8/9/2012 FINAL EXAM **Maths 21a, O. Knill, Summer 2012**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F The dot product between $\langle 1,1,2\rangle \cdot \langle 2,3,4\rangle$ is 13.

Solution:
True by definition.

2) T F The distance between a line $\vec{r}(t) = \vec{Q} + t\vec{v}$ with unit vector $\vec{v}$ and a point P is given by $|\vec{PQ} \times \vec{v}|$.

Solution:
Yes, this is the distance formula in the case $|\vec{v}| = 1$.

3) T F The TNB frame consists of the vectors $\vec{r}_u, \vec{r}_v, \vec{r}_u \times \vec{r}_v$.

Solution:
The TNB frame is attached to a curve not a surface.

4) T F The surface $x^2 + y^2 - z^2 = -1$ is a one sheeted hyperboloid.

Solution:
It is a two sheeted hyperboloid.

5) T F The equation $u_{tt} + uu_x = u_{xx}$ is called the Burgers equation.

Solution:
The Burgers equation is $u_t + uu_x = u_{xx}$.

6) T F The fundamental theorem of line integrals assures that the line integral of any vector field along a closed loop is zero.

Solution:
It needs to be a gradient field.

7) T **F** If $\vec{r}(t)$ is a curve in space then $\nabla \vec{r}(t)$ is a vector perpendicular to the curve.

Solution:
We have not defined the gradient of a vector valued function.

8) T **F** If $\nabla f(3,1) = \langle 0,0 \rangle$ and $f_{xx}(3,1) > 0$, then $(3,1)$ is a local minimum of f.

Solution:
We need also $D > 0$.

9) **T** F Given two points P, Q in space and two lines L, M where L goes through P and M goes through Q. The distance between P, Q is larger or equal than the distance between the two lines.

Solution:
Indeed, the distance between two lines is defined as the minimal distance which two points on the line can have.

10) T **F** The equation $u_t - u_x = u_{xx}$ is called the heat equation.

Solution:
This is a fantasy equation

11) T **F** The flux of the curl of $\vec{F}$ through the surface S is positive, where S is the surface $x^2 + y^2 + z^2 = 1$ oriented outwards.

Solution:
It is zero

12) T **F** The dot product between two parallel vectors is always zero.

Solution:
It is the cross product which is zero for parallel vectors.

13) T **F** $\vec{r}(u,v) = \langle u, u, 0 \rangle$ parametrizes a surface S which is a cylinder.

Solution:
This is a curve not a surface.

14) **T** F The length of the gradient $|\nabla f|$ is always minimal at critical points.

Solution:
Indeed, it is zero at critical points and positive away from critical points.

15) T **F** The triple integral $\int\int\int_E \text{div}(\vec{F}(x,y,z))\ dxdydz$ over a sphere E is always zero since the flux of $\vec{F}$ through the boundary surface is zero.

Solution:
The explanation sounds convincing since stated with confidence but it is false. If F were the curl of a vector field, then this would be true.

16) **T** F Assume $\vec{r}(t)$ is a path of length 1 parametrized on $[a,b]$, then $\int_a^b |\vec{r}\,'(t)|\ dt = 1$.

Solution:
This is the definition of arc length.

17) **T** F If $\vec{r}\,'(t) = \langle 2t, 1-2t \rangle$ and $\vec{r}(0) = \langle 2, 3 \rangle$, then $\vec{r}(t) = \langle 2+t^2, 3+t-t^2 \rangle$.

Solution:
Yes, just differentiate $\vec{r}(t)$ and check the initial condition.

18) **T** F For any two unit vectors $\vec{v}, \vec{w}$ we have $|\vec{v} \times \vec{w}|^2 + (\vec{v} \cdot \vec{w})^2 = 1$.

Solution:
Use the formulas for the length of the cross product and the dot product.

19) T F The directional derivative $D_v(f)$ is always perpendicular to the vector $\vec{v}$ and to the surface $f = c$.

Solution:
The directional derivative is a scalar.

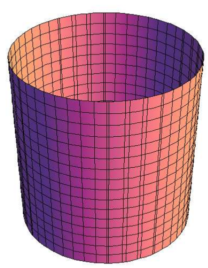
20) T F $\langle 1,0,0 \rangle \cdot (\langle 1,1,0 \rangle \times \langle 0,0,1 \rangle) = 1/6$.

