7/28/2011 SECOND HOURLY **Maths 21a, O.Knill, Summer 2011**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [✓ T] F The partial differential equation $u_t = 5u_x$ is called a transport equation.

Solution:
This was a knowledge question. Since this year in lecture, I had introduced the transport equation without a constant, also "false" was marked correct here if $u_t = u_x$ was taken as the definition of the transport equation.

2) T [✓ F] If $f(x, y)$ has a saddle point at $(0, 0)$ and $g(x, y)$ has a saddle point at $(0, 0)$ then the function $f(x, y) + g(x, y)$ has a saddle point at $(0, 0)$.

Solution:
Take $f(x, y) = 2x^2 - y^2$ and $g(x, y) = 2y^2 - x^2$. Then $f(x, y) + g(x, y) = x^2 + y^2$ has a local minimum. There are also cheaper counter examples like $g = -f$ in which case $f + g = 0$.

3) T [✓ F] A function $f(x, y)$ with a two local maxima also also a local minimum.

Solution:
Take the 'camel function' with two hills. There are two maxima and a saddle point but no other critical points.

4) T [✓ F] If $f_{xyy} = f_{yxx}$ holds everywhere in the plane then f is constant.

Solution:
Take $f(x, y) = x + y$ for example. It is not constant but satisfies the partial differential equation.

5) [✓ T] F The tangent line to the function $x^2 + y^2 = 5$ at the point $(2, 1)$ is $4x + 2y = 10$.

Solution:
The gradient is $\langle 2x, 2y \rangle$ which is $\langle 4, 2 \rangle$ at the point. The tangent line is $4x + 2y = d$ with some constant d which we can get by plugging in the point $(2, 1)$ showing $d = 10$.

6) T [✓ F] Fubini's theorem implies that for any function $f(x,y)$ of two variables, we have $\int_0^1 \int_3^4 f(x,y)\,dxdy = \int_0^1 \int_3^4 f(y,x)\,dxdy$.

Solution:
A counter example is $f(x,y) = y$.

7) [✓ T] F If a function $f(x,y)$ has a local maximum at $(0,0)$ then $D_{\vec{v}}f(0,0) = 0$ for every unit vector v.

Solution:
The directional derivative is $\nabla f(0,0) \cdot \vec{v}$ and $\nabla f(0,0)$ is the zero vector.

8) T [✓ F] If $f(x,y) = \sqrt{x^2+y^2}$ then $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dxdy$ is the area of the unit disc.

Solution:
Only if we write this in polar coordinates, we would have to add a $r = \sqrt{x^2+y^2}$ integration factor.

9) [✓ T] F If $f(x,y,z) = c$ contains all the coordinate planes, then $(0,0,0)$ is a critical point of $f(x,y,z)$.

Solution:
The gradient must be perpendicular to the three surfaces. It must therefore be zero. This was similar to one of the toblerone chocolate questions in the review.

10) T [✓ F] If $f_{xx}(x_0,y_0)$ is positive at a critical point (x_0,y_0) then f has a local minimum at (x_0,y_0).

Solution:
This can also happen at a saddle point.

11) T [✓ F] If $\nabla g = 0$, then f has is a local maximum or minimum of f under the constraint $g = 0$.

Solution:
Take $f(x,y) = x$ and $g(x,y) = x^2 + y^2$ which gives us a counter example.

12) [✓ T] F

By writing $\nabla f = \langle a,b \rangle$, we see the inequality $-\sqrt{a^2+b^2} \leq a\cos(\theta) + b\sin(\theta) \leq \sqrt{a^2+b^2}$.

Solution:
This follows from steepest decent.

13) T [✓ F]

If $f(x,y,z)$ has a critical point under the constraint $g(x,y,z) = 0$, then we have infinitely many solutions. Only if we add two constraints like $g(x,y,z) = 0, h(x,y,z) = 0$ we get finitely many solutions.

Solution:
No, we can have Lagrange problems with arbitrarily many variables and only one constraint and finitely many solutions.

14) [✓ T] F

If the entropy $-x\log(x) - y\log(y) - z\log(z) = S(x,y,z)$ is maximal under the constraint $x+y+z = 1$, then $x = y = z$.

