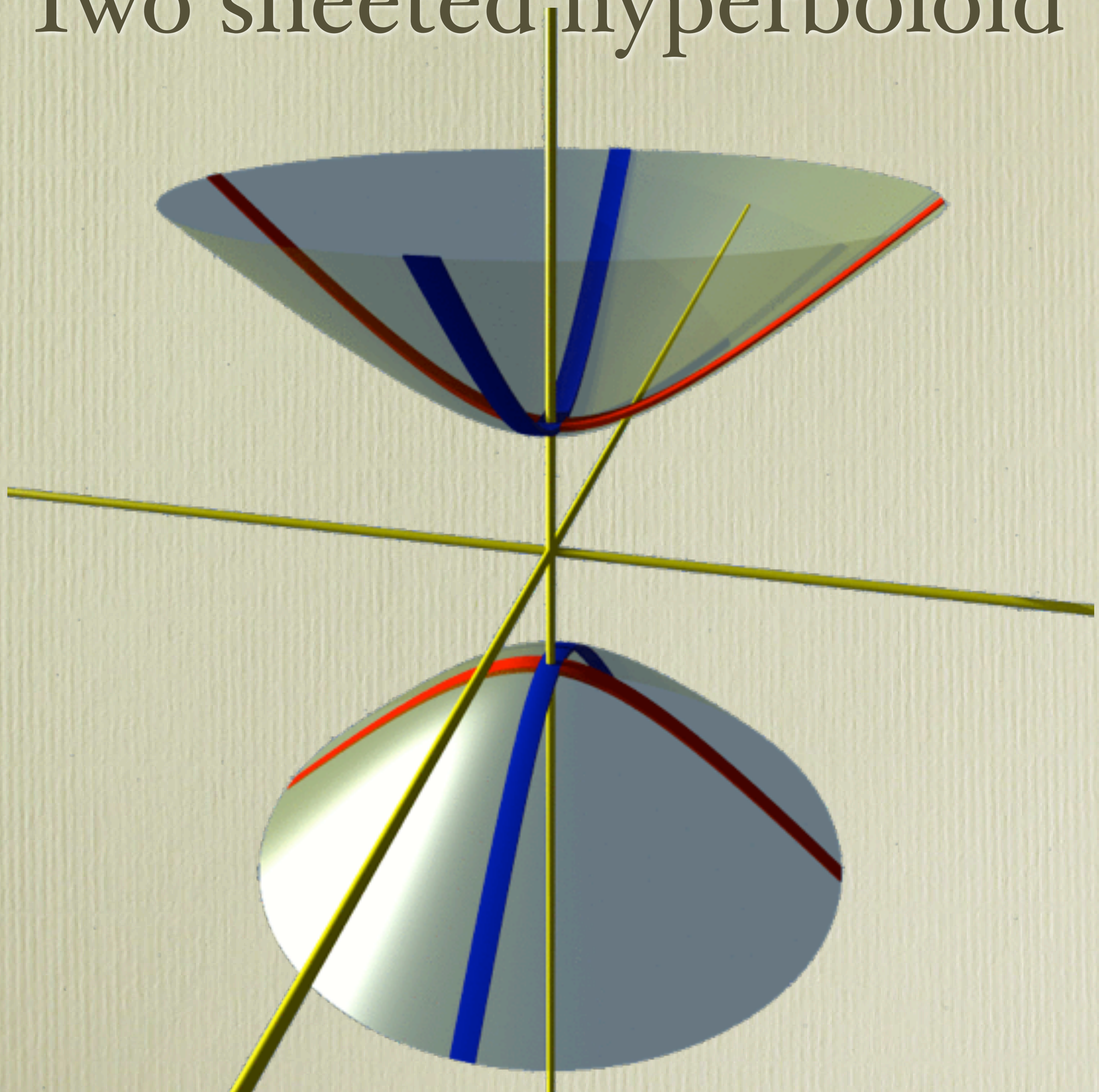
Ellipsoid



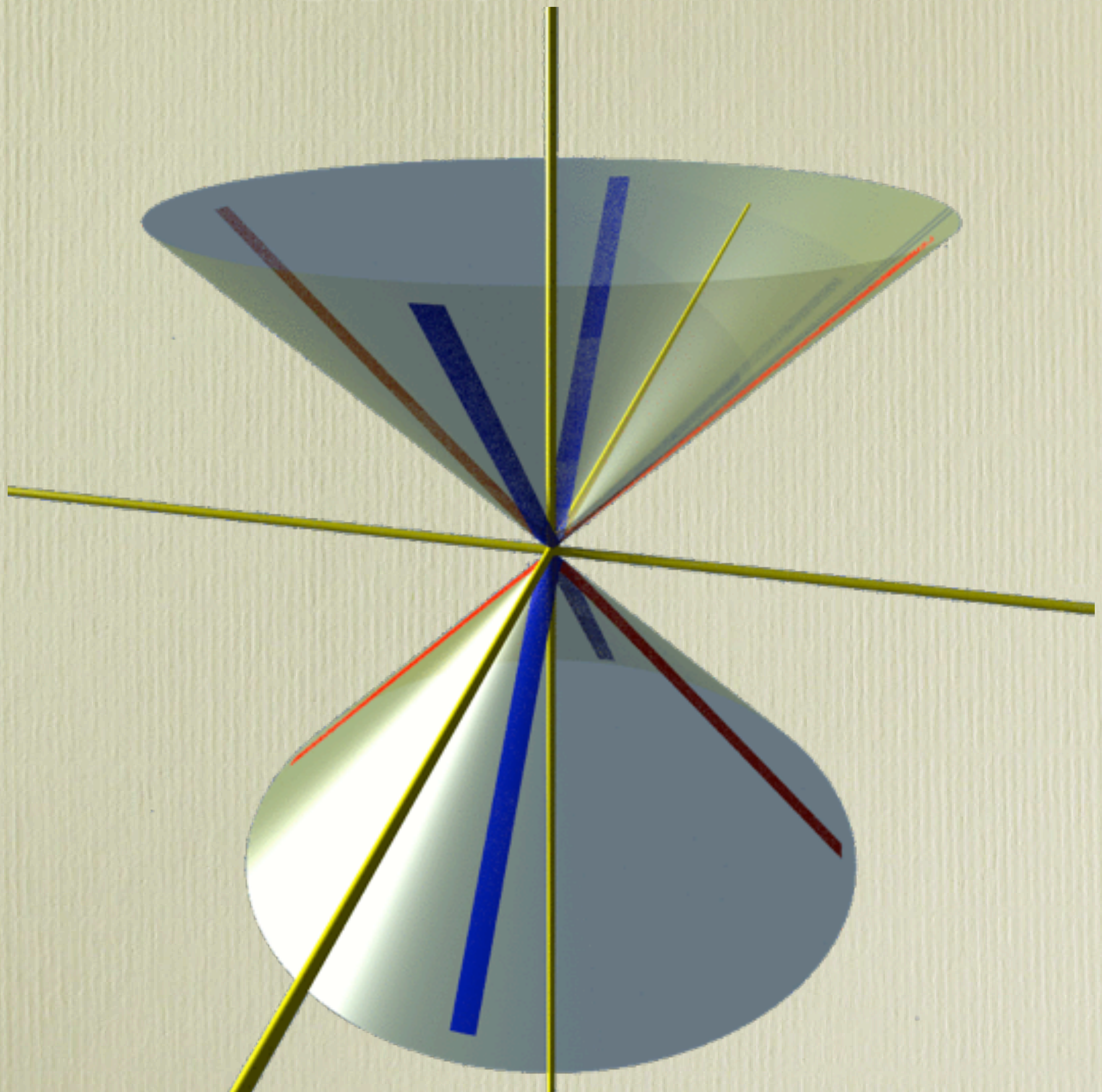
One sheeted hyperboloid



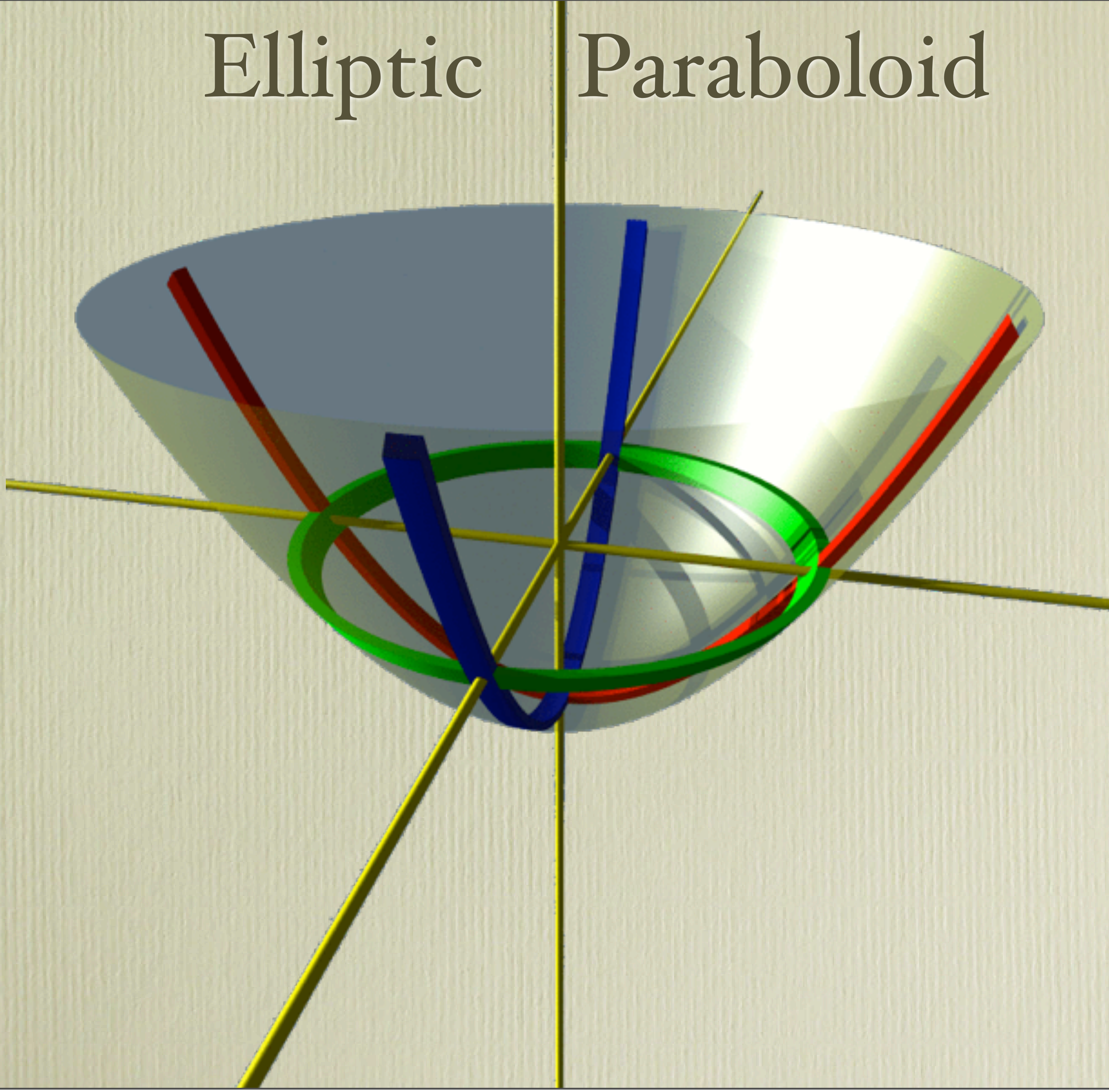
Two sheeted hyperboloid



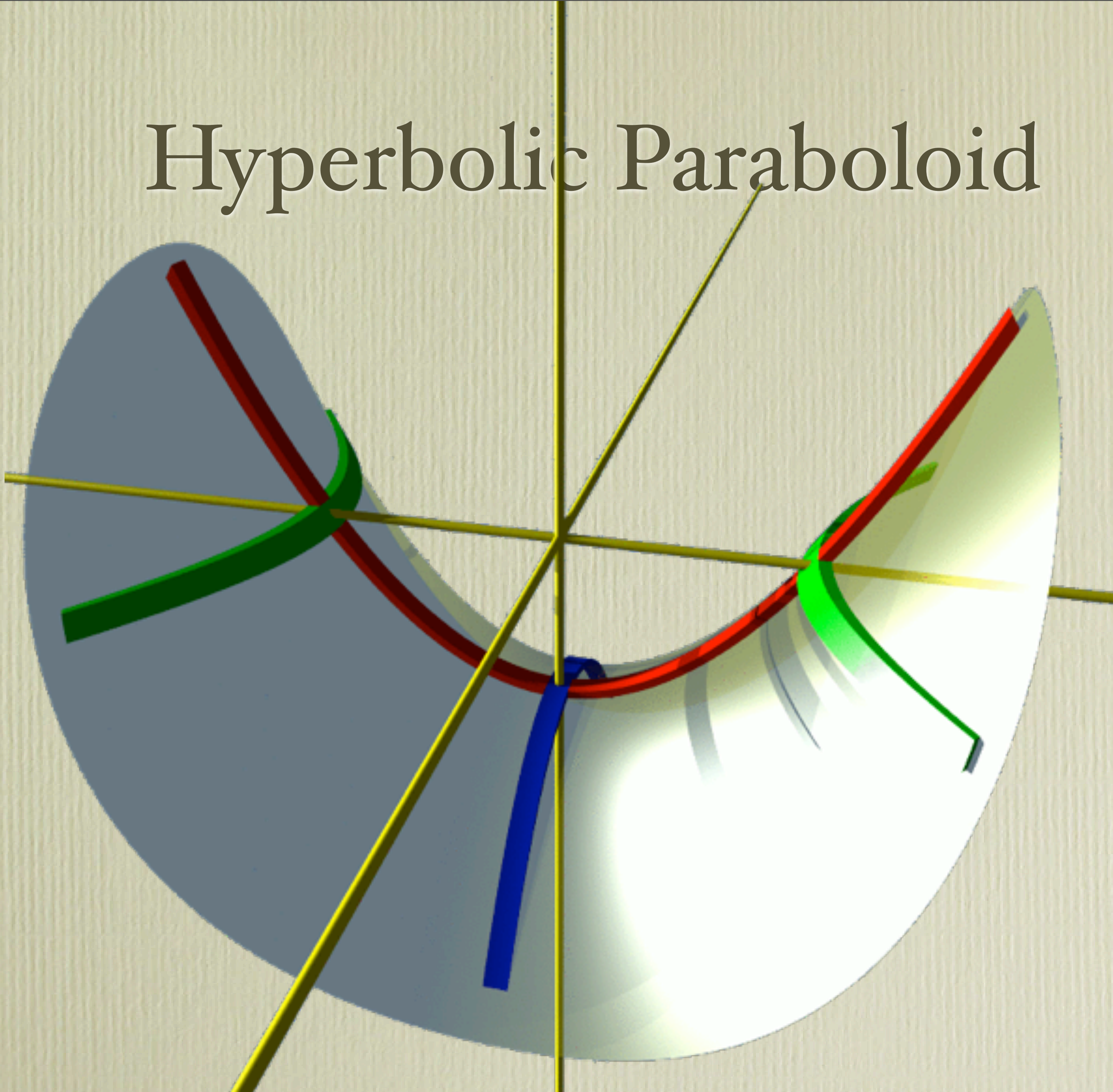
Cone