Solution:
Direct computation or seeing that this is 1.

Problem 2) (10 points) No justifications are necessary.

Match the following objects.

Formula	Enter 1-9
$\rho \leq (1+\sin(\phi+\theta))$	
$x^2+y^2+z^2+\sin(x^2y^2z^2)=1.$	
$r=t+(2\pi-t)+\cos(5\theta), z=0.$	
$\vec{F}(x,y)=\langle x,y^2\rangle$	
$(x-5)^2+(y-3)^2+(z+1)^2=1$	
$x^2+y^2=3$	
$\vec{r}(t)=\langle\cos(t),\sin(3t),\sin(2t)\rangle$	
$\vec{r}(u,v)=\langle(3+\sin(u)\cos(v))\cos(u)),(3+\sin(u)\cos(v))\sin(u)),(1+\sin(u))\sin(v)\rangle$	
$\vec{F}(x,y,z)=\langle x,y^2,z^3\rangle$	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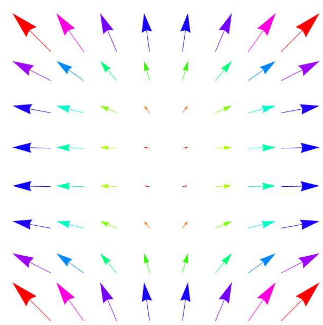
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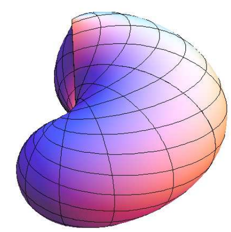
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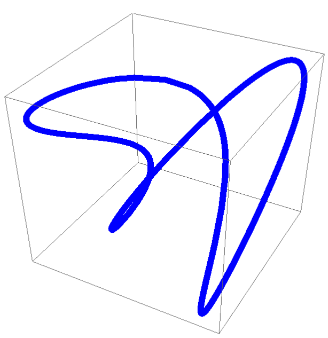
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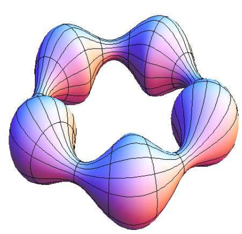
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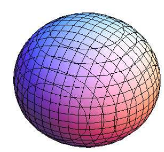
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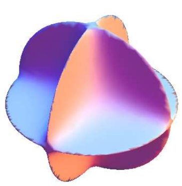
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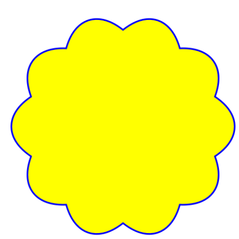
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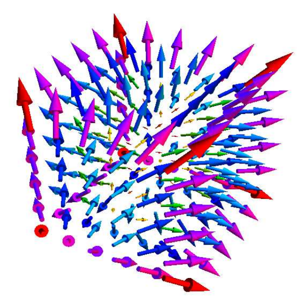
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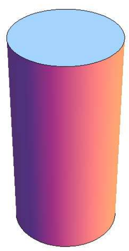
Solution:
3,7,8,2,6,1,4,5,9, Since 7 was an unusual cut, also O (no match) was accepted there.

Problem 3) (10 points) No justifications are necessary

a) (3 points) Matching solids.

Formula	Enter A,B,C,D
$x^2 + y^2 + z^2 \leq 1$	
$x^{40} + y^{40} + z^{40} \leq 1$	
$x^2 y^2 z^2 + x^2 + y^2 + z^2 \leq 1$	
$x^2 + y^2 \leq 1, z^2 \leq 1$	

A


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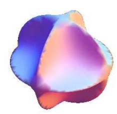

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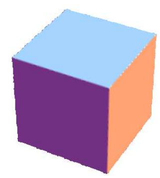
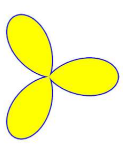


D


b) (3 points) Matching polar regions

Formula	Enter E,F,G,H
$r = \theta$	
$r = \lvert\sin(3\theta)\rvert$	
$r = \sin(\theta)^6$	
$r = 1 + \cos(3\theta)$	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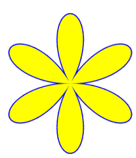
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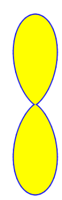
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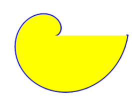
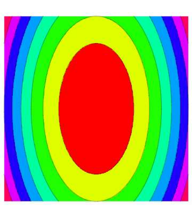