Solution:
We have seen this in class in more generality.

15) T [✓ F]

The equation $u_{tt} = u^2 u_t$ is a partial differential equation, not an ordinary differential equation.

Solution:
This is an ordinary differential equation because it involves only derivatives with respect to one variable.

16) T [✓ F]

The integral $\int\int_R |\nabla f(x,y)| \; dxdy$ is equal to the surface area of the graph of f above the region R.

Solution:
Take the constant function $f = 0$ for example. The surface area is the area of R but the given integral is zero.

17) T [✓ F]

Fubini's theorem assures that $\int_0^1 \int_0^\theta r dr d\theta = \int_0^{\pi/2} \int_0^r r \; d\theta dr$.

Solution:
It is not only the upper boundary point $\pi/2$ which is wrong but it is also the wrong triangle. Make a picture of the type I and type II region in the r-θ plane to see it more clearly.

18) [✓ T] F The region $x^2 + y^2 \leq 1, x > 0$ is both a type I and type II region.

Solution:
It is a half circle which can both be treated as a type I and type II region.

19) T [✓ F] The partial differential equation $u_t u_x + u_{xx} = 0$ is called the Burgers equation.

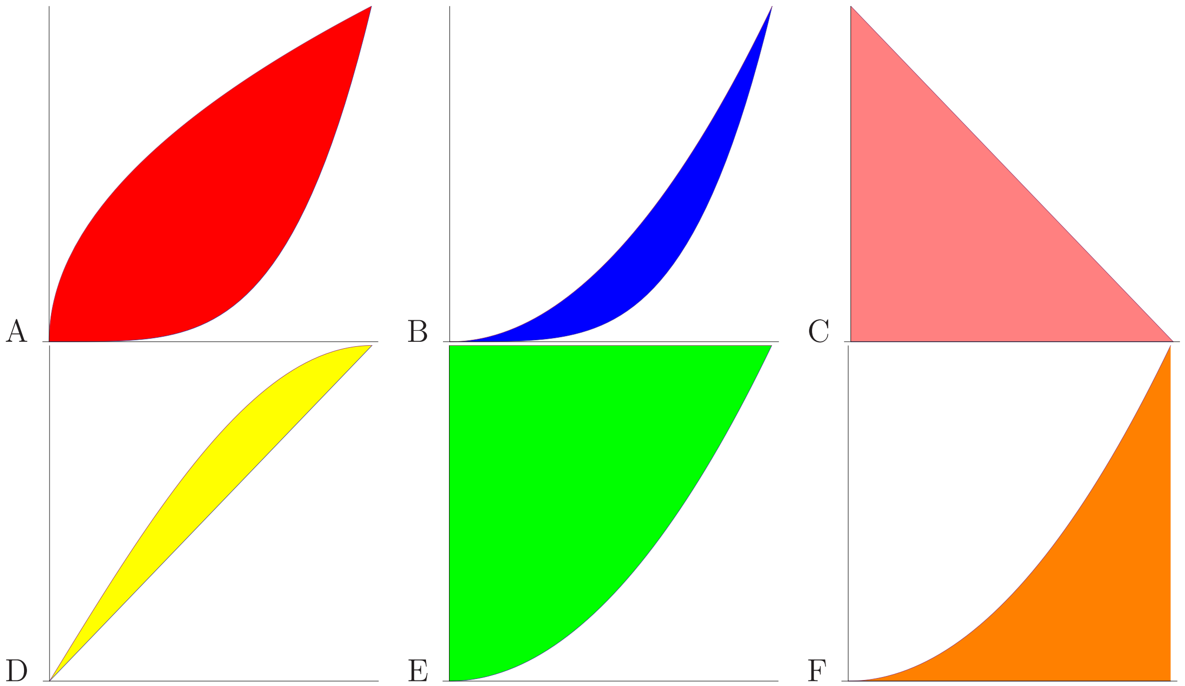
Solution:
The Burger's equation is $u_t = uu_x + u_{xx}$.