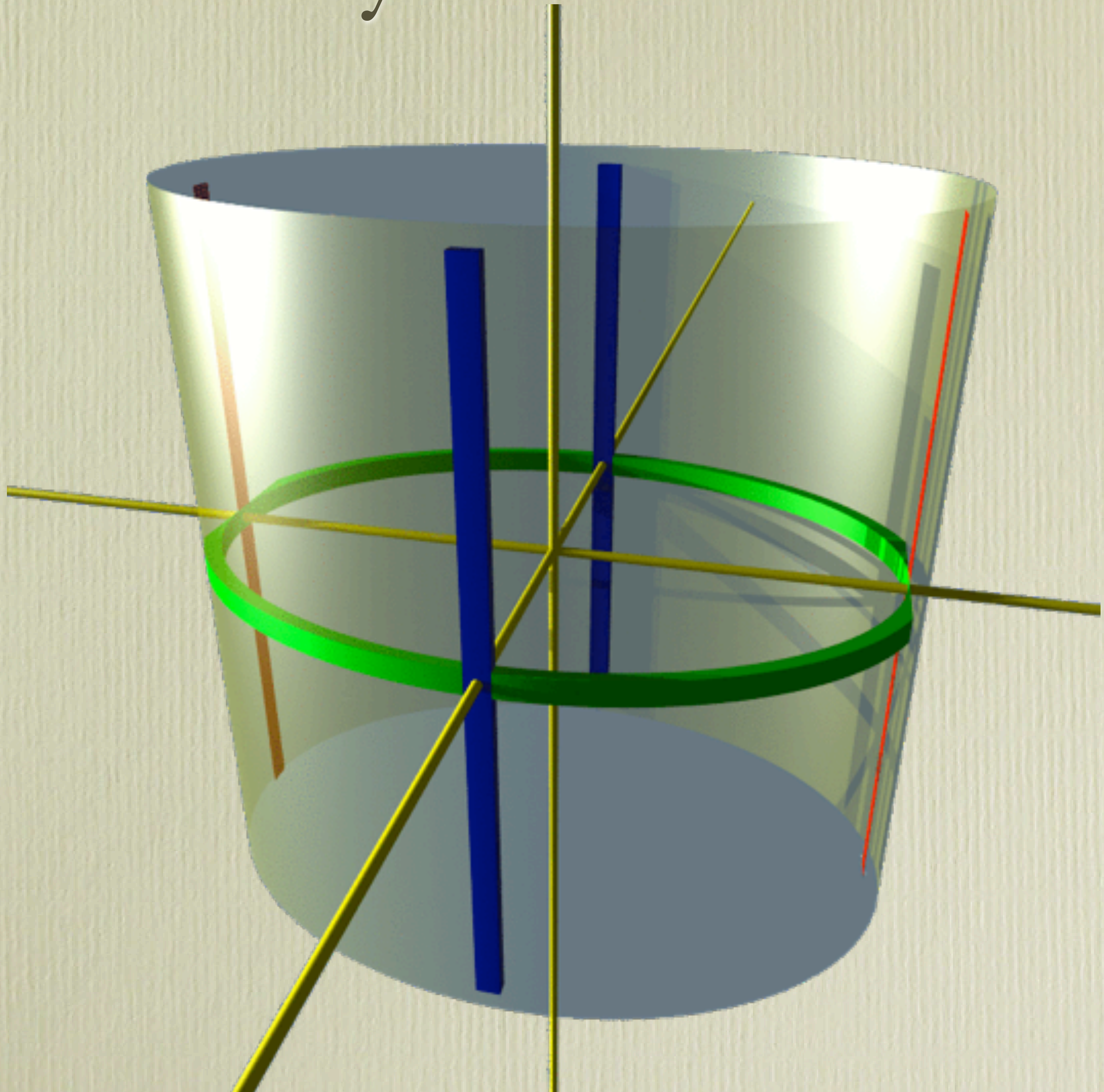
Elliptic Paraboloid



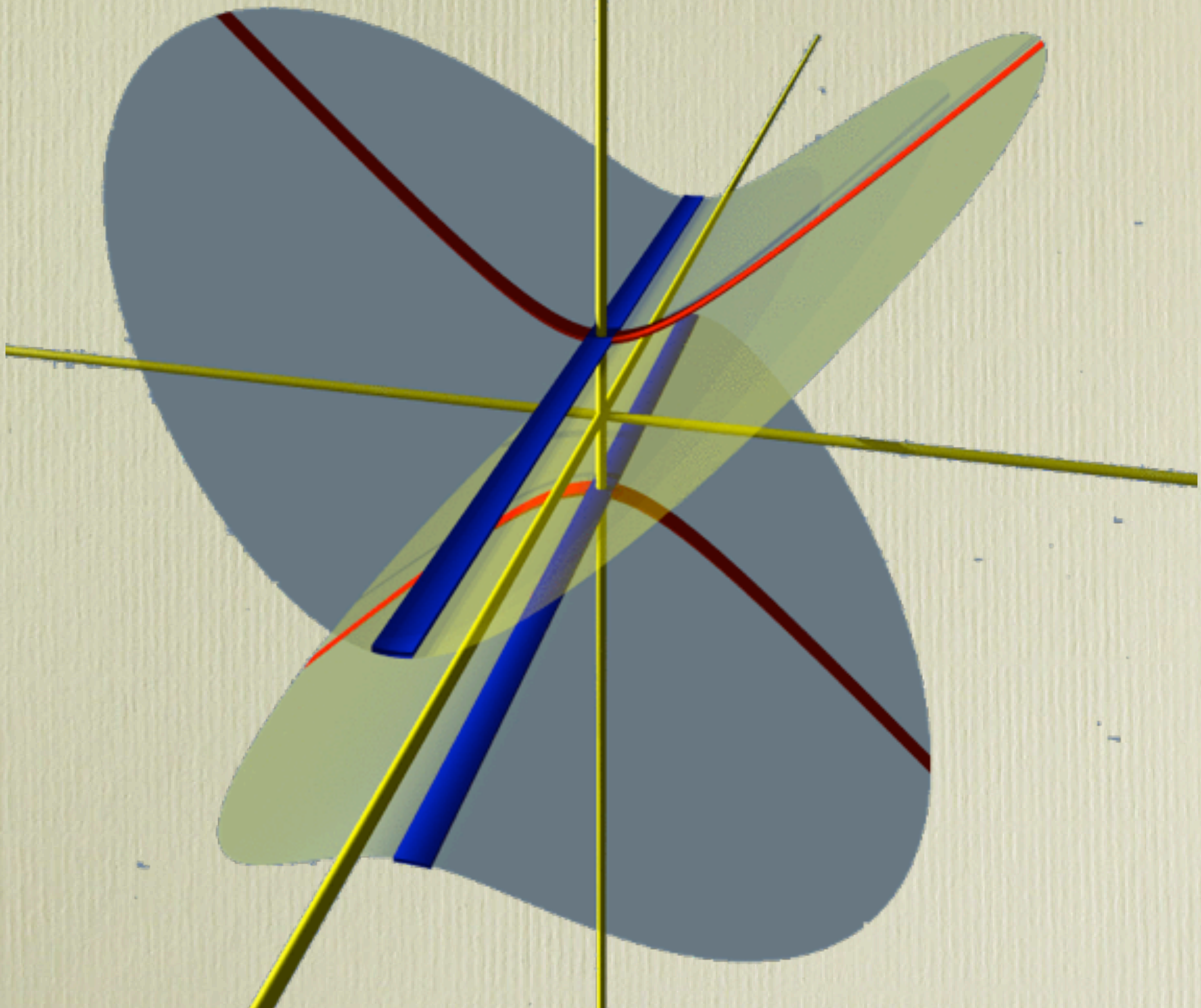
Hyperbolic Paraboloid



Cylinder



Cylindrical hyperboloid



Polar coordinates

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Problem:

What surface is this:

$$x^2 - y^2 - z^2 = 2y$$

Cylindrical coordinates

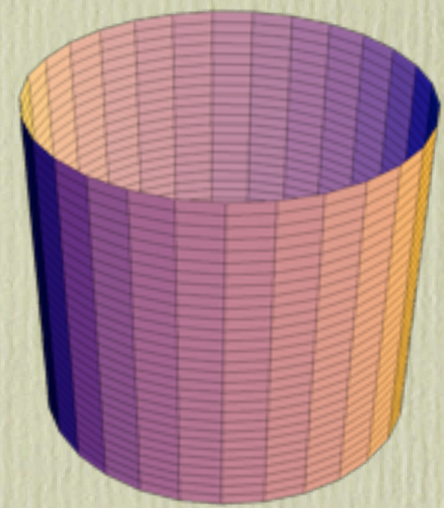
$$x = r\cos(\theta)$$

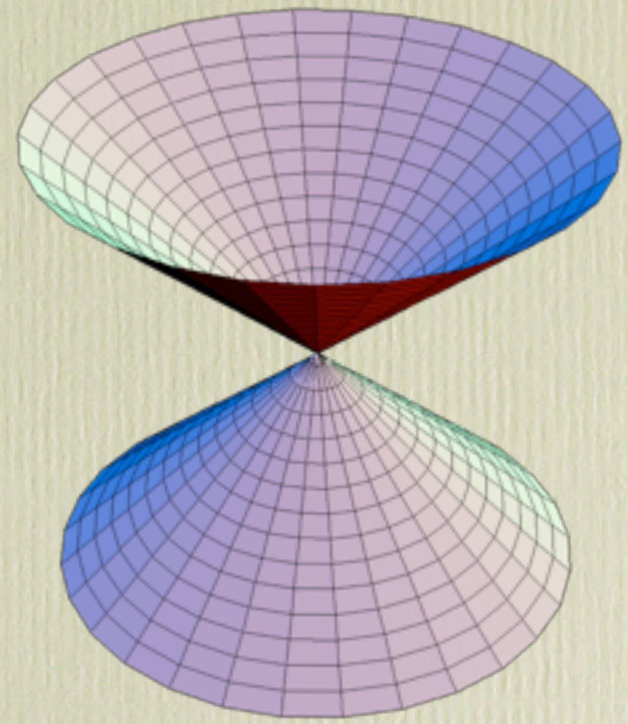
$$y = r\sin(\theta)$$

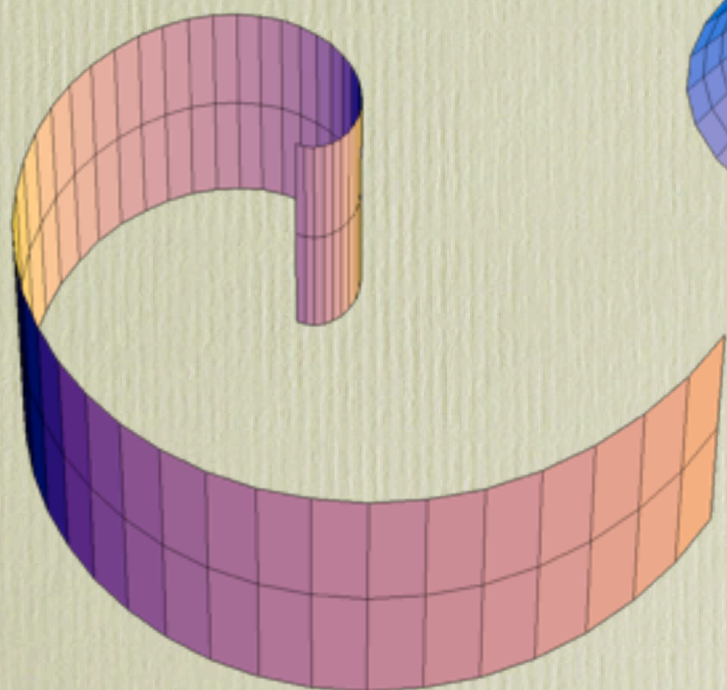
$$z = z$$

Surfaces in Cylindrical coordinates

- $r=1$
- $r=z$
- $r = \theta$







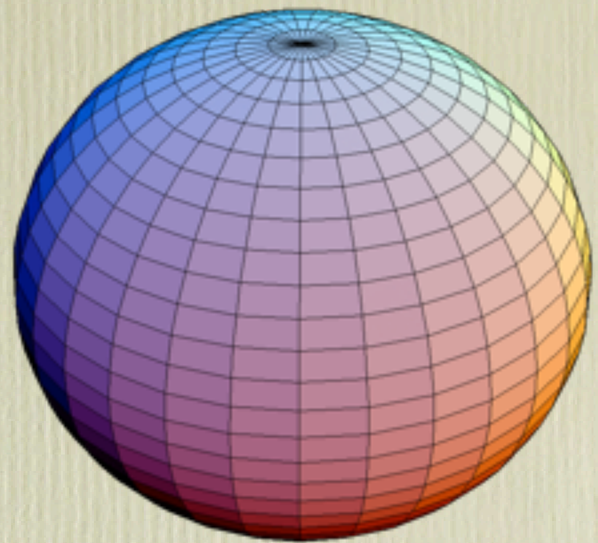
Spherical coordinates