H

c) (3 points) Matching level curves

Formula	Enter I,J,K,L
$f(x,y) = 5y - 1$	
$f(x,y) = x^4 - y^4$	
$f(x,y) = 3x^2 + y^2$	
$f(x,y) = y^2 - x^3$	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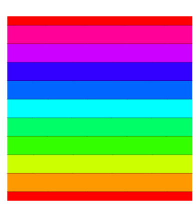
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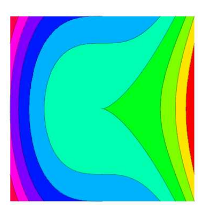
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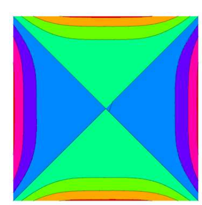
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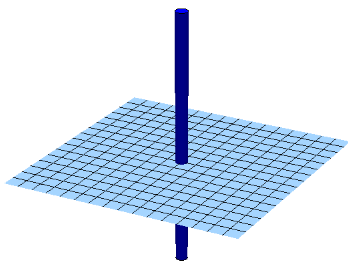
d) (1 point) Partial differential equation

The equation $f_{tt} = f_{xx}$ is called

Heat equation	
Wave equation	
Transport equation	
Burgers equation	

Solution:
a) DCBA
b) HFGE
c) JLIK
d) Wave equation

Problem 4) (10 points)



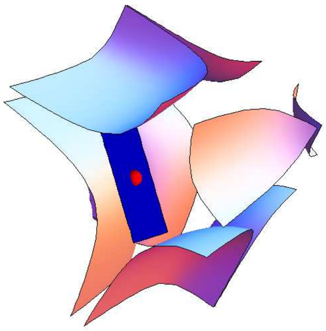
a) (3 points) Find a formula for the distance of a point (x, y, z) to the xy-plane.

b) (3 points) Find a formula for the distance of a point (x, y, z) to the z-axes.

c) (4 points) Find the surface consisting of all points (x, y, z) for which the distance to the z-axes is the same than the distance to the xy plane.

Solution:
a) You can use the distance formula $|\vec{PQ} \cdot \langle 0, 0, 1\rangle|/|\langle 0, 0, 1\rangle|$ with $P = (x, y, z)$ and a point like $Q = (0, 0, 0)$. but also just see that $\boxed{d(x, y, z) = |z|}$.
b) Again you can use a distance formula $\vec{PQ} \times \langle 0, 0, 1\rangle|/|\langle 0, 0, 1\rangle|$ with $P = (x, y, z)$ and $Q = (0, 0, 0)$ and use that $\langle x, y, z\rangle \times \langle 0, 0, 1\rangle = \langle y, -x, 0\rangle$ but its possible to see that $\boxed{d(x, y, z) = \sqrt{x^2 + y^2}}$ directly since the distance from the z axes is the r variable in polar coordinates.
c) $z = \sqrt{x^2 + y^2}$ is equivalent to $g(x, y, z) = z^2 - x^2 + y^2 = 0$ which is a $\boxed{\text{double cone}}$.

Problem 5) (10 points)



a) (5 points) Estimate $2.001^3 \cdot 0.9999^4 \cdot 0.999^2$ using linearization.

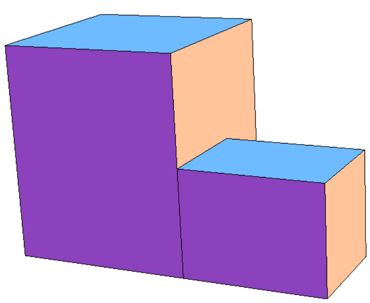
b) (5 points) Find the tangent plane to the surface $x^3y^4z^2 = 8$ at the point $(2, 1, 1)$.

Solution:
a) The gradient is $\nabla f(x, y, z) = \langle 3x^2y^4z^2, 4x^3y^3z^2, 2x^3y^4z \rangle$. At the point $(2, 1, 1)$, it becomes $\nabla f(2, 1, 1) = \langle 12, 32, 16 \rangle$. We have $f(2, 1, 1) = 8$. And so

$$L(x, y, z) = 8 + 12(x - 2) + 32(y - 1) + 16(z - 1) .$$

At the given point this is $8 + 12 * 0.001 - 32 * 0.0001 - 16 * 0.001 = \boxed{7.9928}$. The real value is 7.99279208.... Pretty impressive, you could compute that fast without calculator.
b) The tangent plane has the equation $3x + 8y + 4z = d$ where d can be determined by filling in the given point $(2, 1, 1)$. We have $\boxed{3x + 8y_4z = 18}$.