20) [✓ T] F Let (x_0, y_0) be a maximum of $f(x, y)$ under the constraint $g(x, y) = 0$. Then$|\nabla f(x_0, y_0) \cdot \nabla g(x_0, y_0)| = |\nabla f(x_0, y_0)| \cdot |\nabla g(x_0, y_0)|$.

Solution:
The two vectors are parallel. The cos-formula now assures that the dot product absolute value is the product of the lengths of the vectors because $\cos(\alpha) = -1$ or $\cos(\alpha) = 1$

Problem 2) (10 points) No justifications are needed

a) (6 points) Match the regions with the double integrals. If none applies, put O.



Enter A-F	Integral of Function $f(x,y)$
	$\int_0^1 \int_x^{\sin(\pi x/2)} f(x,y)\ dydx$
	$\int_0^1 \int_0^{x^2} f(x,y)\ dxdy$
	$\int_0^1 \int_{x^4}^{\sqrt{x}} f(x,y)\ dydx$

Enter A-F	Integral of Function $f(x,y)$
	$\int_0^1 \int_{x^2}^1 f(r,\theta) r\ drd\theta$
	$\int_0^1 \int_{x^4}^{x^2} f(x,y)\ dxdy$
	$\int_0^1 \int_0^{1-x} f(x,y)\ dydx$

b) (4 points) Various concepts are related to the gradient. Match them. Each of the formulas $A-F$ match exactly one spot:

Concept related to gradient	Enter A-F
critical point	
estimation	
directional derivative	
linearization	
tangent space	
chain rule	

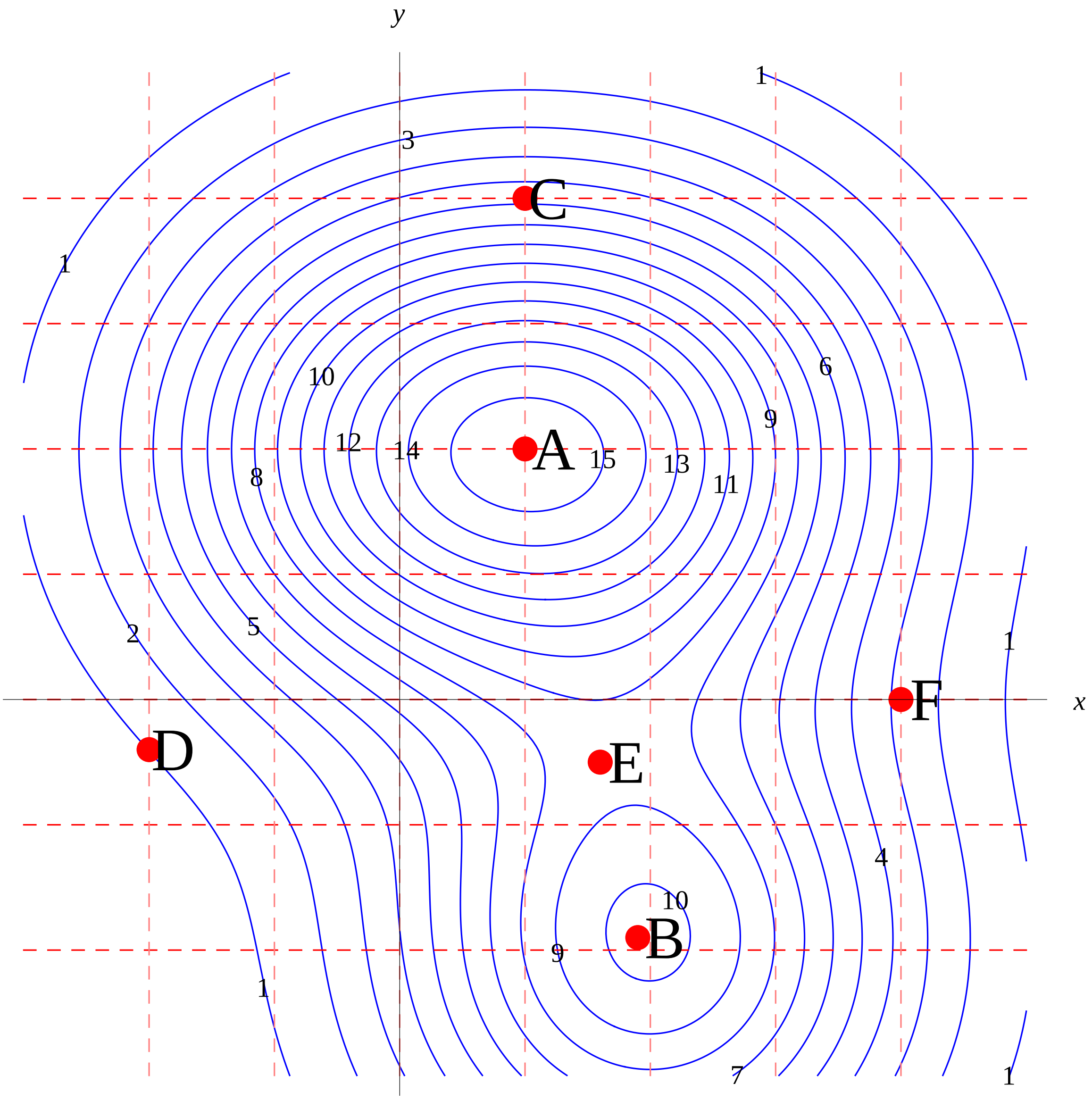
	Formula or notation
A	$L(x,y) = L(x_0,y_0)$
B	$f(x+a),y+b) \sim L(x+a,y+b)$
C	$\nabla f(x,y) = (0,0)$
D	$D_{\vec{v}} f$
E	$\frac{d}{dt} f(\vec{r}(t))$
F	$L = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot \langle x-x_0, y-y_0 \rangle$