$$x = \rho \sin(\phi) \cos(\theta)$$

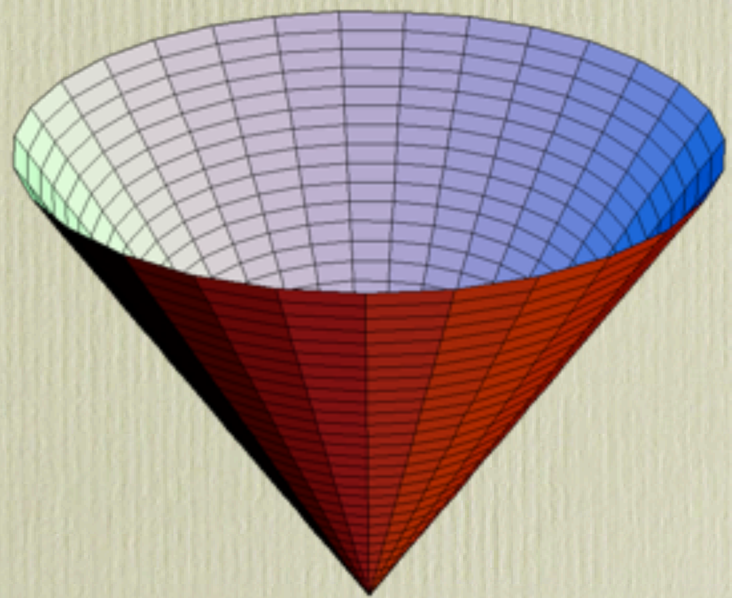
$$y = \rho \sin(\phi) \sin(\theta)$$

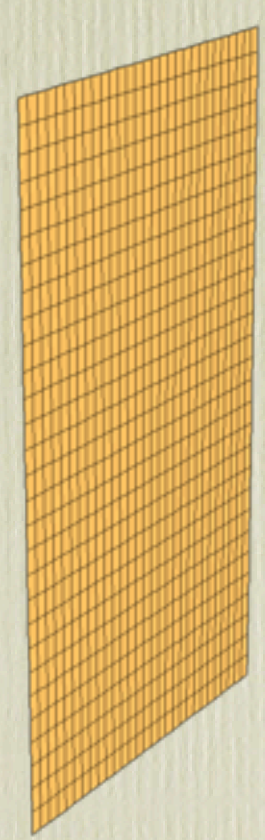
$$z = \rho \cos(\phi)$$

Surfaces in Spherical coordinates



- $\rho = 1$
- $\phi = 1$
- $\theta = 1$





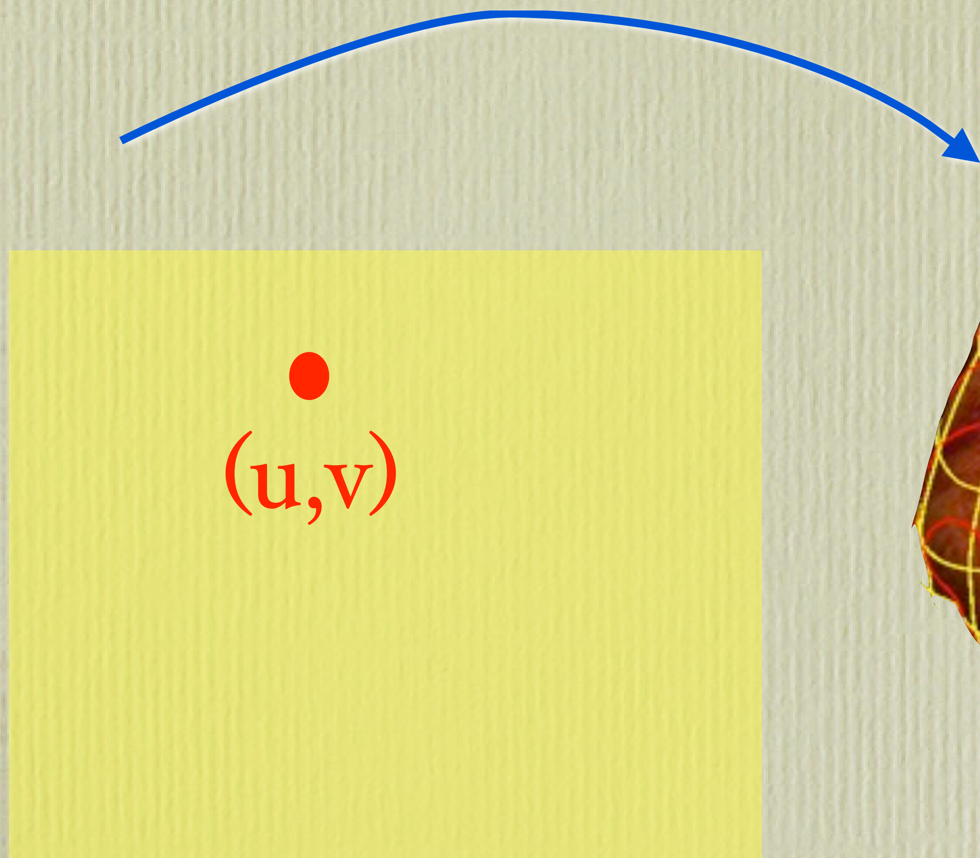
Problem:

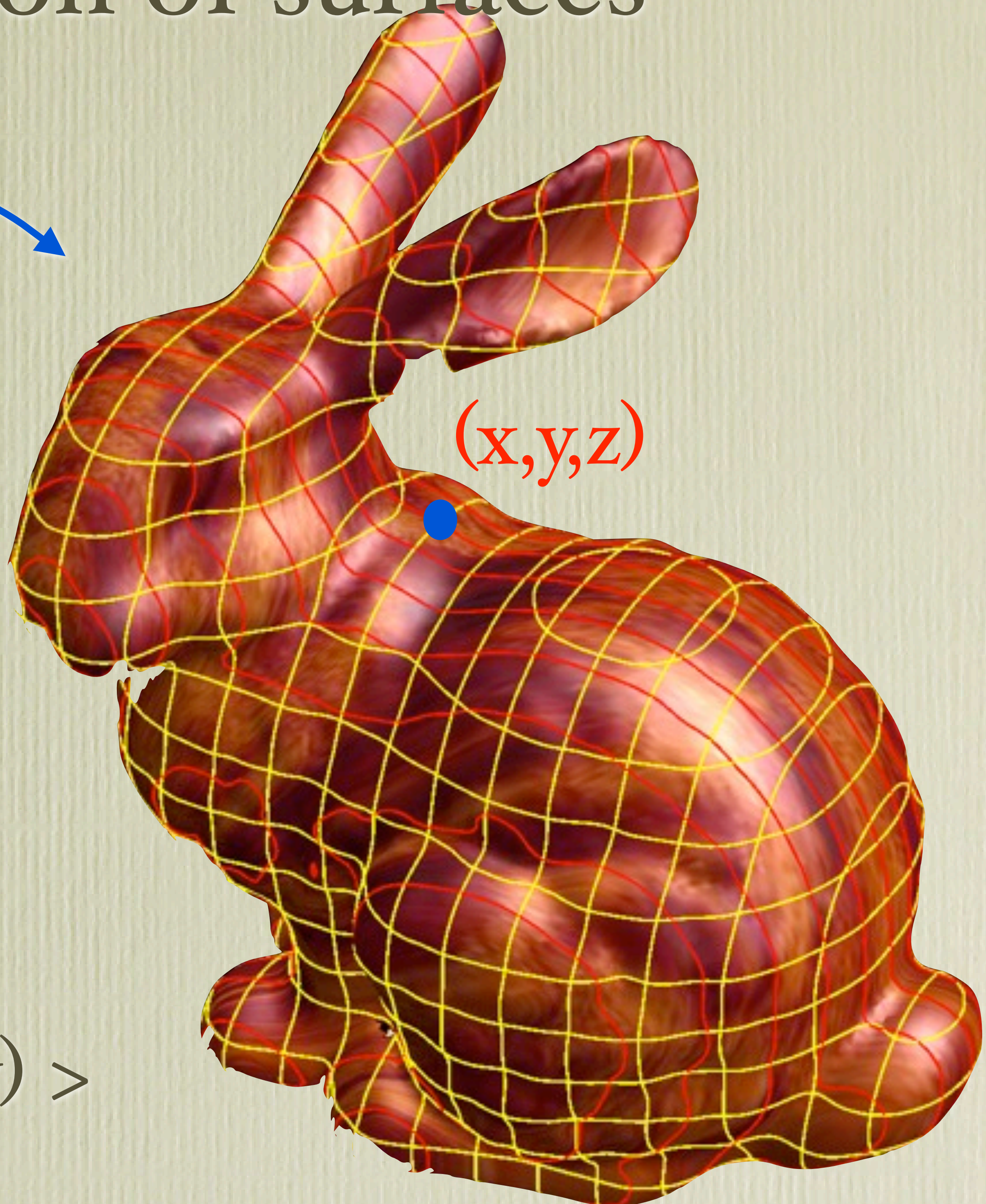
What surface is this:

$$r \cos(2\theta) = z$$

Parametrized Surfaces

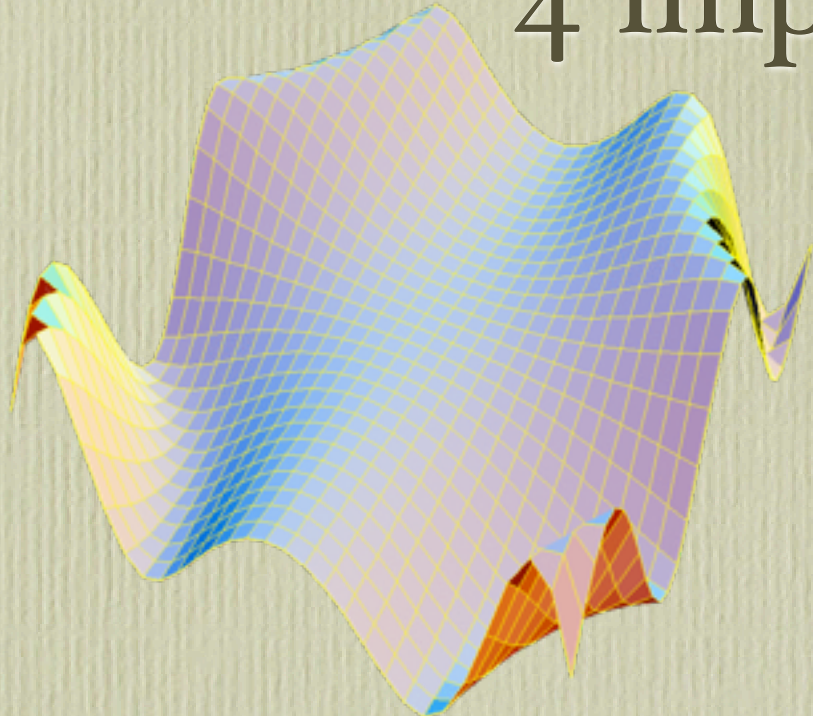
Parametrization of surfaces

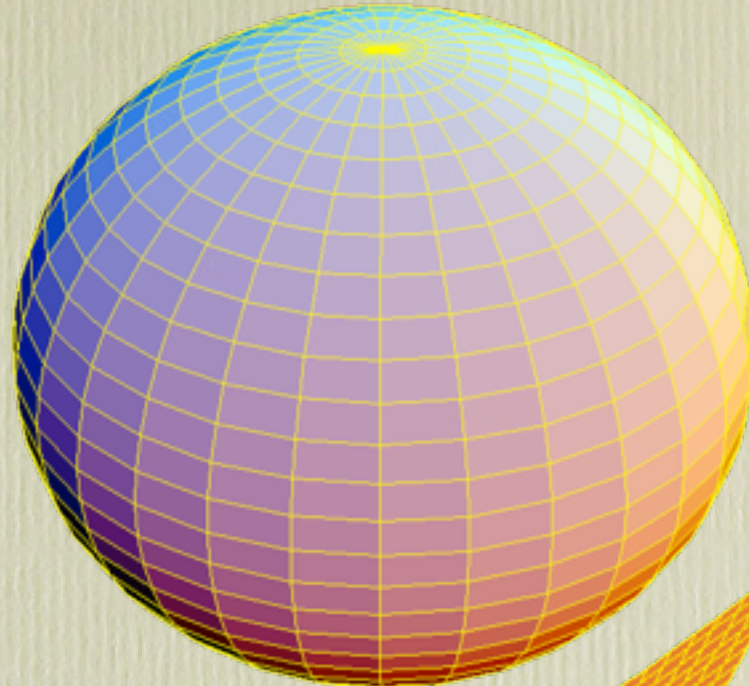


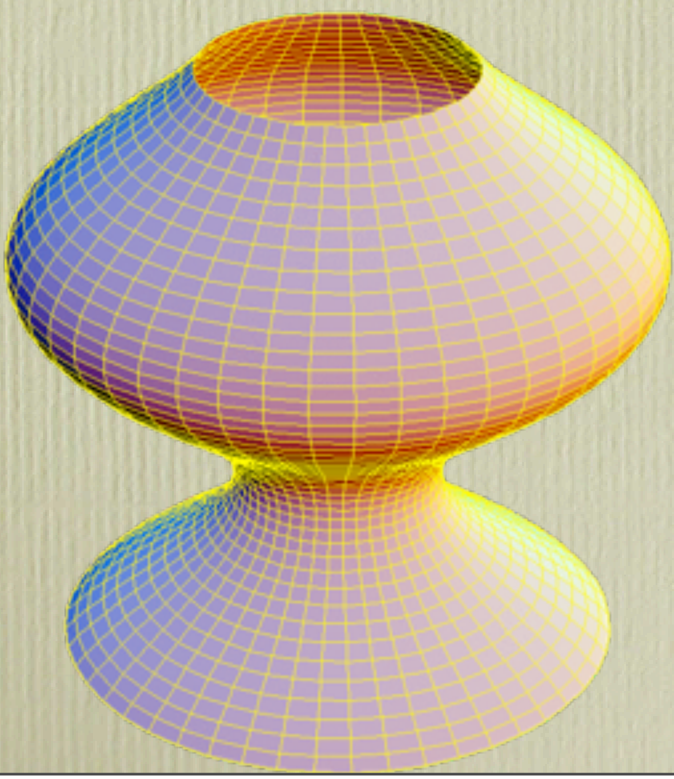


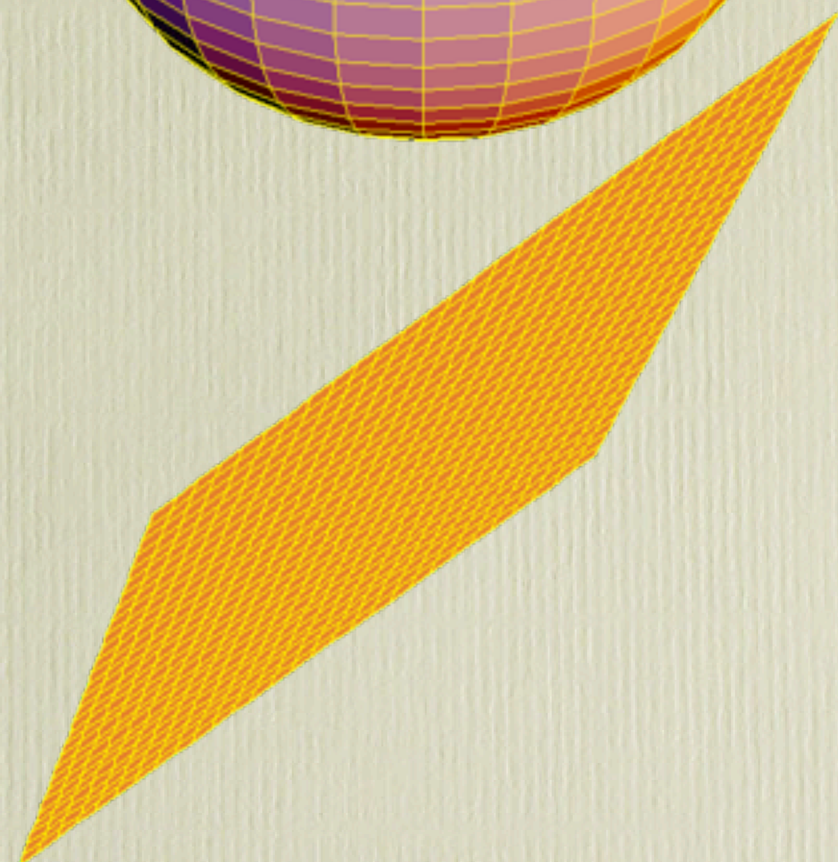
$$\vec{r}(u,v) = \langle\, x(u,v), y(u,v), z(u,v)\,\rangle$$

4 important classes:



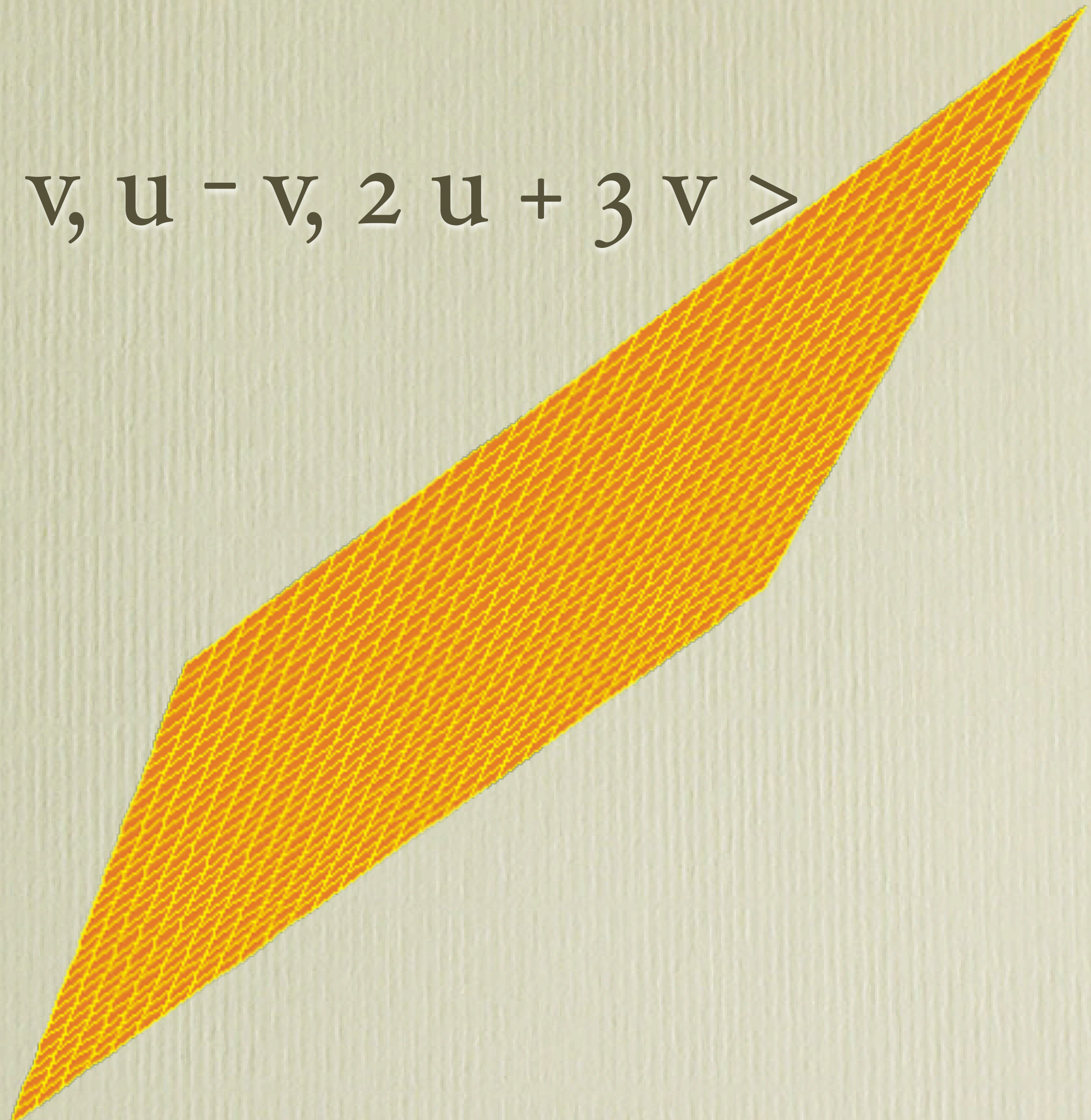




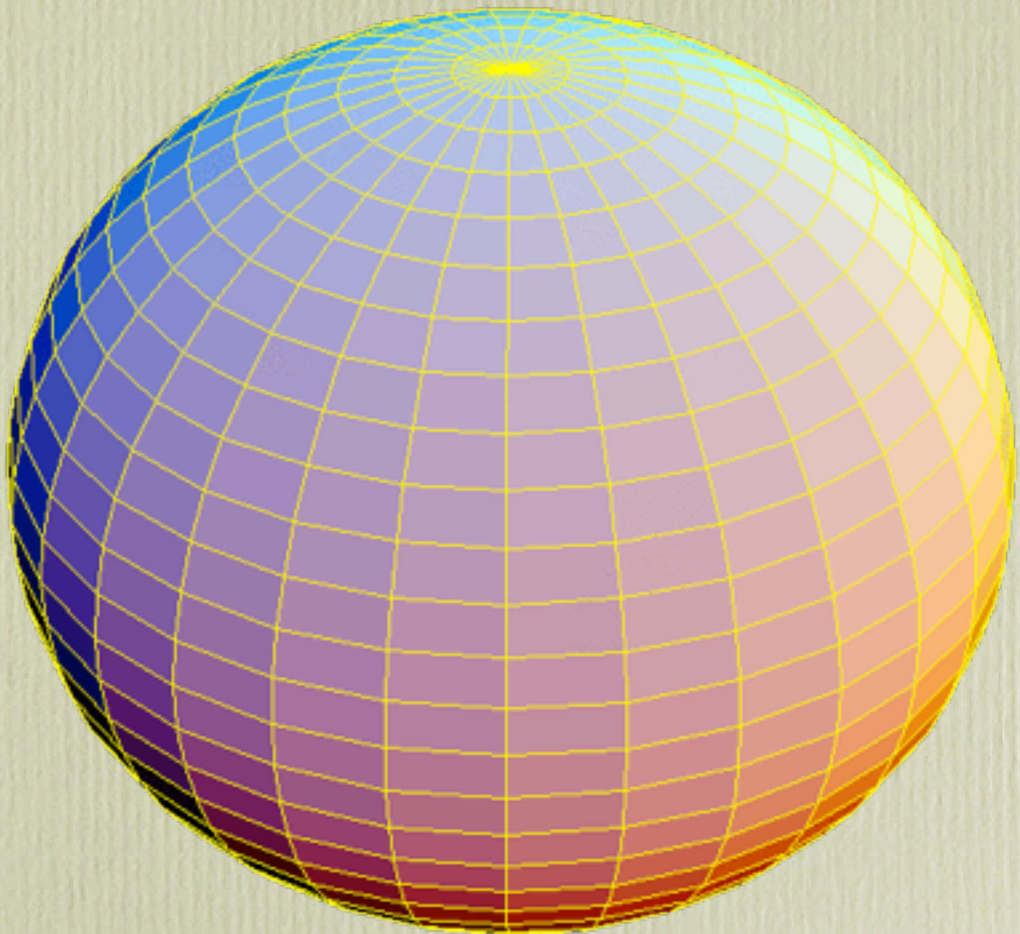


Planes

$\vec{r}(u,v) = \langle 1 + u + v,\ u - v,\ 2u + 3v \rangle$

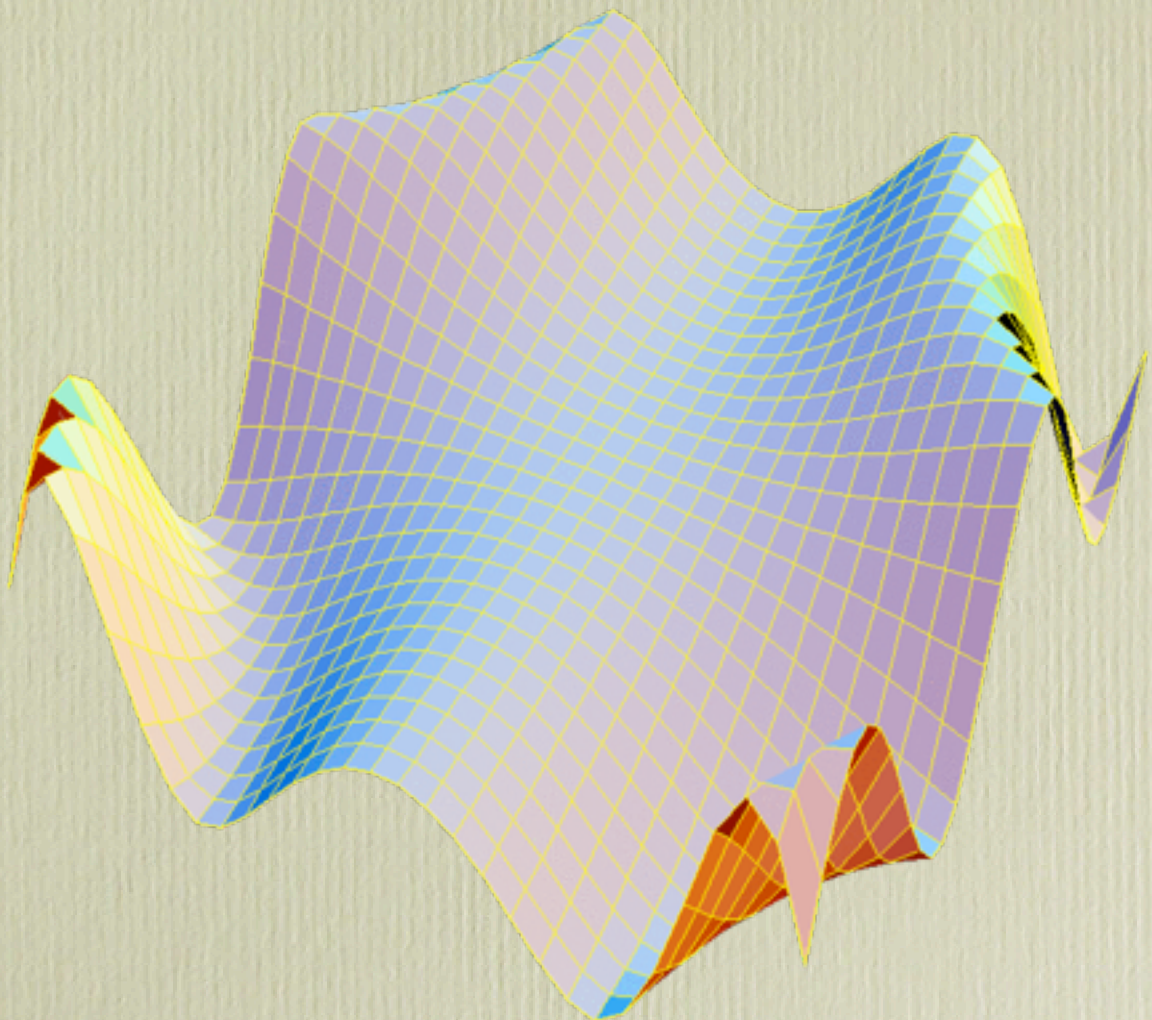


Spheres



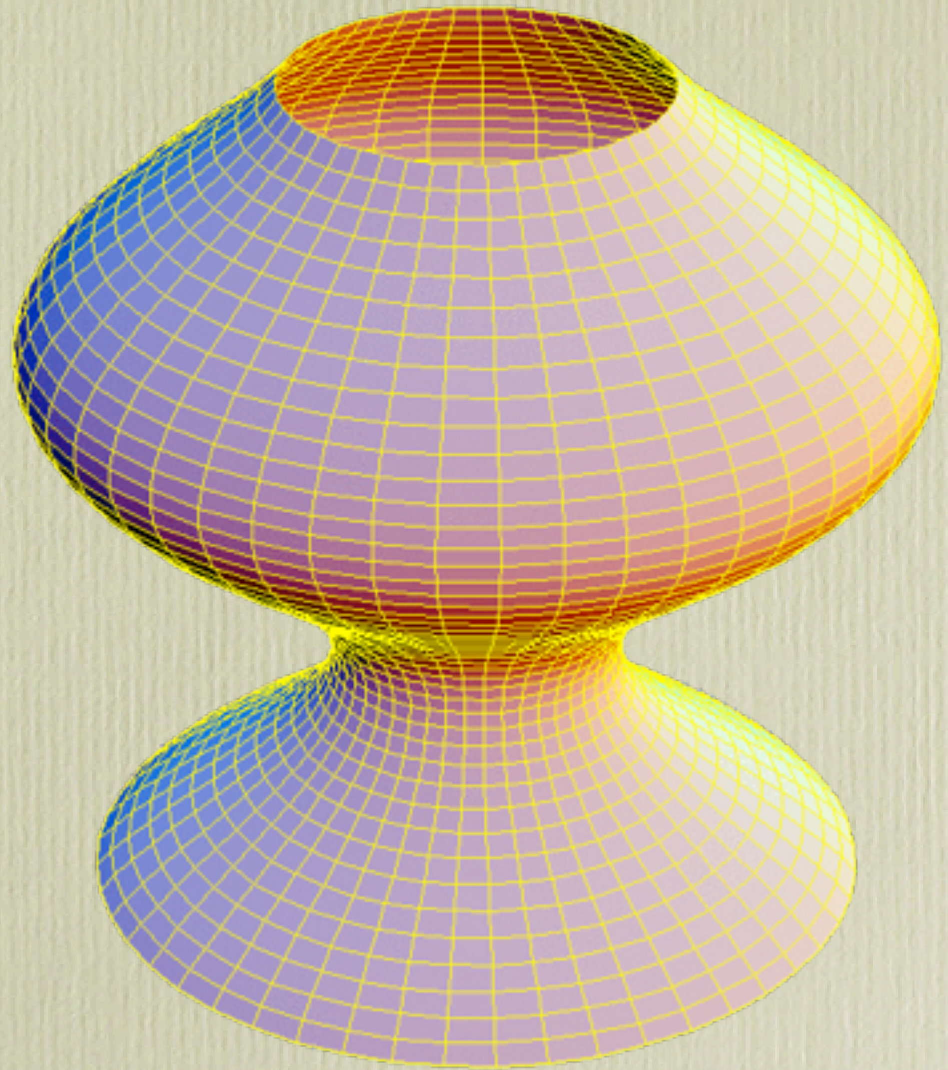
$$\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$$