Problem 6) (10 points)



A **chicken coop** is made of two cubes of length x and y. The volume of the house is $f(x, y) = x^3 + y^3$. The surface area is $g(x, y) = 5x^2 + 3y^2$. Using Lagrange, find the coop of maximal volume if the constraint is $g = 38$.

Solution:
The Lagrange equations are

$$3x^2 = \lambda 10x, 3y^2 = \lambda 6y, 5x^2 + 3y^2 = 38 .$$

Eliminating λ gives $x = (5/3)y$. Plug this into the constraint to get $\boxed{x = 5/2, y = 3/2}$. The maximal coop volume is 19. Of course, also expressions like $y = \sqrt{342/152} = \sqrt{171/76}$ etc are correct.

Problem 7) (10 points)

The roof of the tower of the Harvard **Lovell house** has height

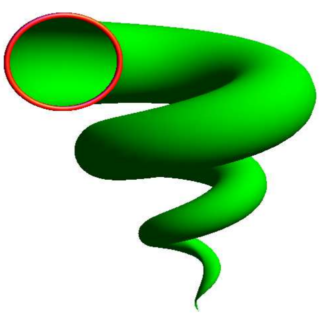
$$f(x,y) = 1 - (x^2 + y^2)^7 .$$

Find the volume under the roof above the disc $x^2 + y^2 \leq 1$ in the xy-plane.

Solution:
Use polar coordinates

$$\int_0^{2\pi} \int_0^1 (1 - r^1 5) r \; dr d\theta = 2\pi(1/2 - 1/6) = 7\pi/8 .$$

Problem 8) (10 points)



What is the flux of the curl of the field $\vec{F}(x,y,z) = \langle 0, z^2 + x^4, x \rangle$ through the shell

$$\langle s(2+\cos(t))\cos(s), s(2+\cos(t))\sin(s), 6s + s\sin(t) \rangle ,$$

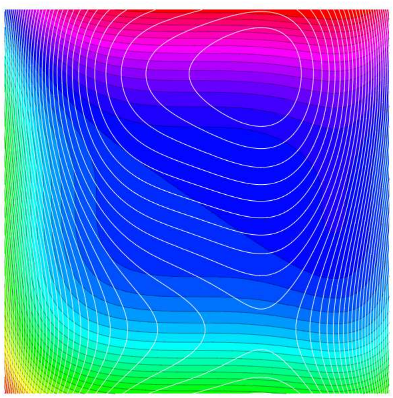
where $0 \leq t \leq 2\pi$ and $0 \leq s \leq 6\pi$. The shell has the boundary curve

$$\vec{r}(t) = \langle 6\pi(2+\cos(t)), 0, 36\pi + 6\pi\sin(t) \rangle .$$

Solution:
This is a Stokes problem. The flux of the curl through the surface is the line integral along the boundary curve. We can apply Stokes again and get the flux $\text{curl}(\vec{F})$ through the disc. The curl is $\langle -2z, -1, 0\rangle$ and the disc is parametrized by $\langle u, 0, v\rangle$ with normal vector $\langle 0, -1, 0\rangle$. We see that the flux is the area of the disc which is $\pi(6\pi)^2 = \boxed{36\pi^3}$. Of course, one can also compute the line integral along the given curve which gives

$$\int_0^{2\pi} 6\pi(2+\cos(t))6\pi\cos(t)\;dt = 36\pi^3 \; .$$

Problem 9) (10 points)



At which point does the function

$$u(x, y) = \frac{2x^3}{3} + 2y^3 - \frac{x^6}{30} - \frac{y^5}{20}$$

have the property that

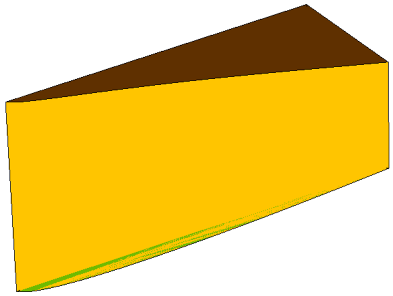
$$f(x, y) = u_{xx}(x, y) + u_{yy}(x, y)$$

is extremal. Find and classify all the critical points of $f(x, y)$.
Remark you can ignore: this problem appears in physics. If the function u is the **electric potential**, then f is the **charge density**. You find the place where the charge density is maximal.