Solution:
a) DOA, OOC.
The formulas which lead to O do not make sense. They either involve different variables or the wrong variables.
b) $CBDFAE$.

Problem 3) (10 points) (no justifications are needed)

A function $f(x, y)$ of two variables is shown as a contour map. You see equally spaced level curves. Check whatever applies. In each line there could be multiple points to be checked or zero checks. In the last question, there is only one check.

	A	B	C	D	E	F
points which are local maxima						
points which are local minima						
points which are saddle points						
points, where $f_x > 0$						
points, where $f_x < 0$						
points, where $f_y < 0$						
points, where $f_y > 0$						
global maxima on the region shown						
point among $\{A, B, C, D, E, F\}$ with maximal $\|\nabla f\|$						



Solution:
It is helpful in this type of problems to draw the gradients at the points using the fact that the gradient points into the direction of steepest increase.

	A	B	C	D	E	F
points which are local maxima	*	*				
points which are local minima						
points which are saddle points					*	
points, where $f_x > 0$				*		
points, where $f_x < 0$						*
points, where $f_y < 0$				*		
points, where $f_y > 0$			*			
global maxima on the region shown	*					
point among $\{A, B, C, D, E, F\}$ with maximal $\|\nabla f\|$			*			

Problem 4) (10 points)

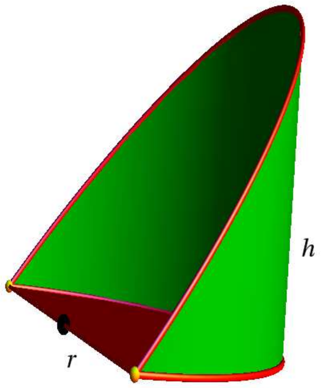
The **hoof of Archimedes** has the volume

$$V(r, h) = 2r^2h/3 \ .$$

The surface area without the top is

$$A(r, h) = \pi r^2/2 + 2rh \ .$$

Find the hoof which has minimal volume $V(r, h)$ under fixed surface area $A(r, h) = \pi/2$.