Graphs



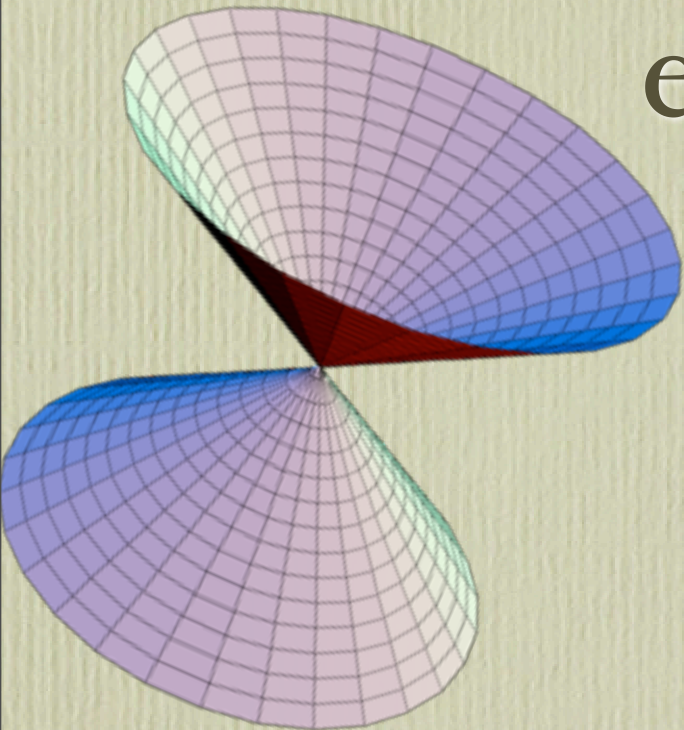
$$\vec{r}\,(u,v) = \langle u,v,f(u,v)\rangle$$

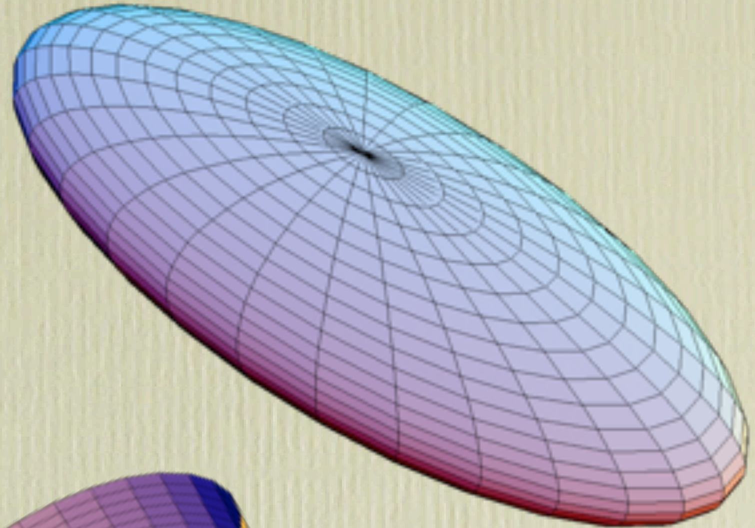
Surfaces of revolution



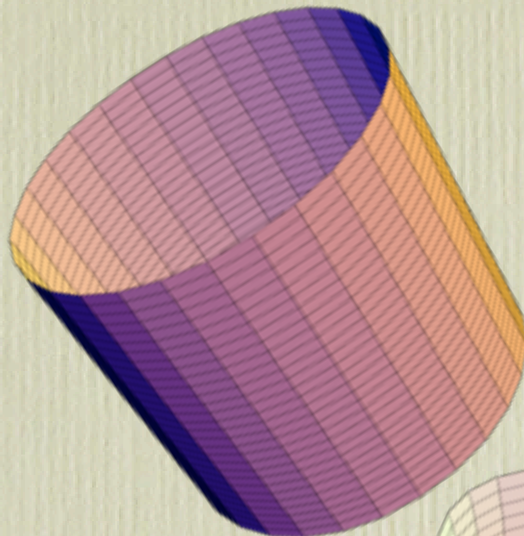
$$\vec{r}(u,v) = \langle r(v)\cos(u),\ r(v)\sin(u),\ v \rangle$$

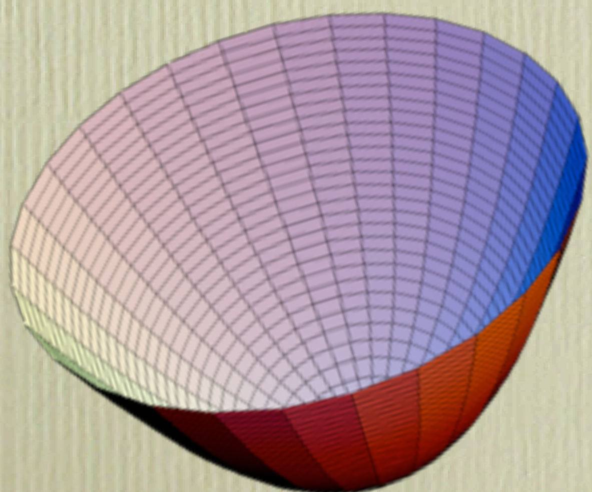
More examples

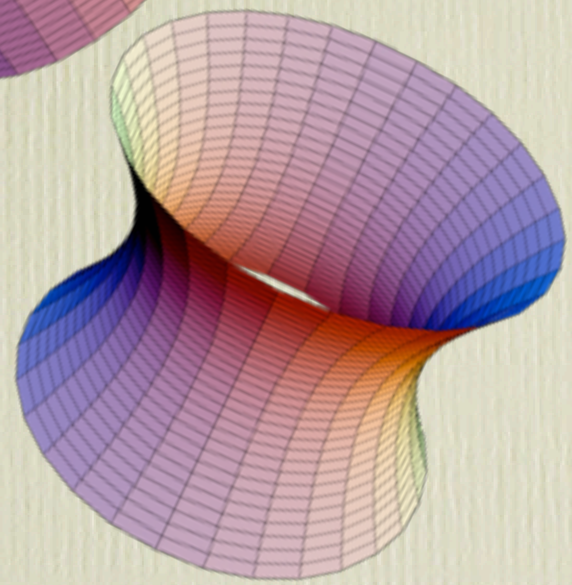




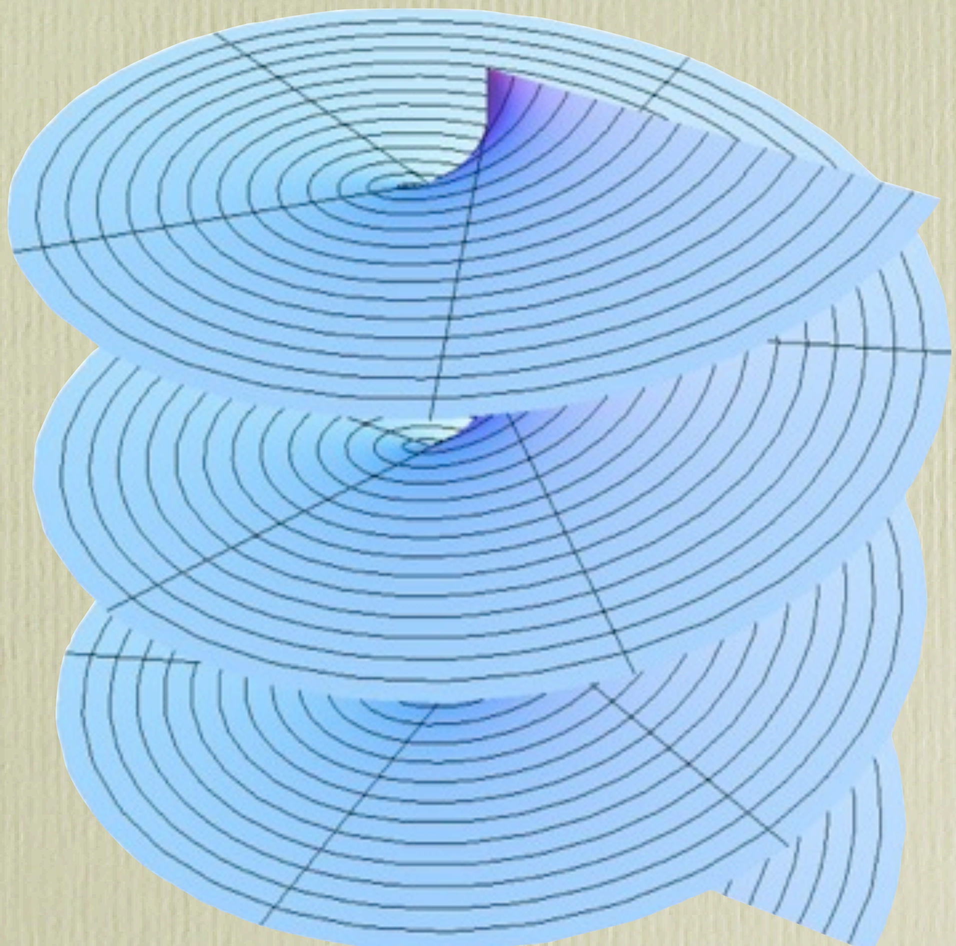
- Ellipsoids
- Cone
- Cylinder
- Paraboloids
- Hyperboloids