Solution:
The function is $f(x, y) = 4*x - x^4 + 12*y - 3*y^3$. We have to compute the gradient of this and solve $f_x = 0, f_y = 0$ to get the critical points. There are two, one is a maximum, the other is a saddle. There are no global extrema.

x	y	D	f_{xx}	type	f
1	-2	-144	-12	saddle	-13
1	2	144	-12	maximum	19

Problem 10) (10 points)



The following integral gives the volume of a piece of **Swiss cheese**

$$\int_0^3 \int_{\sqrt{y/3}}^1 \int_0^{e^{-x^3}} 1 \, dz \, dx dy \; .$$

Find it.

Solution:
Write it as a type I integral: Because $x = \sqrt{y}/3$ we have $y = 3x^2$ and so

$$\int_0^1 \int_0^{3x^2} e^{-x^3} dy \, dx = \int_0^1 3x^2 e^{-x^3} \, dx = -e^{-x^3}|_0^1 = 1 - \frac{1}{e} \; .$$

Problem 11) (10 points)

While Mars rover **"Curiosity"** was landing on Mars, a force

$$\vec{F}(x, y, z) = \langle \sin(x), y, -30z \rangle$$

acted on the rover while it was 'descending on the path

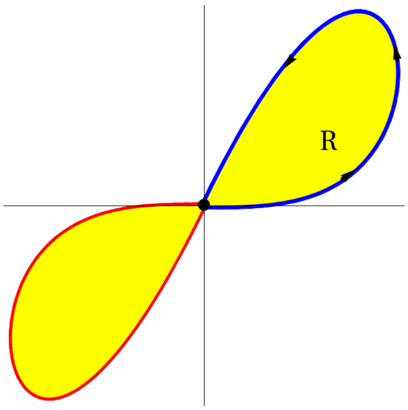
$$\vec{r}(t) = \langle 1, 2t, 10 - t^2 \rangle \; .$$

Find the line integral

$$\int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt \; .$$

Solution:
It was possible to compute the line integral directly. Much easier is to find the potential $f(x, y, z) = -\cos(x) + y^2/2 - 30z^2/2$ and evaluate it at $\vec{r}(2) = \langle 1, 4, 6 \rangle$ and $\vec{r}(0) = \langle 1, 0, 10 \rangle$ and take the difference. The result is $\boxed{968}$.

Problem 12) (10 points)



Find the area of the **propeller** shaped region enclosed by the figure 8 curve

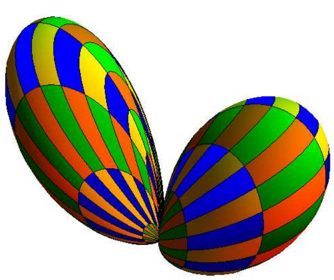
$$\vec{r}(t) = \langle t - t^3, 2t^3 - 2t^5 \rangle \ ,$$

parametrized by $-1 \leq t \leq 1$. To find the total area compute the area of the region R enclosed by the right loop $0 \leq t \leq 1$ and multiply by 2.

Solution:
This is a typical Green problem. We compute the line integral of the vector field $\vec{F}(x, y) = \langle 0, x \rangle$ along the curve

$$2 \int_0^{2\pi} \langle 0, t - t^3 \rangle \cdot \langle 1 - 3t^2, 6y^2 - 10t^4 \ dt = 1/6 \ .$$

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x, y, z) = \langle -y^7, -x^8, -z + x^5 \rangle$$

through the surface given in spherical coordinates as

$$\rho \leq (\sin(\phi)\cos(\phi)\cos^2(\theta))^{1/3}$$

with $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$. The surface is oriented outwards.

Solution:
The divergence is -1. We have therefore to compute the volume of the solid and multiply by -1:

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{(\sin(\phi)\cos(\phi)\cos^2(\theta))^{1/3}} \rho^2 \sin(\phi) \ d\rho d\phi d\theta$$

This gives $\int_0^{\pi/2} [\cos(\phi)\sin^2(\phi)]/3 \ d\phi \int_0^{2\pi} \cos^2(\theta) \ d\theta$ which is $\boxed{\pi/9}$.