Solution:
The Lagrange equations are

$$\begin{aligned} 4rh/3 &= \lambda(\pi r + 2h) \\ 2r^2/3 &= \lambda 2r \\ \pi r^2/2 + 2rh &= \pi/2 \end{aligned}$$

Because r can not be zero we have

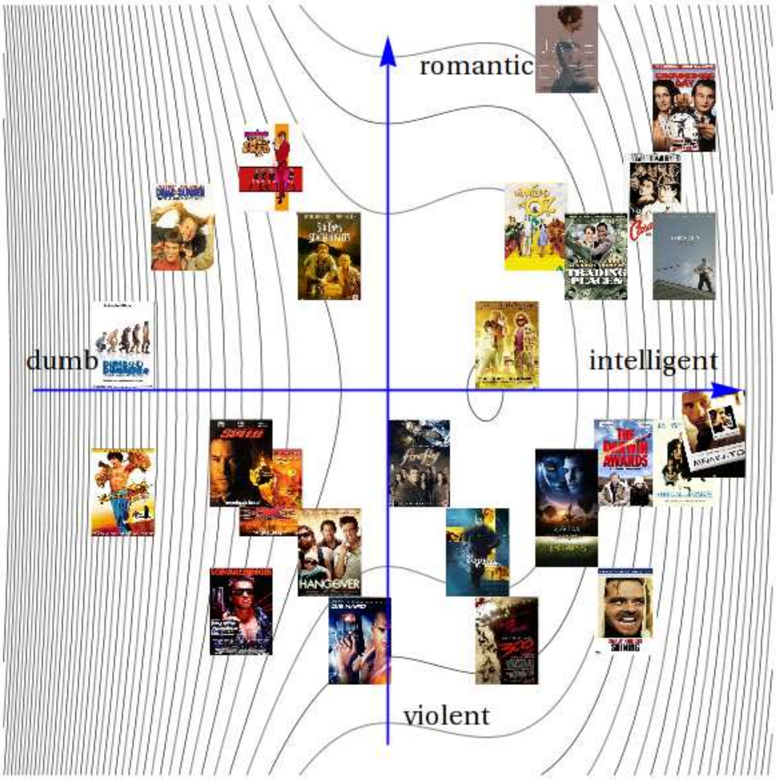
$$\begin{aligned} 4rh/3 &= \lambda(\pi r + 2h) \\ 2r/3 &= 2\lambda \\ \pi r^2/2 + 2rh &= \pi/2 \end{aligned}$$

Divide the first equation by the second to get $2h = \pi r/2 + h$ showing that $h = \pi r/2$. Plug this into the last equation to get $\boxed{r = 1/\sqrt{3}}$ and so $\boxed{h = \pi/(2\sqrt{3})}$.

Solution:
$1/\sqrt{3}$.

Problem 5) (10 points)

Companies like **Netflix** or **Hulu** track movie preferences. One can visualize preferences on parameter spaces. Let the x axes be the **intelligence - emotion** plane. Based on viewing habits, the service decides what you want to see. Your profile is a function $f(x, y)$. Maximizing this function allows the company to pick movies for you. Assume that your user profile is the function $f(x, y) = -2x^3 + 9x^2 - 12x - y^2$. Find and classify all the critical points and especially find the local maxima of f.



Solution:

x	y	D	f_{xx}	Type	f
1	0	-12	6	saddle	-5
2	0	12	-6	maximum	-4

The point $(1, 0)$ is a saddle point and the maximum is at $\boxed{(2, 0)}$.

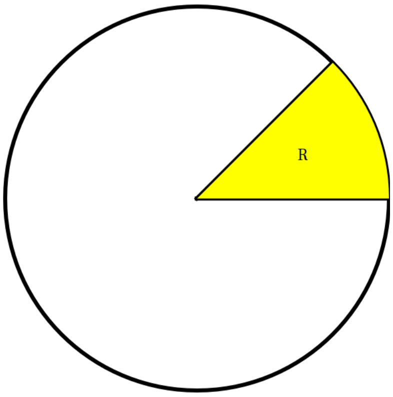
Remark. This problem was motivated by an article in "What's new in Mathematical Sciences, AMS, 2010". The graphics was produced in Mathematica which allows to include and place pictures in graphics (probably with less time effort than any photoshop job could do). The pictures files were obtained from DVD covers as displayed in the IMBd database.