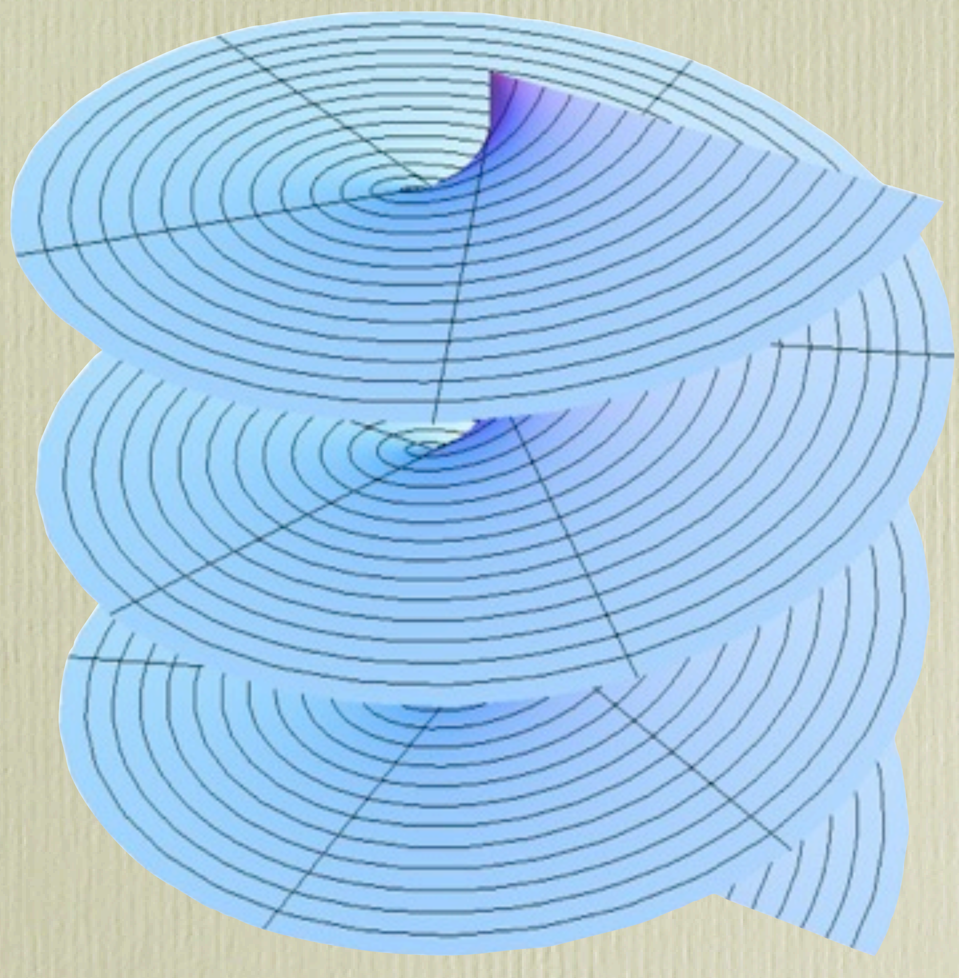


Problem: What is this?

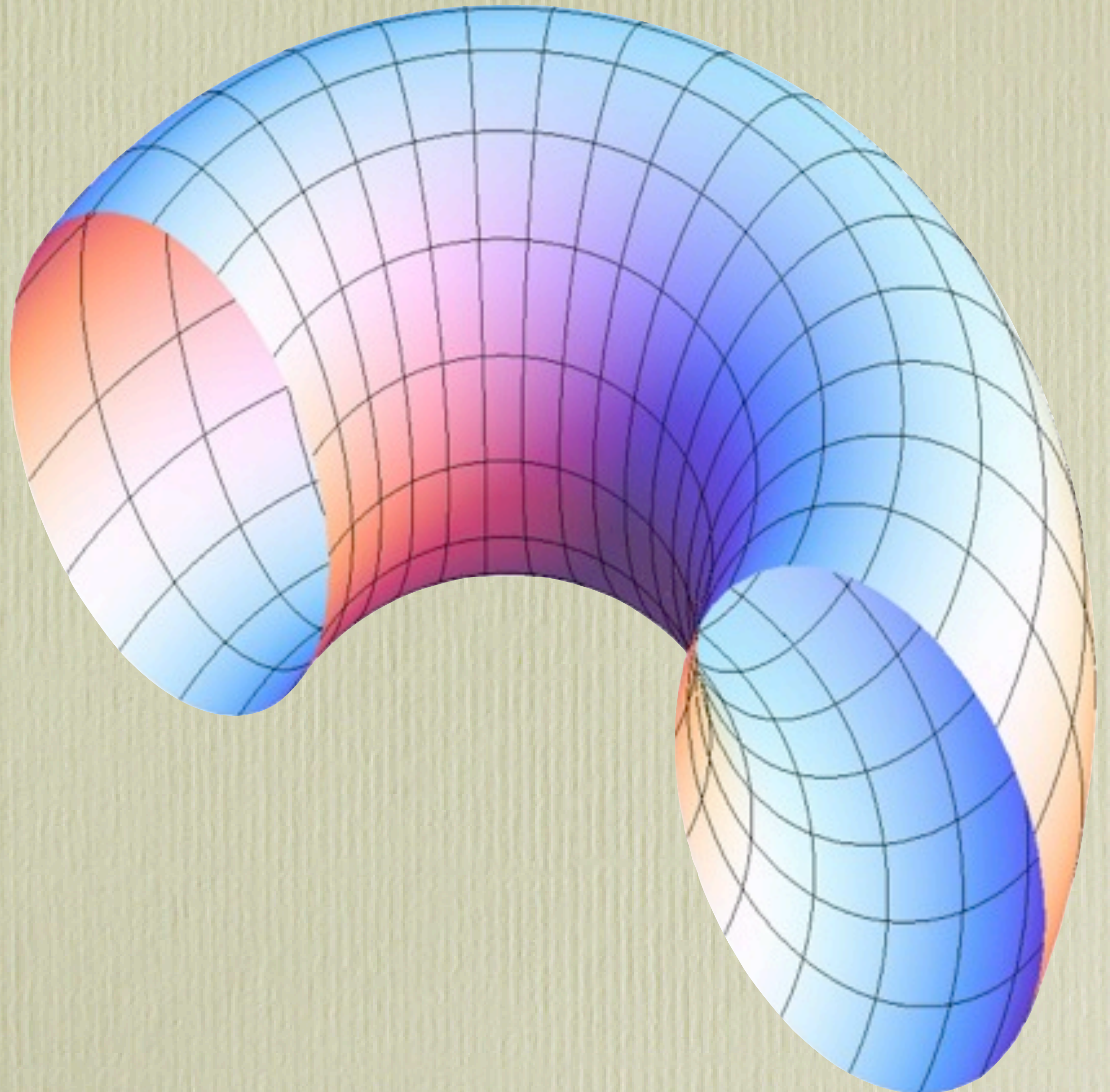


Problem: What is this?

$$\vec{r}\,(u,v) = \langle v\cos(u),\ v\sin(u),\ u \rangle$$

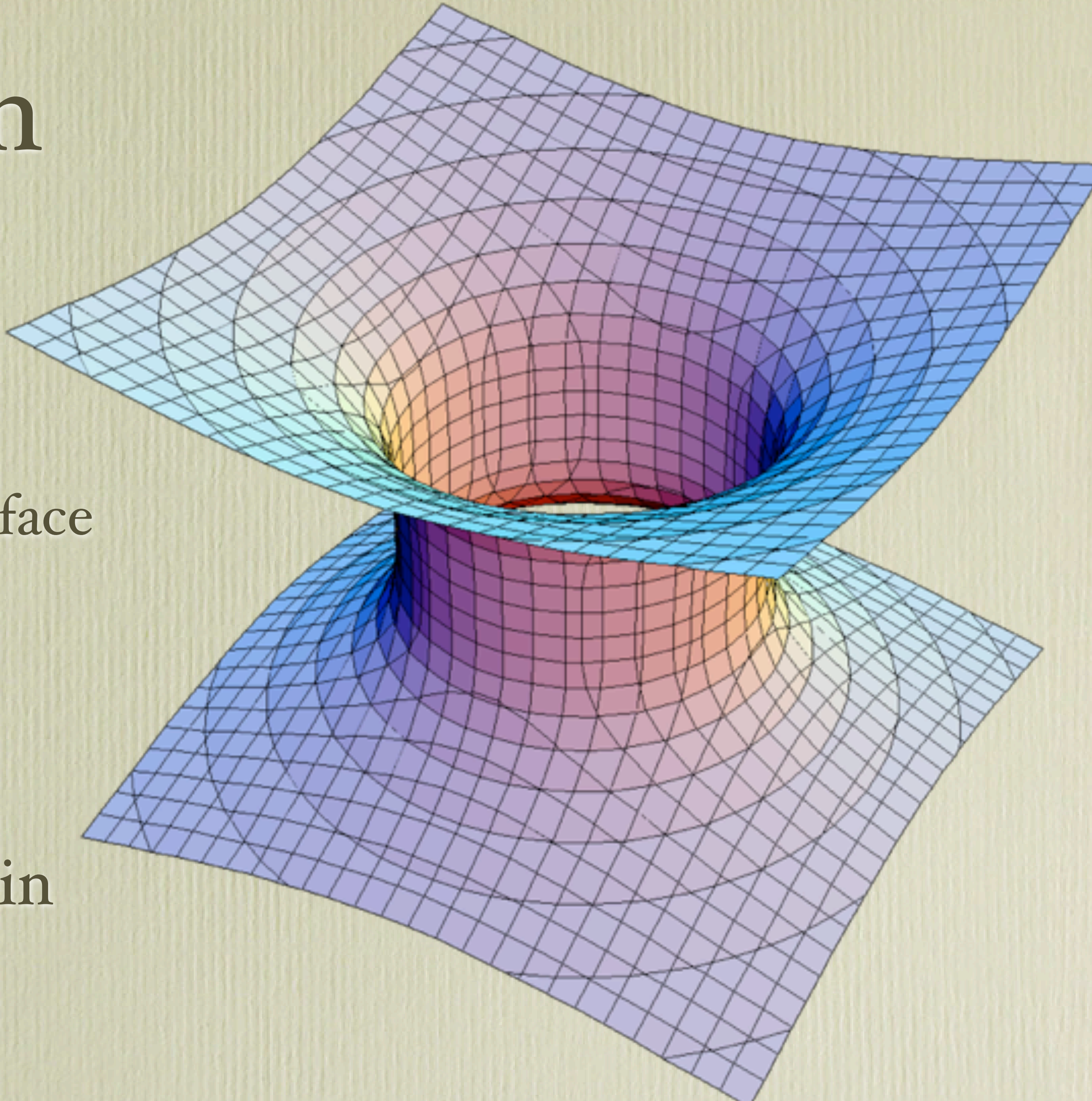


Example:



$\vec{r}(u,v) = \langle (2+\cos(v))\cos(u), (2+\cos(v))\sin(u), \sin(v) \rangle$

Problem



Parametrize the surface

$$x^2 + y^2 - z^4 = 1$$

and describe it in cylindrical coordinates.