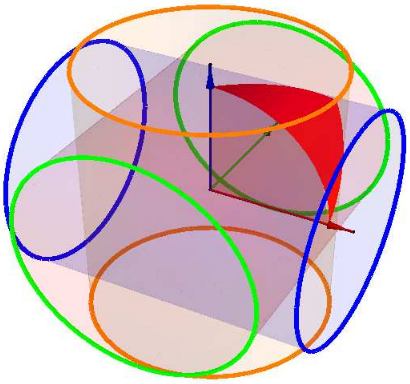
Problem 6) (10 points)

Find the surface area of the graph of $f(x,y) = \sqrt{1-x^2}$ over the sector region

$R = \{x^2 + y^2 \leq 1, y > 0, y \leq x\}$.

Remark: Multiply this result by 16 and you have computed the surface of the solid which is obtained by intersecting the three cylinders $x^2+y^2 \leq 1, x^2+z^2 \leq 1$ $y^2+z^2 \leq 1$.





Solution:
We parametrize the surface as $r(u,v) = \langle u, v, \sqrt{1-u^2}\rangle$. Then compute $r_u(u,v) = \langle 1, 0, -u/\sqrt{1-u^2}\rangle$. $r_v(u,v) = \langle 0,1,0\rangle$ and $r_u \times r_v = \langle u/\sqrt{1-u^2}, 0, 1\rangle$. The length of this is the surface area $|r_u \times r_v| = 1 + u^2/(1-u^2) = 1/(1-u^2)$. We have to integrate this function over the sector region R. One can solve this integral both in Cartesian and Polar coordinates. Both have been found by the class:
Here is a solution with Cartesian coordinates:

$$\begin{aligned} I &= \int_0^{1/\sqrt{2}} \int_0^x \frac{1}{1-x^2} \, dydx + \int_{1/\sqrt{2}}^1 \int_0^{\sqrt{1-x^2}} \frac{1}{1-x^2} \, dydx \\ &= \int_0^{1/\sqrt{2}} \frac{x}{(1-x^2} \, dx + \int_{1/\sqrt{2}}^1 1 \, dx = (1 - \frac{1}{\sqrt{2}}) + (1 - \frac{1}{\sqrt{2}}) = 2 - \sqrt{2} \, . \end{aligned}$$

Here is a solution using polar coordinates:

$$\begin{aligned} I &= \int_0^{\pi/4} \int_0^1 \frac{r}{\sqrt{1 - r^2\cos^2(\theta)}} \, drd\theta = \int_0^{\pi/4} \frac{(-1)}{\cos^2(\theta)} \sqrt{1 - r^2\cos^2(\theta)}|_0^1 \, d\theta \\ &= \int_0^{\pi/4} \frac{1}{\cos^2(\theta)} - \frac{\sin(\theta)}{\cos^2(\theta)} \, d\theta = \tan(\theta) - \frac{1}{\cos(\theta)}|_0^{\pi/4} = (1-0) - (\sqrt{2}-1) = 2 - \sqrt{2} \, . \end{aligned}$$

The answer is $\boxed{2-\sqrt{2}}$. Unfortunately, most were hung up much earlier and computed the volume by integrating $f(u,v) = \sqrt{1-u^2}$, or $|\nabla f(u,v)|$ which both do not give the surface area. As you see from the computations, finding the integrals was not so easy. A Cartesian approach needed to split up the region R. The polar integration requires some integration skills when considering the time constraints and no computer assistance. Full credit was given here if the integral was written down correctly. The few students who have got through gained no additional points but lots of admiration.

Problem 7) (10 points)

Estimate $f(0.999, 0.01) = 0.999^{10} \cdot (1 + \sin(0.01))$ by linear approximation.

Solution:
The first task is to find the function $f(x, y)$ which allows us to estimate this. The function is

$$f(x, y) = x^{10}(1 + \sin(y)) \ .$$

Next we compute the gradient

$$\nabla f(x, y) = \langle 10x^9(1 + \sin(y)), x^{10} \cos(y) \rangle$$

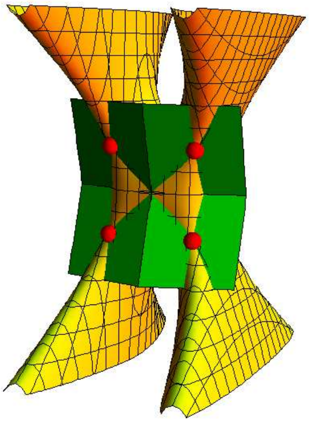
and evaluate it at the point $(1, 0)$ at which we obviously want to make the linear approximation. The gradient is there

$$\nabla f(1, 0) = \langle 10, 1 \rangle \ .$$

We also compute $f(1, 0) = 1$. The linearization is

$$f(1, 0) + \nabla f(1, 0) \cdot \langle 0.999 - 1, 0.01 - 0 \rangle = 1 + 10(0.999 - 1) + 1(0.01 - 0) = 1 \ .$$

Problem 8) (10 points)



Part of a "transformer" action figure given by the surface

$$-2x^2y^2 + x^6 + y^2 + 3z^2 = 3 \ .$$

In order to put some armor around the body, we have to find the tangent planes at the "arm pits" $(1, -1, -1), (1, 1, -1)$ and "kidney regions" $(-1, -1, -1), (-1, 1, -1)$.
Find the tangent plane at the left armpit $(1, -1, -1)$.

Solution:
The surface is given as $f(x,y,z)=3$. We compute the gradient of f:

$$\nabla f(x,y,z) = \langle -4xy^2 + 6x^5, -4x^2y + 2y, 6z \rangle$$

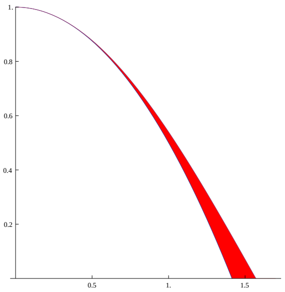
which is evaluated at the point $(1,-1,-1)$ equal to

$$\nabla f(1,-1,-1) = \langle 2,-2,6 \rangle \ .$$

The equation of the tangent plane is therefore $2x + 2y + (-6)z = d$ for some constant d which we can determine by plugging in the point $(1,-1,-1)$. We get $\boxed{2x + 2y - 6z = 6}$.

Problem 9) (10 points)

a) (5 points) Since the graph of $1-x^2/2$ is always below the graph of $y = \cos(x)$, we can ask about the area of the "eye lash" region between the two graphs and the x axes. Write down a double integral which gives the area of the region above the positive x axes. You do not have to evaluate the integral in this part of the problem.



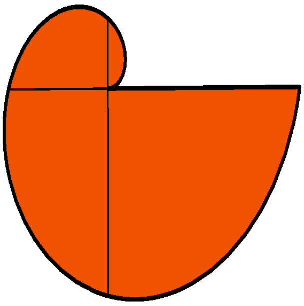
Solution:
a) If you look closely you see that when writing the integral as a type I region, we would have to break it up. As a type II integral it is

$$\int_0^1 \int_{\sqrt{(2-2y)}}^{\arccos(y)} 1 \ dxdy \ .$$

b) (5 points) Find the "mini" moment of inertia

$$\int \int_R \sqrt{x^2+y^2} \ dxdy$$

of the region R bound by the **Archimedean spiral** given in polar coordinates as $r \leq \theta$ and the x axes. You **should** of course evaluate the integral in this part of the problem.



Solution:
b) This is an integral we do in polar coordinates

$$\int_0^{2\pi} \int_0^{\theta} r^2 \, d\theta dr = \int_0^{2\pi} \theta^3/3 \, d\theta = \frac{\theta^4}{12}|_0^{2\pi} = \frac{16\pi^4}{12} = \frac{4\pi^4}{3} \, .$$

Remark: The actual moment of inertia would have been $\int\int_R x^2 + y^2 \, dxdy$ and could have been computed in the same way:

$$\int_0^{2\pi} \int_0^{\theta} r^3 \, d\theta dr = \int_0^{2\pi} \theta^4/4 \, d\theta = \frac{\theta^5}{20}|_0^{2\pi} = \frac{32\pi^4}{20} = \frac{8\pi^5}{5} \, .$$